Ordering trees by their Laplacian spectral radii

Aimei Yu\textsuperscript{a,b,*}, Mei Lu\textsuperscript{c}, Feng Tian\textsuperscript{a}

\textsuperscript{a}Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China
\textsuperscript{b}Graduate School of the Chinese Academy of Sciences, Beijing 100039, China
\textsuperscript{c}Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Received 2 June 2004; accepted 23 February 2005
Available online 10 May 2005
Submitted by S. Kirkland

Abstract

Denote by \(\mathcal{T}_n\) the set of trees on \(n\) vertices. Zhang and Li [X.D. Zang, J.S. Li, The two largest eigenvalues of Laplacian matrices of trees (in Chinese), J. China Univ. Sci. Technol. 28 (1998) 513–518] and Guo [J.M. Guo, On the Laplacian spectral radius of a tree, Linear Algebra Appl. 368 (2003) 379–385] give the first four trees in \(\mathcal{T}_n\), ordered according to their Laplacian spectral radii. In this paper, we determine the fifth to eighth trees in the above ordering.

\(\text{AMS classification: 05C50; 15A18}\)

\textit{Keywords:} Tree; Laplacian matrix; Spectral radius

1. Introduction

Let \(G = (V, E)\) be a simple undirected graph on \(n\) vertices. For \(v \in V\), the degree of \(v\), written by \(d(v)\), is the number of edges incident with \(v\). Let \(A(G)\) be the

\textsuperscript{*}This work is partially supported by National Natural Science Foundation of China.
\textsuperscript{*}Corresponding author.

\textit{E-mail addresses:} yuaimimath@yeah.net (A. Yu), mlu@math.tsinghua.edu.cn (M. Lu), ftian@iss.ac.cn (F. Tian).

0024-3795/ - see front matter \(\text{© 2005 Elsevier Inc. All rights reserved.}\)
adjacency matrix of $G$ and $D(G)$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. Denote by $p(G, x)$ the characteristic polynomial of $L(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin’s theorem, it follows that its eigenvalues, the roots of $p(G, x)$, are nonnegative real numbers. We denote the largest eigenvalue of $L(G)$ by $\lambda_1(G)$ and call it the Laplacian spectral radius of $G$.

Two edges of a graph are said to be independent if they are not incident with a common vertex. A matching $M$ of $G$ is a set of mutually independent edges. Denote by $|M|$ the number of the edges in $M$. If $G$ has no matching $M'$ with $|M'| > |M|$, then $M$ is a maximum matching. We call the number of edges in a maximum matching of $G$ the edge-independence number of $G$ and denote it by $\alpha'(G)$.

Theorem 1.1 [3,7]. Let $T \in \mathcal{T}_n$, then $\lambda_1(T) \leq n$, and equality holds if and only if $T \cong S^1_n \cong K_{1,n-1}$.

Theorem 1.2 [3,7]. Let $T \in \mathcal{T}_n \setminus \{K_{1,n-1}\}$, then $\lambda_1(T) \leq c_2$, where $c_2$ is the largest root of the equation

$$x^3 - (n + 2)x^2 + (3n - 2)x - n = 0.$$

Equality holds if and only if $T \cong S^2_n$ (see Fig. 1).

Theorem 1.3 [3,7]. Let $T \in \mathcal{T}_n \setminus \{K_{1,n-1}, S^2_n\}$ and $n \geq 6$, then $\lambda_1(T) \leq c_3$, where $c_3$ is the largest root of the equation

$$x^3 - (n + 2)x^2 + (4n - 7)x - n = 0.$$

Equality holds if and only if $T \cong S^3_n$ (see Fig. 1).

Theorem 1.4 [3]. Let $T \in \mathcal{T}_n \setminus \{K_{1,n-1}, S^2_n, S^3_n\}$ and $n \geq 6$, then $\lambda_1(T) \leq c_4$, where $c_4$ is the largest root of the equation

$$x^3 - (n + 1)x^2 + (3n - 5)x - n = 0.$$

Equality holds if and only if $T \cong S^4_n$ (see Fig. 1).
Theorems 1.1–1.4 give the first four trees in $T_n$, ordered according to their Laplacian spectral radii. In this paper, we refined the above results and determine the fifth to eighth trees in the above ordering.

2. Lemmas and results

Lemma 2.1 [1]. Let $G$ be a graph. Then $\lambda_1(G) \leq \max\{d(u) + d(v) : uv \in E(G)\}$.

Lemma 2.2 [2]. If $G$ is a graph with at least one edge, then $\lambda_1(G) \geq \max\{d(v) : v \in V(G)\} + 1$. For a connected graph on $n > 1$ vertices, equality holds if and only if $\max\{d(v) : v \in V(G)\} = n - 1$.

Lemma 2.3 [3]. Let $T$ be a tree on $n$ vertices with $\alpha'(T) = m$. Then $\lambda_1(T) \leq \lambda_1(T(n, m))$, where $\lambda_1(T(n, m))$ is the largest root of the equation

$$x^3 - (n - m + 4)x^2 + (3n - 3m + 4)x - n = 0.$$ $\lambda_1(T)$ is the largest root of the equation $p(G; x) = 0$, we have $p(G; x) > 0$ for all $x > \lambda_1(G)$. Then we immediately get the following elementary but useful result.

Lemma 2.4. Let $G_1$ and $G_2$ be two graphs. If $p(G_1; x) < p(G_2; x)$ for $x \geq \lambda_1(G_1)$, then $\lambda_1(G_1) > \lambda_1(G_2)$.

Let $P = v_0v_1 \cdots v_k$ ($k \geq 1$) be a path of $T \in T_n$. If $d_T(v_0) \geq 3$, $d_T(v_k) \geq 3$ and $d_T(v_i) = 2$ ($0 < i < k$) when $k > 1$, then we call $P$ an internal path of $T$. If $d_T(v_0) \geq 2$, $d_T(v_k) = 1$ and $d_T(v_i) = 2$ ($0 < i < k$) when $k > 1$, we call $P$ a pendant path of $T$. For example, we consider the tree in Fig. 2(b). $v_3v_4v_5v_6$ and $v_9v_{10}$ are two internal paths of $T$, while $v_6v_7v_8$ and $v_7v_8$ are two pendant paths of $T$. It is easy to see that any edge of $T$ is either on an internal path or on a pendant path of $T$.

Now we define three kinds of operations of $T \in T_n$.

![Fig. 2.](image_url)
(i) If \( e = uv \) is an edge on an internal path of \( T \) and \( T' \) is obtained from \( T \) by contracting \( uv \), i.e., identifying vertices \( u \) and \( v \) in \( T - uv \), we say that \( T' \) is obtained from \( T \) by Operation I.

(ii) If \( T' \) is obtained from \( T \) by connecting a new vertex \( v \) with some vertex of \( T \) by an edge, we say that \( T' \) is obtained from \( T \) by Operation II.

(iii) If \( uv \) is a nonpendant edge of \( T \) and \( T' \) is obtained from \( T \) by contracting \( uv \) into a new vertex \( w \) and attaching a pendant edge to \( w \), we say that \( T' \) is obtained from \( T \) by Operation III.

Lemma 2.5

(1) If \( T' \) is obtained from \( T \) by Operation I or II, then \( \lambda_1(T') > \lambda_1(T) \) [8].

(2) Denote by \( T_{k, l}^n(u) \) the tree obtained from a tree \( T^* \) by attaching paths of length \( k \) and \( l \) at a vertex \( u \). If \( k \geq l \geq 1 \), then \( \lambda_1(T_{k,l}^n(u)) > \lambda_1(T_{k+1,l-1}^n(u)) \) [8].

(3) If \( T' \) is obtained from \( T \) by Operation III, then \( \lambda_1(T') > \lambda_1(T) \).

Proof. (1) and (2) have been shown in [8].

Let \( T' \) be a tree obtained from \( T \) by contracting a nonpendant edge \( uv \) into a new vertex \( w \) and attaching a pendant edge to \( w \). If \( uv \) is an edge on an internal path of \( T \), then (3) holds immediately by (1). Otherwise, we can assume that \( uv \) is a nonpendant edge of the pendant path \( P = v_0 (= u) v_1 (= v) \cdots v_k (k \geq 2) \) with \( d_G(v_0) \geq 2 \), \( d_G(v_i) = 2 \) \((0 < i < k)\) and \( d_G(v_k) = 1 \). Denote \( T^* = T - \{v_1, v_2, \ldots, v_k\} \). Then \( T \equiv T_{k,l}^n(u) \) and \( T' \equiv T_{k-1,l}^n(u) \). From (2), (3) holds immediately. \( \square \)

According to the number of the nonpendant edges in a tree, we give a partition of trees in \( \mathcal{F}_n \) as follows:

\[
\mathcal{F}_n = \bigcup_{i=0}^{n-3} \mathcal{F}_i^n,
\]

where \( \mathcal{F}_i^n = \{ T | T \in \mathcal{F}_n, \text{ and there exists exactly } i \text{ nonpendant edges in } T \} \).

Obviously, \( \mathcal{F}_0^n \) and \( \mathcal{F}_1^n \) contain only \( K_{1,n-1} \) and \( P_n \), the path on \( n \) vertices, respectively. It is easy to see that any tree \( T \in \mathcal{F}_i^n \) \((1 \leq i \leq n-3)\) can be transformed into a tree in \( \mathcal{F}_j^n \) \((j < i)\) by carrying out Operation III repeatedly.

Let \( P_k = v_1v_2 \cdots v_k \) be a path of order \( k \). We denote by \( P_{k_1,l_1}, \ldots, k_l \) \((k \geq 1)\) the tree obtained from \( P_k \) by attaching \( l_i \) \((1 \leq i \leq k)\) pendant edges to the vertex \( v_i \). Denote by \( K_{1,3}^{q_1,q_2,q_3} \) \((q_3 \geq q_2 \geq q_1)\) the tree obtained from \( K_{1,3} \) by attaching \( q_0 \) pendant edges to the vertex of degree 3 of \( K_{1,3} \) and attaching \( q_1 \) and \( q_2 \) pendant edges to the three pendant vertices of \( K_{1,3} \), respectively. \( P_{1,2}^{1,2} \), \( P_{1,3}^{1,2,1} \), \( P_{1,4}^{1,2,1,1} \), and \( K_{1,3}^{1,2,1,1} \) are shown in Figs. 3 and 4.

We notice that

\[
\mathcal{F}_1^n = \left\{ P_{l_1,l_2}^{l_1,l_2}, l_1 + l_2 = n - 2, \ 1 \leq l_1 \leq l_2 \leq n - 3 \right\}
\]
and
\[ T_n^2 = \left\{ P_3^{l_1,l_2,l_3}, \quad l_1 + l_2 + l_3 = n - 3, \quad 1 \leq l_1 \leq l_3, \quad l_2 \geq 0 \right\}. \]

Clearly, \( S_n^2, S_n^3 \in T_n^1 \), and \( S_n^4 \in T_n^2 \) (\( S_n^2 \cong P_2^{1,n-3}, S_n^3 \cong P_2^{2,n-4}, S_n^4 \cong P_3^{1,n-5,1} \)).

The set \( T_n^3 \) consists of two kinds of trees \( K_{13}^{q_0,q_1,q_2,q_3} \) and \( P_4^{l_1,l_2,l_3,l_4} \), where \( q_0 + q_1 + q_2 + q_3 = n - 4, \ q_0 \geq 0, \ q_3 \geq q_2 \geq q_1 \geq 1 \) and \( l_1 + l_2 + l_3 + l_4 = n - 4, l_2, l_3 \geq 0, l_4 \geq l_1 \geq 1 \).

Guo showed the following two results in [3].

**Lemma 2.6** [3]. \( \lambda_1(P_2^{l_1,l_2}) < \lambda_1(P_2^{l_1-1,l_2+1}) \), where \( l_1 \leq l_2 \).

**Lemma 2.7** [3]. \( \lambda_1(P_3^{l_1,l_2}) < \lambda_1(P_3^{l_1-1,l_2+1}) \), where \( l_1 \leq l_3 \).

Hence, we have
\[
\lambda_1(P_2^{1,n-3}) > \lambda_1(P_2^{2,n-4}) > \lambda_1(P_2^{3,n-5}) > \lambda_1(P_2^{4,n-6})
\]
\[ > \cdots > \lambda_1 \left( P_2^{\lceil \frac{n-2}{2} \rceil, \lceil \frac{n-2}{2} \rceil} \right) \]
and
\[
\lambda_1(P_3^{1,0,n-4}) > \lambda_1(P_3^{2,0,n-5}) > \lambda_1(P_3^{3,0,n-6}) > \cdots > \lambda_1 \left( P_3^{\lceil \frac{n-4}{2} \rceil, 0, \lceil \frac{n-4}{2} \rceil} \right),
\]
where \( \lfloor x \rfloor, \lceil x \rceil \) denote the largest integer not greater than \( x \) and the smallest integer no less than \( x \), respectively.
Now we begin to show our results.

**Lemma 2.8.** Let $T$ be a tree in $\mathcal{T}_n$ $(n \geq 7)$, and $T \notin \mathcal{T}^1_n$, $T \notin \mathcal{T}^2_n$. Then

$$\lambda_1(T) \leq \lambda_1(K^{n-7,1,1,1}_{1,3}),$$

where $\lambda_1(K^{n-7,1,1,1}_{1,3})$ is the largest root of the equation $x^3 - nx^2 + (3n - 8)x - n = 0$, and equality holds if and only if $T \cong K^{n-7,1,1,1}_{1,3}$.

**Proof.** By an elementary calculation, we have

$$p(K^{n-7,1,1,1}_{1,3}, x) = x(x - 1)^{n-8}(x^2 - 2x + 1)^2 \left[ x^3 - nx^2 + (3n - 8)x - n \right].$$

Recall that $\lambda_1(K^{n-7,1,1,1}_{1,3})$ is the largest root of the equation

$$x^3 - nx^2 + (3n - 8)x - n = 0.$$

Let $T$ be a tree in $\mathcal{T}_n$ $(n \geq 7)$, and $T \notin \mathcal{T}^1_n$, $T \notin \mathcal{T}^2_n$. Then $T \in \mathcal{T}^i_n$ for some $i \geq 3$.

If $T \in \mathcal{T}^3_n$, we distinguish the following two cases:

**Case 1.** $T \cong K^{q_0,q_1,q_2,q_3}_{1,3}$, where $q_0 + q_1 + q_2 + q_3 = n - 4$, $q_0 \geq 0$, $q_3 \geq q_2 \geq q_1 \geq 1$.

If $q_0 \neq 0$, then $\alpha'(T) = 4$. By Lemma 2.3, $\lambda_1(T) < \lambda_1(T(n, 4))$. Since $T(n, 4) \cong K^{n-7,1,1,1}_{1,3}$, it follows that $\lambda_1(T) \leq \lambda_1(K^{n-7,1,1,1}_{1,3})$ with equality holding if and only if $T \cong K^{n-7,1,1,1}_{1,3}$.

If $q_0 = 0$, then $\alpha'(T) = 4$. By Lemma 2.5, $\lambda_1(T) < \lambda_1(K^{n-7,1,1,1}_{1,3})$. Note that $K^{0,1,1,n-6}_{1,3} \cong K^{n-7,1,1,1}_{1,3}$, when $n = 7$. Hence if $q_0 = 0$, $T \cong K^{0,1,1,n-6}_{1,3}$ and $n \geq 7$, then $\lambda_1(T) \leq \lambda_1(K^{n-7,1,1,1}_{1,3})$ with equality holding if and only if $T \cong K^{n-7,1,1,1}_{1,3}$.

Now we suppose $q_0 = 0$ and $T \cong K^{0,1,1,n-6}_{1,3}$, which implies that $q_2 \geq 2$. By Lemmas 2.1 and 2.2, we have

$$\lambda_1(K^{n-7,1,1,1}_{1,3}) > (n - 4) + 1 = (n - 7) + 4 \geq (q_3 + 1) + 3 \geq \lambda_1(T).$$

**Case 2.** $T \cong F^{l_1,l_2,l_3,l_4}_{4,4}$, where $l_1 + l_2 + l_3 + l_4 = n - 4$, $l_2, l_3 \geq 0$, $l_4 \geq l_1 \geq 1$. 

![Fig. 5.](image-url)
If \( l_2 \neq 0 \) and \( l_3 \neq 0 \), then \( \alpha'(T) = 4 \). By Lemma 2.3, \( \lambda_1(T) \leq \lambda_1(K_{1,3}^{n-7,1,1,1}) \), and equality holds if and only if \( T \cong K_{1,3}^{n-7,1,1,1} \).

If \( l_2 = l_3 = 0 \) and \( l_4 \geq l_1 \geq 2 \), by Lemmas 2.1 and 2.2, we have

\[
\lambda_1(K_{1,3}^{n-7,1,1,1}) > (n - 4) + 1 = (n - 6) + 3 \geq (l_4 + 1) + 2 \geq \lambda_1(T).
\]

Finally, we consider the cases when \( l_1 \neq 0 \) and \( l_4 \neq 0 \) or \( l_2 \neq 0 \) and \( l_3 = 0 \).

When \( l_1 = 1 \), then \( l_2 = 1 \) by \( l_4 \geq l_1 \geq 1 \). Thus when \( l_2 = 0 \) and \( l_3 \neq 0 \), \( T \cong P_4^{1,0,n-6,1} \); and \( l_2 \neq 0 \) and \( l_3 = 0 \), \( T \cong P_4^{1,0,n-6,0,1} \). Since \( P_4^{1,0,n-6,0,1} \cong P_4^{1,n-6,0,1} \) for \( l_4 = 1 \) by (S1), \( \lambda_1(K_{1,3}^{n-7,1,1,1}) > \lambda_1(P_4^{1,0,n-6,1}) \).

Thus, in the following proof, we assume that \( l_4 \geq 2 \).

(1) \( l_2 = 0, l_3 \neq 0 \).

When \( l_1 \geq 2 \), by Lemmas 2.1 and 2.2, we have

\[
\lambda_1(K_{1,3}^{n-7,1,1,1}) > (n - 4) + 1 \geq \max\{l_1 + 3, l_3 + 3, l_3 + l_4 + 3\} \geq \lambda_1(T).
\]

When \( l_1 = 1 \), we have \( T \cong P_4^{1,0,1/3,l_4} \). Note that \( l_4 \geq 2 \). It is easy to see that \( P_4^{1,0,1/3,l_4} \) can be transformed into \( P_4^{1,0,0,n-5} \) by Operations I and II (see Fig. 7). By Lemma 2.5 and (S1), \( \lambda_1(P_4^{1,0,1/3,l_4}) < \lambda_1(P_4^{1,0,0,n-5}) < \lambda_1(K_{1,3}^{n-7,1,1,1}) \).

(2) \( l_2 \neq 0, l_3 = 0 \).

Since \( l_4 \geq 2 \), by Lemmas 2.1 and 2.2, we have

\[
\lambda_1(K_{1,3}^{n-7,1,1,1}) > (n - 4) + 1 \geq \max\{l_2 + 4, l_4 + 3, l_1 + l_2 + 3\} \geq \lambda_1(T).
\]
Fig. 7.

If $T \in \mathcal{T}_n^i$, $i \geq 4$, $T$ can be transformed into $T' \in \mathcal{T}_n^3$ by Operation III. So $\lambda_1(T) < \lambda_1(K_{n-7,1,1,1})$. This completes the proof of Lemma 2.8. □

**Lemma 2.9.** Let $T \in \mathcal{T}_n^2 \setminus \{S_4^n\}$ and $n \geq 6$. Then $\lambda_1(T) \leq \lambda_1(P_{3,1,0,n-4})$, where $\lambda_1(P_{3,1,0,n-4})$ is the largest root of the equation

$$x^4 - (n + 3)x^3 + (5n - 4)x^2 - (6n - 10)x + n = 0,$$

and equality holds if and only if $T \cong P_{3,1,0,n-4}$.

**Proof.** By an elementary calculation, we have

$$p(P_{3,1,0,n-4}, x) = x(x - 1)^{n-3}[x^4 - (n + 3)x^3 + (5n - 4)x^2 - (6n - 10)x + n]. \quad (S2)$$

Recall that $\lambda_1(P_{3,1,0,n-4})$ is the largest root of the equation

$$x^4 - (n + 3)x^3 + (5n - 4)x^2 - (6n - 10)x + n = 0.$$

Let $T \cong P_{3,1,1,l_1,1} \in \mathcal{T}_n^2 \setminus \{S_4^n\}$ and $n \geq 6$. If $l_2 = 0$, then $\lambda_1(T) \leq \lambda_1(P_{3,1,0,n-4})$ by Lemma 2.7, and equality holds if and only if $T \cong P_{3,1,0,n-4}$.

If $l_2 \neq 0$ and $l_3 \geq l_1 \geq 2$, then $\lambda_1(P_{3,1,0,n-4}) > n - 2 \geq l_2 + l_3 + 3 \geq \lambda_1(T)$ by Lemmas 2.1 and 2.2. Now we suppose $l_2 \neq 0$ and $l_1 = 1$. Then $T \cong P_{3,1,1/l_1,1}$. Noting that $T \not\cong S_4^n(\cong P_{3,1,n-5,1})$, we have $l_3 \geq 2$. Thus $P_{3,1,1/l_1,1}$ can be transformed into $P_{3,1,0,n-4}$ by Operations I and II (see Fig. 8) and $\lambda_1(P_{3,1,1/l_1,1}) < \lambda_1(P_{3,1,0,n-4})$ by Lemma 2.5. □

Fig. 8.
Lemma 2.10. Let $T \in \mathcal{F}_n^2 \setminus \{S_n^4, P_3^{1,0,n-4}\}$ and $n \geq 9$. Then $\lambda_1(T) \leq \lambda_1(P_3^{1,n-6,2})$, where $\lambda_1(P_3^{1,n-6,2})$ is the largest root of the equation
\[ x^5 - (n + 4)x^4 + (7n - 7)x^3 + (32 - 14n)x^2 + (7n - 10)x - n = 0, \]
and equality holds if and only if $T \equiv P_3^{1,n-6,2}$.

Proof. By an elementary calculation, we have
\[
p(P_3^{1,n-6,2}, x) = x(x - 1)^{n-6}[x^5 - (n + 4)x^4 + (7n - 7)x^3 + (32 - 14n)x^2 + (7n - 10)x - n].
\]
(S3)
Recall that $\lambda_1(P_3^{1,n-6,2})$ is the largest root of the equation
\[ x^5 - (n + 4)x^4 + (7n - 7)x^3 + (32 - 14n)x^2 + (7n - 10)x - n = 0. \]

Let $T \equiv P_3^{1,l_2,l_3} \in \mathcal{F}_n^2 \setminus \{S_n^4, P_3^{1,0,n-4}\}$ and $n \geq 9$. If $l_2 = 0$, then $\lambda_1(T) \leq \lambda_1(P_3^{1,0,n-5})$ by Lemma 2.7 and $T \not\equiv P_3^{1,0,n-4}$. Since $n \geq 9$, $P_3^{2,0,n-5}$ can be transformed into $P_3^{1,n-6,2}$ by Operations I and II (see Fig. 9). By Lemma 2.5, $\lambda_1(T) \leq \lambda_1(P_3^{2,0,n-5}) < \lambda_1(P_3^{1,n-6,2})$. Thus in the following proof, we will assume that $l_2 \neq 0$ and distinguish the following three cases:

Case 1. $l_3 \geq l_1 \geq 3$.
By Lemmas 2.1 and 2.2, $\lambda_1(P_3^{1,n-6,2}) > (n - 4) + 1 \geq l_2 + l_3 + 3 \geq \lambda_1(T)$.

Case 2. $l_1 = 2$, i.e., $T \equiv P_3^{2,l_3}$.
Since $l_3 \geq l_1$, we have $T \not\equiv P_3^{1,n-6,2}$. Thus $P_3^{2,l_3}$ can be transformed into $P_3^{1,n-6,2}$ by Operations I and II (see Fig. 10) and then $\lambda_1(P_3^{2,l_3}) < \lambda_1(P_3^{1,n-6,2})$ by Lemma 2.5.

Case 3. $l_1 = 1$, i.e., $T \equiv P_3^{1,l_3}$.
By an elementary calculation, we have
\[
p(P_3^{1,l_3}, x) = x(x - 1)^{l_3-2}c(x),
\]
(S4)
where
\[
c(x) = x^5 - (l_2 + l_3 + 8)x^4 + (23 + 6l_3 + 5l_2 + l_2l_3)x^3
- (30 + 11l_3 + 8l_2 + 3l_2l_3)x^2
+ (18 + 7l_3 + 5l_2 + l_2l_3)x - (4 + l_2 + l_3).
\]

Fig. 9.
In particular,
\[ p(P_{1,n-6,2}, x) = x(x-1)^{n-6}d(x), \]
\[ p(P_{1,1,n-5}, x) = x(x-1)^{n-6}e(x), \]
where
\[ d(x) = x^5 - (n+4)x^4 + (7n-7)x^3 + (32-14n)x^2 \]
\[ + (7n-10)x - n, \]
\[ e(x) = x^5 - (n+4)x^4 + (7n-7)x^3 + (32-14n)x^2 \]
\[ + (8n-17)x - n. \]
We have \( e(x) - d(x) = (n-7)x > 0 \), when \( x > 0 \). This implies \( p(P_{1,1,n-5}, x) > p(P_{1,n-6,2}, x) \) when \( x \geq \lambda_1(P_{1,n-6,2}) > n-3 \geq 6 \). Hence, by Lemma 2.4, we have \( \lambda_1(P_{1,1,n-5}) > \lambda_1(P_{1,n-6,2}). \)

Now we just need to show that \( \lambda_1(P_{3,1,n-5}) > \lambda_1(T) \) if \( T \cong P_{3,l_2,l_3} \) \((l_2 \neq 0)\) and \( T \approx S_{l_2}^1, P_{3,1,n-6,2}, P_{3,1,n-5} \).

By an elementary calculation,
\[ c(x) = e(x) + x[(l_3 - 2)(n-l_3 - 5)(x^2 - 3x) + (l_3 - 3)(n-l_3 - 5)]. \]

Since \( T \approx S_{l_2}^1(\cong P_{3,1,n-5}) \), \( P_{3,1,n-6,2} \), \( P_{3,1,n-5} \), we have that \( 3 \leq l_3 \leq n-6 \) and then \( \lambda_1(P_{3,1,l_2,l_3}) > l_3 + 1 + 1 \geq 5 \). When \( x = \lambda_1(P_{3,1,l_2,l_3}) \), we have \( c(x) = 0, x^2 - 3x > 0, (l_3 - 2)(n-l_3 - 5) > 0 \) and \( (l_3 - 3)(n-l_3 - 5) \geq 0 \). So \( e(\lambda_1(P_{3,1,l_2,l_3})) < 0 \) by (S5).

(*) By Theorem 1.1, we have \( \lambda_1(P_{3,1,n-5}) < n \) and \( \lambda_1(P_{3,1,l_2,l_3}) < n \). Noting that \( \lambda_1(P_{3,1,n-5}) \) is the largest root of \( e(x) = 0 \), we have \( e(n) > 0 \). Combining \( e(\lambda_1(P_{3,1,l_2,l_3})) < 0 \) with \( e(n) > 0 \), it follows that the equation \( e(x) = 0 \) has a root \( a \in (\lambda_1(P_{3,1,l_2,l_3}), n) \). So \( \lambda_1(P_{3,1,n-5}) > a > \lambda_1(P_{3,1,l_2,l_3}). \)
Hence if $T \cong P^1_{3,4} (l_2 \neq 0)$ and $T \not\cong S^4_n, P^1_{3,1,n-6,2}, P^1_{3,1,n-5}$, we have $\lambda_1(P^1_{3,1,n-5}) > \lambda_1(T)$ (by (*)). This completes the proof of the Lemma 2.10. \hfill \Box

By a similar argument to that in the proof of Lemma 2.10, we have the following result.

**Lemma 2.11.** Let $T \in \mathcal{T}^2_n \setminus \{S^4_n, P^1_{3,1,0,n-4}, P^1_{3,1,n-6,2}\}$ and $n \geq 9$. Then $\lambda_1(T) \leq \lambda_1(P^1_{3,1,n-5})$, where $\lambda_1(P^1_{3,1,n-5})$ is the largest root of the equation

$$x^5 - (n+4)x^4 + (7n-7)x^3 + (32-14n)x^2 + (8n-17)x - n = 0,$$

and equality holds if and only if $T \cong P^1_{3,1,n-5}$.

**Lemma 2.12**

(1) $\lambda_1(P^1_{3,1,0,n-4}) > \lambda_1(P^1_{3,1,n-6,2})$ for $n \geq 8$;
(2) $\lambda_1(P^1_{3,1,n-6,2}) > \lambda_1(K^1_{3,1,1})$ for $n \geq 7$;
(3) $\lambda_1(K^1_{3,1,1}) > \lambda_1(P^4_{2,1,1})$ for $n \geq 15$;
(4) $\lambda_1(K^1_{3,1,1}) > \lambda_1(P^1_{3,1,n-5})$ for $n \geq 10$.

**Proof.** (1) By an elementary calculation, we have

$$p(P^1_{2,1}, x) = x(x - 1)^{n-4}[x^3 - (n+2)x^2 + (2n + l_1 l_2 + 1)x - n]. \quad (S6)$$

So we have

$$p(P^3_{2,1,n-5}, x) = x(x - 1)^{n-4} f(x),$$

where

$$f(x) = x^3 - (n+2)x^2 + (5n-14)x - n.$$

By (S2) and (S3), we have

$$p(P^1_{3,1,0,n-4}, x) = x(x - 1)^{n-5} g(x),$$

$$p(P^1_{3,1,n-6,2}, x) = x(x - 1)^{n-6} h(x),$$

where

$$g(x) = x^4 - (n+3)x^3 + (5n-4)x^2 - (6n-10)x + n,$$

$$h(x) = x^5 - (n+4)x^4 + (7n-7)x^3 + (32-14n)x^2 + (7n-10)x - n.$$  

Since $(x - 1)f(x) - g(x) = (n-8)x^2 + 4x > 0$ for $x > 0$ and $n \geq 8$, we have $p(P^3_{2,1,n-5}, x) > p(P^1_{3,1,0,n-4}, x)$ when $x \geq \lambda_1(P^1_{3,1,0,n-4}) > n - 2 \geq 6$. Thus $\lambda_1(P^1_{3,1,0,n-4}) > \lambda_1(P^3_{2,1,n-5})$ by Lemma 2.4.
Comparing $f(x)$ and $h(x)$, we have

$$h(x) = (x^2 - 2x + 3) f(x) + 2[5x^2 + (16 - 5n)x + n].$$  \hspace{1cm} (S7)

Note that $5x^2 + (16 - 5n)x + n = 5x(x - n + 3) + n + x$. By $n \geq 8$, $\lambda_1(P_3^{1,n-6,2}) > n - 3 \geq 5$. Thus when $x = \lambda_1(P_3^{1,n-6,2}) > n - 3$, we have $h(x) = 0$, $x^2 - 2x + 3 = (x - 1)^2 + 2 > 0$ and $5x^2 + (16 - 5n)x + n = (5x + 1)(x - (n - 3)) + 2n - 3 > 0$. So $f(\lambda_1(P_3^{1,n-6,2})) < 0$ by (S7). By a similar argument as that in case 3 of Lemma 2.10 (*), we have $f(n) > 0$ and $\lambda_1(P_3^{1,n-6,2}) < n$. Combining $f(\lambda_1(P_3^{1,n-6,2})) < 0$ with $f(n) > 0$ and noting that $\lambda_1(P_3^{2,n-3})$ is the largest root of $f(x) = 0$, we have, $\lambda_1(P_3^{2,n-3}) > \lambda_1(P_3^{1,n-6,2})$ for $n \geq 8$.

(2) By (S0),

$$p(K_{1,3}^{n-7,1,1,1}, x) = (x - 1)^n - (x^2 - 3x + 1)^2 b(x),$$

where $b(x) = x^3 - nx^2 + (3n - 8)x - n$.

Then $h(x) + 2x = (x^2 - 2x + 3)b(x)$. When $x = \lambda_1(K_{1,3}^{n-7,1,1,1})$, we have $b(x) = 0$ and then $h(x) < 0$ by $n \geq 7$ and $\lambda_1(K_{1,3}^{n-7,1,1,1}) > n - 3$. By a similar argument as that in case 3 of Lemma 2.10 (*), we have $h(n) > 0$ and $\lambda_1(K_{1,3}^{n-7,1,1,1}) < n$. Combining $b(\lambda_1(K_{1,3}^{n-7,1,1,1})) < 0$ with $h(n) > 0$ and noting that $\lambda_1(P_3^{1,n-6,2})$ is the largest root of $h(x) = 0$, we have $\lambda_1(P_3^{1,n-6,2}) > \lambda_1(K_{1,3}^{n-7,1,1,1})$ for $n \geq 7$.

(3) From equality (S6), we have

$$p(K_{1,3}^{d,n-6,6}, x) = x(x - 1)^{n-4}[x^3 - (n + 2)x^2 + (6n - 23)x - n].$$

Let \(i(x) = x^3 - (n + 2)x^2 + (6n - 23)x - n\), then \(i(x) = b(x) + (-2x + 3n - 15)x\). Note that $n \geq 15$ and $n - 4 < \lambda_1(P_3^{d,n-6}) < n$. When $x = \lambda_1(P_3^{d,n-6})$, we have $i(x) = 0$ and $(-2x + 3n - 15)x > (-2m + 3n - 15)(n - 4) = (n - 15)(n - 4) > 0$. So $b(\lambda_1(P_3^{d,n-6})) < 0$. By a similar argument as that in case 3 of Lemma 2.10 (*), we have $b(n) > 0$. Combining $b(\lambda_1(P_3^{d,n-6})) < 0$ with $b(n) > 0$ and noting that $\lambda_1(K_{1,3}^{n-7,1,1,1})$ is the largest root of $b(x) = 0$, we have $\lambda_1(K_{1,3}^{n-7,1,1,1}) > \lambda_1(P_3^{d,n-6})$.

(4) By an elementary calculation, we have

$$p(K_{1,3}^{1,n-5}, x) = x(x - 1)^{-e(x)},$$

where $e(x) = x^5 - (n + 4)x^4 + (7n - 7)x^3 + (32 - 14n)x^2 + (8n - 17)x - n$. Then $e(x) = (x^2 - 4x + 1)b(x) + (n - 9)x$. Note that $n \geq 10$ and $\lambda_1(K_{1,3}^{n-7,1,1,1}) \geq n - 3$. When $x = \lambda_1(P_3^{1,n-5})$, we have $e(x) = 0$, $x^2 - 4x + 1 > 0$ and $(n - 9)x > 0$. So $b(\lambda_1(P_3^{1,n-5})) < 0$. By a similar argument as that in case 3 of Lemma 2.10 (*), we have $b(n) > 0$ and $\lambda_1(P_3^{1,n-5}) < n$. Combining $b(\lambda_1(P_3^{1,n-5})) < 0$ with $b(n) > 0$ and noting that $\lambda_1(K_{1,3}^{n-7,1,1,1})$ is the largest root of $b(x) = 0$, we have $\lambda_1(P_3^{1,n-5}) < \lambda_1(K_{1,3}^{n-7,1,1,1})$ for $n \geq 10$. \(\square\)
Lemma 2.13. Let $T \in \mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, P_2^{3,n-5}\}$ and $n \geq 15$. Then $\lambda_1(T) \leq \lambda_1(K_{1,3}^{n-7,1,1,1})$ and equality holds if and only if $T \cong K_{1,3}^{n-7,1,1,1}$.

Proof. By Lemmas 2.6 and 2.12, the result holds immediately. □

Lemma 2.14. Let $T \in \mathcal{F}_n \setminus \{S_n^4, P_2^{1,0,n-4}, P_3^{1,n-6,2}\}$ and $n \geq 10$. Then $\lambda_1(T) \leq \lambda_1(K_{1,3}^{n-7,1,1,1})$ and equality holds if and only if $T \cong K_{1,3}^{n-7,1,1,1}$.

Proof. By Lemmas 2.9–2.12, the result holds immediately. □

Now we present our main results.

Theorem 2.1. Let $T \in \mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, S_n^4\}$ and $n \geq 8$. Then $\lambda_1(T) \leq \lambda_1(P_3^{1,0,n-4})$, where $\lambda_1(P_3^{1,0,n-4})$ is the largest root of the equation

$$x^4 - (n + 3)x^3 + (5n - 4)x^2 - (6n - 10)x + n = 0,$$

and equality holds if and only if $T \cong P_3^{1,0,n-4}$.

Proof. Let $T \in \mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, S_n^4\}$ and $n \geq 8$. Note that

$$\mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, S_n^4\} = (\mathcal{F}_n \setminus \{S_n^2, S_n^3\}) \cup (\mathcal{F}_n \setminus \{S_n^4\}) \cup \left( \bigcup_{i=3}^{n-3} \mathcal{F}_n^i \right).$$

If $T \in \mathcal{F}_n \setminus \{S_n^2, S_n^3\}$, by Lemmas 2.6 and 2.12(1), we have

$$\lambda_1(P_3^{1,0,n-4}) > \lambda_1(P_2^{3,n-5}) \geq \lambda_1(T).$$

If $T \in \mathcal{F}_n \setminus \{S_n^4\}$, by Lemma 2.9, we have $\lambda_1(P_3^{1,0,n-4}) \geq \lambda_1(T)$ and equality holds if and only if $T \cong P_3^{1,0,n-4}$.

If $T \in \bigcup_{i=3}^{n-3} \mathcal{F}_n^i$, by Lemmas 2.8, 2.12(1) and 2.12(2), we have

$$\lambda_1(P_3^{1,0,n-4}) > \lambda_1(P_2^{3,n-5}) > \lambda_1(P_3^{1,n-6,2}) > \lambda_1(K_{1,3}^{n-7,1,1,1}) \geq \lambda_1(T).$$

So Theorem 2.1 holds. □

Theorem 2.2. Let $T \in \mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, S_n^4, P_3^{1,0,n-4}\}$ and $n \geq 9$. Then $\lambda_1(T) \leq \lambda_1(P_2^{3,n-5})$, where $\lambda_1(P_2^{3,n-5})$ is the largest root of the equation

$$x^3 - (n + 2)x^2 + (5n - 14)x - n$$

and equality holds if and only if $T \cong P_2^{3,n-5}$.

Proof. Let $T \in \mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, S_n^4, P_3^{1,0,n-4}\}$ and $n \geq 9$. Note that

$$\mathcal{F}_n \setminus \{S_n^1, S_n^2, S_n^3, S_n^4, P_3^{1,0,n-4}\}$$
So Theorem 2.2 holds.

Let \( T \in \mathcal{F}_n \setminus \{ S^2_n, S^3_n \} \), by Lemma 2.6, we have \( \lambda_1(P^{3,\pi-5}_2) \geq \lambda_1(T) \), and equality holds if and only if \( T \cong P^{3,\pi-5}_2 \).

If \( T \in \mathcal{F}_n \setminus \{ S^2_n, S^3_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2 \} \), by Lemmas 2.10 and 2.12(1), we have \( \lambda_1(P^{3,\pi-5}_2) > \lambda_1(P^{1,\pi-6,2}_3) \geq \lambda_1(T) \).

If \( T \in \bigcup_{i=3}^{n-3} \mathcal{F}_n \), by Lemmas 2.8, 2.12(1) and 2.12(2), we have \( \lambda_1(P^{3,\pi-5}_2) > \lambda_1(P^{1,\pi-6,2}_3) \geq \lambda_1(K^{n-7,1,1,1}_{1,3}) \geq \lambda_1(T) \).

So Theorem 2.2 holds. \( \square \)

Theorem 2.3. Let \( T \in \mathcal{F}_n \setminus \{ S^1_n, S^2_n, S^3_n, S^4_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2 \} \) and \( n \geq 15 \). Then \( \lambda_1(T) \leq \lambda_1(P^{1,\pi-6,2}_3) \), where \( \lambda_1(P^{1,\pi-6,2}_3) \) is the largest root of the equation

\[
x^5 - (n + 4)x^4 + (7n - 7)x^3 + (32 - 14n)x^2 + (7n - 10)x - n = 0,
\]

and equality holds if and only if \( T \cong P^{1,\pi-6,2}_3 \).

Proof. Let \( T \in \mathcal{F}_n \setminus \{ S^1_n, S^2_n, S^3_n, S^4_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2 \} \) and \( n \geq 15 \). Note that

\[
\mathcal{F}_n \setminus \{ S^1_n, S^2_n, S^3_n, S^4_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2 \} \Rightarrow (\mathcal{F}_n \setminus \{ S^2_n, S^3_n, S^4_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2 \}) \cup (\mathcal{F}_n \setminus \{ S^1_n, S^2_n, S^3_n, S^4_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2 \}).
\]

If \( T \in \mathcal{F}_n \setminus \{ S^2_n, S^3_n, P^{3,\pi-5}_2 \} \), by Lemmas 2.6, 2.12(2) and 2.12(3), we have \( \lambda_1(P^{3,\pi-5}_2) > \lambda_1(K^{n-7,1,1,1}_{1,3}) \geq \lambda_1(T) \).

If \( T \in \mathcal{F}_n \setminus \{ S^4_n, P^{1,0,\pi-4}_3 \} \), by Lemma 2.10, we have \( \lambda_1(P^{1,\pi-6,2}_3) \geq \lambda_1(T) \), and equality holds if and only if \( T \cong P^{1,\pi-6,2}_3 \).

If \( T \in \bigcup_{i=3}^{n-3} \mathcal{F}_n \), by Lemmas 2.8 and 2.12(2), we have \( \lambda_1(P^{1,\pi-6,2}_3) > \lambda_1(K^{n-7,1,1,1}_{1,3}) \geq \lambda_1(T) \).

So Theorem 2.3 holds. \( \square \)

Theorem 2.4. Let \( T \in \mathcal{F}_n \setminus \{ S^1_n, S^2_n, S^3_n, S^4_n, P^{1,0,\pi-4}_3, P^{3,\pi-5}_2, P^{1,\pi-6,2}_3 \} \) and \( n \geq 15 \). Then

\[
\lambda_1(T) \leq \lambda_1(K^{n-7,1,1,1}_{1,3}),
\]

where \( \lambda_1(K^{n-7,1,1,1}_{1,3}) \) is the largest root of the equation \( x^3 - nx^2 + (3n - 8)x - n = 0 \), and equality holds if and only if \( T \cong K^{n-7,1,1,1}_{1,3} \).
Proof. Let $T \in \mathcal{F}_n \setminus \{S^1_n, S^2_n, S^3_n, P^1_{3,n-4}, P^3_{n-5}, P^1_{3,n-6,2}\}$ and $n \geq 15$. Note that
\[
\mathcal{F}_n \setminus \{S^1_n, S^2_n, S^3_n, P^1_{3,n-4}, P^3_{n-5}, P^1_{3,n-6,2}\} = \mathcal{F}_n \setminus \{S^2_n, P^3_{n-5}\} \cup \mathcal{F}_n \setminus \{S^4_n, P^1_{3,n-4}, P^1_{3,n-6,2}\} \cup \left(\bigcup_{i=3}^{n-3} \mathcal{F}_n^i\right).
\]
If $T \in \mathcal{F}_n \setminus \{S^2_n, P^3_{n-5}\}$, by Lemmas 2.6 and 2.12(3), we have
\[
\lambda_1(K^{n-7,1,1,1}_{1,3}) > \lambda_1(P^4_{n-6}) \geq \lambda_1(T).
\]
If $T \in \mathcal{F}_n \setminus \{S^4_n, P^1_{3,n-4}, P^1_{3,n-6,2}\}$, by Lemmas 2.11 and 2.12(4), we have
\[
\lambda_1(K^{n-7,1,1,1}_{1,3}) > \lambda_1(P^4_{3,n-5}) \geq \lambda_1(T).
\]
If $T \in \bigcup_{i=3}^{n-3} \mathcal{F}_n^i$, by Lemma 2.8, we have
\[
\lambda_1(K^{n-7,1,1,1}_{1,3}) \geq \lambda_1(T),
\]
and equality holds if and only if $T \cong K^{n-7,1,1,1}_{1,3}$. So Theorem 2.4 holds. □

Acknowledgments

The authors would like to thank anonymous referees for their valuable suggestions.

References