Characterization on graphs which achieve a Das’ upper bound for Laplacian spectral radius

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Abstract

Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For any vertex $v_i$, we denote by $N(v_i)$ the set of the vertices adjacent to $v_i$ in $G$. Das got the following upper bound for Laplacian spectral radius:

$$\lambda_1(G) \leq \max\{|N(v_i) \cup N(v_j)| : 1 \leq i < j \leq n, v_i v_j \in E\}.$$

In this paper, we characterize all the connected graphs which achieve the above upper bound.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite simple undirected graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $v_i \in V(G)$, we denote by $N_G(v_i)$ the set of the vertices adjacent to $v_i$ in $G$. The degree of $v_i$, written by $d_G(v_i)$, is the number of vertices in $N_G(v_i)$. If $W \subseteq V(G)$, we denote $N_W(v_i) = N_G(v_i) \cap W$ and $d_W(v_i) = |N_W(v_i)|$. For short, we will write $N(v_i)$ and $d(v_i)$ instead of $N_G(v_i)$ and $d_G(v_i)$, respectively. A bipartite graph is called semiregular if each vertex in the same part of a bipartition has the same degree.

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin’s Theorem, it follows that its eigenvalues are non-negative real numbers. We denote by $\lambda_1(G)$ the largest eigenvalue of $L(G)$ and call it the Laplacian spectral radius of $G$.

Anderson and Morley [1] gave the following well known upper bound for the Laplacian spectral radius of the graph $G$:

$$\lambda_1(G) \leq \max\{d(v_i) + d(v_j) : v_i v_j \in E(G)\}. \quad (1)$$

About 15 years later, Rojo et al. [4] got

$$\lambda_1(G) \leq \max\{|N(v_i) \cup N(v_j)| : 1 \leq i < j \leq n\}. \quad (2)$$

Recently, Das [2] improved the upper bound in (1) and (2) and got

$$\lambda_1(G) \leq \max\{|N(v_i) \cup N(v_j)| : 1 \leq i < j \leq n, v_i v_j \in E(G)\}. \quad (3)$$

But in [2], Das did not characterize the graphs which achieve the above upper bound. In another paper [3], Das proposed a conjecture on the graphs which achieve the upper bound in (2). Here we confirm his conjecture.

2. Lemmas and results

Suppose that $G$ has at least one edge. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the eigenvector corresponding to $\lambda_1(G)$. We may assume that $x_i = 1$ for some $1 \leq i \leq n$, and $|x_k| \leq 1$ for all $1 \leq k \leq n$.

**Definition.** If $x_i = 1$ and $x_j = \min\{x_k : v_k \in N(v_i)\}$, then we call $(v_i, v_j)$ a standard pair.

By Das’ proof of the upper bound in (3) (see [2]), we can obtain the following result.

**Lemma 2.1** [2]. Suppose that $(v_i, v_j)$ is a standard pair. Then

$$\lambda_1(G) \leq |N(v_i) \cup N(v_j)|.$$
Moreover, the equality holds if and only if $x_k = 1$ for each $v_k \in N(v_j) \setminus N(v_i)$, and $x_k = x_j 
eq 1$ for each $v_k \in N(v_i) \setminus N(v_j)$.

From Lemma 2.1 we obtain the following result due to Das:

**Theorem 2.1 [2].** If $G$ is a graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$\lambda_1(G) \leq \max\{|N(v_i) \cup N(v_j)|: 1 \leq i < j \leq n, \ v_iv_j \in E(G)\}.$$ 

Suppose that $F$ is a semiregular bipartite graph with bipartition $\{U, W\}$. Denote by $F^+$ the supergraph of $F$ with the following property: if $uv \in E(F)$ or $u, v \in U$ (respectively $W$) with $N_U(u) = N_U(v)$ (respectively $N_W(u) = N_W(v)$).

Set

$$F^+ = \{F^+: F \text{ is a semiregular bipartite graph}\}.$$ 

In this paper, we prove the following theorem which was conjectured by Das in [3].

**Theorem 2.2.** Let $G$ be a connected graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then

$$\lambda_1(G) = \max\{|N(v_i) \cup N(v_j)|: 1 \leq i < j \leq n, \ v_iv_j \in E(G)\},$$ 

if and only if $G \in F^+$. 

**Proof.** Let $G = F^+$, where $F$ is a semiregular bipartite graph with bipartition $\{U, W\}$ such that for any vertex $u \in U$, $d_F(u) = r$, and for any vertex $w \in W$, $d_F(w) = s$. Since $F$ is a subgraph of $G$, we have $\lambda_1(G) \geq \lambda_1(F) = r + s$.

Let $v_iv_j$ be any edge of $G$. If $v_i \in U$ and $v_j \in W$, then $N_U(v_i) \subseteq N_U(v_j)$ and $N_W(v_i) \subseteq N_W(v_j)$ by the definition of $F^+$. So in this case $|N(v_i) \cup N(v_j)| = r + s$. If $v_i, v_j \in U$ or $v_i, v_j \in W$, say $v_i, v_j \in U$, then $N_W(v_i) = N_W(v_j)$ and $N_U(v_i) \cup N_U(v_j) \subseteq N_U(v_i)$ by the definition of $F^+$, where $v_i \in N_W(v_j)$. Hence

$$|N(v_i) \cup N(v_j)| \leq r + s.$$ 

By Theorem 2.1, we have

$$\lambda_1(G) \leq \max\{|N(v_i) \cup N(v_j)|: 1 \leq i < j \leq n, \ v_iv_j \in E(G)\} \leq r + s.$$ 

Hence $\lambda_1(G) = r + s$.

Now we show that if $G$ is a connected graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ with at least one edge such that

$$\lambda_1(G) = \max\{|N(v_i) \cup N(v_j)|: 1 \leq i < j \leq n, \ v_iv_j \in E(G)\},$$ 

then $G \in F^+$. 

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the eigenvector corresponding to $\lambda_1(G)$. We can assume that $|x_k| \leq 1$ for all $1 \leq k \leq n$ and there is at least one element of $\mathbf{x}$ equal to
1. Let $U = \{v_i \in V(G) : x_i = 1\}$ and $W = V(G) \setminus U$. Since $G$ has at least one edge and sum of the eigencomponents are zero, $U$ and $W$ both sets are non-null. Denote $F = (V(G), E(U, W))$, where $E(U, W)$ is the set of all the edges of $G$ with one end in $U$ and another end in $W$.

Now we show that $F$ is a semiregular bipartite graph with bipartition $\{U, W\}$ and if there is an edge $uv \in E(G)$ such that $u, v \in U$ (respectively $W$), then $N_W(u) = N_W(v)$ (respectively $N_U(u) = N_U(v)$).

First from the assumption 

$$\lambda_1(G) = \max \{|N(v_i) \cup N(v_j)| : 1 \leq i < j \leq n, \; v_i v_j \in E(G)\},$$

and Lemma 2.1, we have the following fact.

Fact 1. Suppose that $(v_i, v_j)$ is a standard pair. Then we have 

$$\lambda_1(G) = |N(v_i) \cup N(v_j)|.$$ 

Furthermore, $x_k = 1$ for each $v_k \in N(v_j) \setminus N(v_i)$, and $x_k = x_j \neq 1$ for each $v_k \in N(v_i) \setminus N(v_j)$.

If $(v_{i_0}, v_{j_0})$ is a standard pair, then $v_{i_0} \in U$ and $v_{j_0} \in W$. We denote 

$$U_1(i_0, j_0) = \{v_k : v_k \in N(v_{j_0}) \setminus N(v_{i_0})\},$$

$$W_1(i_0, j_0) = \{v_k : v_k \in N(v_{i_0}) \setminus N(v_{j_0})\},$$

$$U_2(i_0, j_0) = \{v_k : x_k = 1, v_k \in N(v_{i_0}) \cap N(v_{j_0})\},$$

$$W_2(i_0, j_0) = \{v_k : x_k \neq 1, v_k \in N(v_{i_0}) \cap N(v_{j_0})\}.$$ 

Obviously, $v_{i_0} \in U_1(i_0, j_0)$ and $v_{j_0} \in W_1(i_0, j_0)$. For short, in the following proof, if there is no confusion, we write $U_1, U_2, W_1$ and $W_2$, instead of $U_1(i_0, j_0), U_2(i_0, j_0), W_1(i_0, j_0)$ and $W_2(i_0, j_0)$.

From Fact 1, we have $U_1 \subseteq U$, $W_1 \subseteq W$ and $x_k = x_{j_0}$ for any vertex $v_k \in W_1$. Furthermore, it is easy to see that $U_2 \subseteq U$, $W_2 \subseteq W$ and 

$$N(v_{i_0}) = W_1 \cup W_2 \cup U_2,$$

$$N(v_{j_0}) = U_1 \cup U_2 \cup W_2.$$ 

Hence we have the following result.

Fact 2. For any standard pair $(v_{i_0}, v_{j_0})$, we have 

$$\lambda_1(G) = |N(v_{i_0}) \cup N(v_{j_0})| = d_F(v_{i_0}) + d_F(v_{j_0}).$$

Fact 3. If $v_i \in U_1 \cup U_2$, then $\min\{x_k : v_k \in N(v_i)\} = x_{j_0}$. That is for any vertex $v_i \in U_1 \cup U_2$, $(v_i, v_{j_0})$ is a standard pair.
Proof of Fact 3. Otherwise, \( v_i \neq v_{i_0} \). Let \((v_i, v_j)\) be a standard pair. Since \( U_1 \cup U_2 \subseteq N(v_{j_0}) \), we have \( v_{j_0} \in N(v_i) \), and hence \( x_j < x_{j_0} \). Noting that \( x_{j_0} = \min\{x_k : v_k \in N(v_{i_0})\} \), we get \( v_j \notin N(v_{i_0}) \), and hence \( v_j \in W \setminus (W_1 \cup W_2) \). Thus \( v_{j_0} \in N(v_i) \setminus N(v_j) \), and by Fact 1, we have \( x_{j_0} = x_j \), a contradiction. \( \square \)

Fact 4. If \( v_t \in U_1 \cup U_2 \), \( v_j \in W_2 \), then \( v_t v_j \in E(G) \).

Proof of Fact 4. By Fact 3, \((v_t, v_{j_0})\) is a standard pair. If \( v_t v_j \notin E(G) \), then \( v_j \in N(v_{j_0}) \setminus N(v_t) \). By Fact 1, we get \( x_j = 1 \), which contradicts \( v_j \in W_2 \). \( \square \)

Fact 5. If \( v_t \in U_2 \), \( v_j \in W_1 \), then \( v_t v_j \in E(G) \).

Proof of Fact 5. By Fact 1, \((v_{i_0}, v_j)\) is a standard pair. If \( v_t v_j \notin E(G) \), then \( v_t \in N(v_{i_0}) \setminus N(v_j) \). By Fact 1, we have \( x_t = x_j \neq 1 \), which contradicts \( v_t \in U_2 \). \( \square \)

Fact 6. If \( v_{j_1} \in W_1 \setminus \{v_{j_0}\}, v_{j_2} \in W_2 \), then \( d_F(v_{j_0}) = d_F(v_{j_1}) = d_F(v_{j_2}) \). Moreover, \( N_U(v_{j_0}) = N_U(v_{j_1}) \) and \( N_W(v_{j_1}) \subseteq W_1 \cup W_2 \).

Proof of Fact 6. By Fact 1, \((v_{i_0}, v_{j_0})\) and \((v_{i_0}, v_{j_1})\) are two standard pairs. So by Fact 2, we have

\[
\lambda_1(G) = |N(v_{i_0}) \cup N(v_{j_0})| = d_F(v_{i_0}) + d_F(v_{j_0}),
\]

\[
\lambda_1(G) = |N(v_{i_0}) \cup N(v_{j_1})| = d_F(v_{i_0}) + d_F(v_{j_1}).
\]

So \( d_F(v_{j_0}) = d_F(v_{j_1}) \).

Now we show that \( N_U(v_{j_0}) = N_U(v_{j_2}) \) and \( N_W(v_{j_2}) \subseteq W_1 \cup W_2 \). Let

\[
a = |U_1 \cup U_2 \cup W_1 \cup W_2|,
\]

\[
p = |\{v_k : v_k \in N(v_{j_2}) \text{ and } v_k \notin U \cup (U_1 \cup U_2)\}|,
\]

\[
q = |\{v_k : v_k \in N(v_{j_2}) \text{ and } v_k \notin W \setminus (W_1 \cup W_2)\}|.
\]

By Facts 1 and 4, we have

\[
\lambda_1(G) = \max\{|N(v_i) \cup N(v_j)| : 1 \leq i < j \leq n, v_i v_j \in E(G)\}
\]

\[
= |N(v_{i_0}) \cup N(v_{j_0})| = a
\]

\[
\geq |N(v_{i_0}) \cup N(v_{j_1})| = a + p + q.
\]

Hence \( p = 0 \) and \( q = 0 \), which imply that \( N_U(v_{j_0}) = N_U(v_{j_2}) \) and \( N_W(v_{j_2}) \subseteq W_1 \cup W_2 \). By \( N_U(v_{j_0}) = N_U(v_{j_2}) \), we have \( d_F(v_{j_0}) = d_F(v_{j_2}) \). \( \square \)

Fact 7. If \( v_{i_1} \in U_1 \setminus \{v_{i_0}\}, v_{i_2} \in U_2 \), then \( d_F(v_{i_0}) = d_F(v_{i_1}) = d_F(v_{i_2}) \). Moreover, \( N_W(v_{i_0}) = N_W(v_{i_2}) \).

Proof of Fact 7. By Fact 3, \((v_{i_1}, v_{j_0})\) and \((v_{i_2}, v_{j_0})\) are two standard pairs. Hence by Fact 2, we have
Proof of Fact 8. \[ \lambda_1(G) = d_F(v_{i_0}) + d_F(v_{j_0}), \]
\[ \lambda_1(G) = d_F(v_{i_1}) + d_F(v_{j_0}) \]
and
\[ \lambda_1(G) = d_F(v_{i_2}) + d_F(v_{j_0}). \]
Hence \( d_F(v_{i_0}) = d_F(v_{i_1}) = d_F(v_{i_2}) \). By Facts 4 and 5, \( NW(v_h) = NW(v_l) \). □

Fact 8. For any two vertices \( v_h, v_l \in U \) with \( v_h v_l \in E(G) \), \( d_F(v_h) = d_F(v_l) \) and \( NW(v_h) = NW(v_l) \).

Proof of Fact 8. Let \( (v_h, v_l) \) be a standard pair. Then we have \( v_l \in U_2(h, j) \). Otherwise, by Fact 1, \( v_l \in W \). Hence, Fact 8 holds by Fact 7. □

Fact 9. Let \( v_h, v_l \in U \) with \( v_h v_l \notin E(G) \). If there exists a path \( P \) connecting \( v_h \) and \( v_l \) such that all internal vertices of \( P \) are in \( W \), then \( d_F(v_h) = d_F(v_l) \).

Proof of Fact 9. Assume, without loss of generality, that \( P = v_h v_1 \cdots v_s v_l \) be the shortest path connecting \( v_h \) and \( v_l \) with \( v_i \in W \) for \( 1 \leq i \leq s \).

If \( (v_h, v_l) \) is a standard pair, then \( v_2 \in U_1(h, 1) \cup U_2(h, 1) \cup W_2(h, 1) \). Since \( v_h v_2 \notin E(G) \), we have \( v_2 \notin W_2(h, 1) \), and then \( v_2 \in U_2(h, 1) \). Thus \( s = 1 \) and Fact 9 holds by Fact 7. Hence we can assume that \( (v_h, v_l) \) is a standard pair with \( x_j \neq x_1 \). Since \( v_1 \in N(v_l) \) and \( x_1 \neq x_j \), \( v_1 \in W_2(h, j) \) by Fact 1. If \( s \geq 2 \), we have \( v_2 \notin W_2(h, j) \) by Fact 6. But in this case, we have \( v_h v_2 \in E(G) \), a contradiction. Thus \( s = 1 \), i.e., \( P = v_h v_1 v_l \). By Fact 6, \( N_U(v_j) = N_U(v_l) \). Noting that \( v_h v_l \notin E(G) \), we have \( v_l \in U_1(h, j) \} \{ v_h \}. Thus Fact 9 holds by Fact 7. □

By Facts 8 and 9, we easily have the following result.

Fact 10. For any two vertices \( v_h, v_l \in U \) with \( v_h v_l \notin E(G) \), \( d_F(v_h) = d_F(v_l) \).

Fact 11. For any vertex \( w \in W \), \( N_U(w) \neq \emptyset \).

Proof of Fact 11. Suppose that there exists a vertex \( w \in W \) such that \( N_U(w) = \emptyset \). Since \( G \) is connected, there is a path connecting \( w \) and some vertex \( u \) of \( U \). Assume that \( P = w(= v_0) v_1 \cdots v_l v_{l+1}(= u) \) is the shortest path connecting \( w \) and \( u \) such that \( v_i \in W \) for \( 1 \leq i \leq l \).

If \( (v_{l+1}, v_l) \) is a standard pair, then \( v_{l-1} \in U_1(l + 1, l) \) by \( v_{l-1} v_{l+1} \notin E(G) \), a contradiction. Hence, we can assume that \( (v_{l+1}, v_l) \) is a standard pair with \( x_l \neq x_j \). Since \( v_l \in N(v_{l+1}) \) and \( x_l \neq x_j \), by Fact 1, \( v_l \in W_2(l + 1, j) \). By Fact 6, \( v_{l-1} \in W_1(l + 1, j) \cup W_2(l + 1, j) \). Thus \( v_{l-1} v_{l+1} \in E(G) \), a contradiction. □

Fact 12. For any two vertices \( v_h, v_l \in W \) with \( v_h v_l \in E(G) \), \( d_F(v_h) = d_F(v_l) \) and \( N_U(v_h) = N_U(v_l) \).
Proof of Fact 12. By Fact 11, there exists $v_i \in U$ such that $v_i v_j \in E(G)$. If $(v_i, v_j)$ is a standard pair, then $v_j \in W_2(i, h)$. Thus Fact 12 holds by Fact 6. Hence we can assume that $(v_i, v_j)$ is a standard pair with $x_j \neq x_h$. Since $v_h \in N(v_i)$ and $x_j \neq x_h$, we have $v_h \in W_2(i, j)$ by Fact 1. By Fact 6, $v_j \in W_1(i, j) \cup W_2(i, j)$. Thus Fact 12 holds by Fact 6. \(\square\)

Fact 13. For any two vertices $v_k, v_l \in W$ with $v_k v_l \notin E(G)$, $d_F(v_k) = d_F(v_l)$.

Proof of Fact 13. For any two vertices $v_k, v_l \in W$ with $v_k v_l \notin E(G)$, we have $N_U(v_k) \cap N_U(v_l) = \emptyset$, by Fact 11. If $N_U(v_k) \cap N_U(v_l) \neq \emptyset$, say $v_i \in N_U(v_k) \cap N_U(v_l)$, then $d_F(v_h) = d_F(v_l)$ by Fact 6. Hence we can assume that $N_U(v_k) \cap N_U(v_l) = \emptyset$. Assume $v_i \in N_U(v_k)$ and $v_i \in N_U(v_l)$. Let $(v_i, v_j)$ and $(v_i, v_j)$ be two standard pairs. By Fact 6, $d_F(v_k) = d_F(v_j)$ and $d_F(v_l) = d_F(v_j)$. By Fact 2, we have

$$\lambda_1(G) = d_F(v_i) + d_F(v_j) = d_F(v_j) + d_F(v_i).$$

But by Facts 8 and 10, we have $d_F(v_i) = d_F(v_j)$. Hence we have $d_F(v_j) = d_F(v_j) = d_F(v_j)$ and then $d_F(v_k) = d_F(v_l)$. \(\square\)

By Facts 8, 10, 12 and 13, we have $G \in \mathcal{F}^+$. This completes the proof of our Theorem. \(\square\)

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References