On the spectral radius of graphs with cut edges

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Abstract

We study the spectral radius of graphs with \( n \) vertices and \( k \) cut edges. In this paper, we show that of all the connected graphs with \( n \) vertices and \( k \) cut edges, the maximal spectral radius is obtained uniquely at \( K^k_{n}\), where \( K^k_{n} \) is a graph obtained by joining \( k \) independent vertices to one vertex of \( K_{n-k} \). We also discuss the limit point of the maximal spectral radius.

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1. Introduction

Let \( G = (V, E) \) be a simple undirected graph with \( n \) vertices and \( k \) cut edges. For \( v \in V(G) \), the degree of \( v \), written by \( d_G(v) \), is the number of edges incident with \( v \). We will use \( G - x \) or \( G - xy \) to denote the graph that arises from \( G \) by deleting the vertex \( x \in V(G) \) or the edge \( xy \in E(G) \). Similarly, \( G + xy \) is a graph that arises from \( G \) by adding an edge \( xy \notin E(G) \), where \( x, y \in V(G) \).
A cut edge in a connected graph $G$ is an edge whose deletion breaks the graph into two components. Denote by $G_k^n$ the set of graphs with $n$ vertices and $k$ cut edges. The graph $K_k^n$ is a graph obtained by joining $k$ independent vertices to one vertex of $K_{n-k}$. For example, for $n = 6$, $K_6^0 = K_6$, $K_6^3$ is a star $K_{1,3}$ and $K_6^1$, $K_6^2$, $K_6^3$ are shown in Fig. 1. In general, $K_n^0 = K_n$, $K_{n-1}^{n-1}$ is a star $K_{1,n-1}$ and $K_n^{n-2} \cong K_n^{n-1}$.

Let $A(G)$ be the adjacency matrix of a graph $G$. The spectral radius, $\rho(G)$, of $G$ is the largest eigenvalue of $A(G)$. For results on the spectral radii of graphs, the reader is referred to [3–5] and the references therein. When $G$ is connected, $A(G)$ is irreducible and by the Perron–Frobenius Theorem, the spectral radius is simple and has a unique positive eigenvector. We will refer to such an eigenvector as the Perron vector of $G$. Note that the spectral radius increases if we add an edge to $G$.

In [2], Brualdi and Solheid proposed the following problem concerning spectral radii:

**Problem.** Given a set of graphs $\mathcal{I}$, find an upper bound for the spectral radii of graphs in $\mathcal{I}$ and characterize the graphs in which the maximal spectral radius is attained.

In [1], Berman and Zhang studied this question for graphs with $n$ vertices and $k$ cut vertices, and described the graph that has the maximal spectral radius in this class. In this paper, we investigate the same question for $\mathcal{I} = G_k^n$, the set of connected graphs with $n$ vertices and $k$ cut edges. We show that of all the connected graphs with $n$ vertices and $k$ cut edges, the maximal spectral radius is obtained uniquely at $K_k^n$. Note that the special cases for $k = n - 1$ and $k = n - 3$ are contained in [6,8], respectively.

2. **Lemmas and results**

Denote the characteristic polynomial of a graph $G$ by $p(G; \lambda)$.

**Lemma 1** [7]. Let $v$ be a pendant vertex of a graph $G$ and $vw \in E(G)$. Then

$$p(G; \lambda) = \lambda p(G - v; \lambda) - p(G - v - w; \lambda).$$
The proof of our main result is carried out mainly by the following lemma (given in [9]), which is a stronger version of a similar lemma in [8]. We cite the proof here for reference only.

**Lemma 2** [9]. Let $G$ be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let $u$, $v$ be two vertices of $G$ and $d_v$ be the degree of vertex $v$. Suppose $v_1, v_2, \ldots, v_t \in N(v) \setminus N(u)$, and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of $A(G)$, where $x_i$ corresponds to the vertex $v_i$ ($1 \leq i \leq n$). Let $G^*$ be the graph obtained from $G$ by deleting the edges $(v, v_i)$ and adding the edges $(u, v_i)$ ($1 \leq i \leq t$). If $x_u \geq x_v$, then

$$\rho(G) < \rho(G^*).$$

**Proof.** Since

$$x^T(A(G^*) - A(G))x = 2 \sum_{i=1}^{s} x_i(x_u - x_v) \geq 0,$$

we have

$$\rho(G^*) = \max_{||y||=1} y^T A(G^*) y \geq x^T A(G^*) x \geq x^T A(G) x = \rho(G).$$

(1)

Assume that $\rho(G^*) = \rho(G)$, then the equalities in (1) hold. So

$$\rho(G^*) = x^T A(G^*) x,$$

and then $A(G^*) x = \rho(G^*) x$. Thus

$$\rho(G^*) x_v = (A(G^*) x)_v = \sum_{v_i \in N_G^*(v)} x_i.$$  

(2)

Since $A(G) x = \rho(G) x$, we have

$$\rho(G) x_v = (A(G) x)_v = \sum_{v_i \in N_G(v)} x_i = \sum_{v_i \in N_G^*(v)} x_i + \sum_{i=1}^{s} x_i.$$  

(3)

Note that $x_i > 0$ ($1 \leq i \leq n$) by $x = (x_1, x_2, \ldots, x_n)$ being the Perron vector of $G$, and hence, by (2) and (3), $\rho(G^*) x_v < \rho(G) x_v$. Thus $\rho(G^*) < \rho(G)$, a contradiction. Therefore $\rho(G) < \rho(G^*)$. □

Let $K_{1,k}$ be a star with vertex set $V(K_{1,k}) = \{v_0, v_1, \ldots, v_k\}$, where $v_0$ is the center of the star. Let $K(a_0, \{a_1, \ldots, a_k\})$ be a graph obtained from $K_{1,k}$ by replacing $v_i$ by clique $K_{a_i}$ ($a_i \geq 1$, $i = 0, 1, \ldots, k$) (see Fig. 2). Denote

$$\mathcal{K}_{n}^k = \left\{ K(a_0, \{a_1, \ldots, a_k\}) : a_i \geq 1, \ 0 \leq i \leq k, \ \sum_{i=0}^{k} a_i = n \right\}.$$
Fig. 2. $K(a_0, [a_1, \ldots, a_k])$.

Obviously, $K^k_n = K(n - k, [1, \ldots, 1])$.

**Theorem 3.** Of all the connected graphs with $n$ vertices and $k$ cut edges, the maximal spectral radius is obtained uniquely at $K^k_n$.

**Proof.** We have to prove that if $G \in \mathcal{G}^k_n$, then $\rho(G) \leq \rho(K^k_n)$ with equality only when $G \cong K^k_n$. Let $E_1 = \{e_1, e_2, \ldots, e_k\}$ be the set of the cut edges of $G$. Denote the Perron vector of $A(G)$ by $x = (x_0, x_1, \ldots, x_{n-1})$, where $x_i$ corresponds to the vertex $v_i$ ($0 \leq i \leq n - 1$). Note that if we add an edge $e$ to a connected graph $G$, then $\rho(G + e) > \rho(G)$ as the adjacent matrix of a connected graph is irreducible. So we can have the following assumption.

**Assumption 0.** Each component of $G - E_1$ is a clique.

If $k = 0$, then $G = K_n$ by Assumption 0 and the theorem holds immediately. Therefore we may assume that $k \geq 1$. Again, by Assumption 0, we can denote the components of $G - E_1$ by $K_{a_0}, K_{a_1}, \ldots, K_{a_k}$, where $a_0, a_1, \ldots, a_k$ are the numbers of the vertices of these components, respectively. Then $a_0 + a_1 + \cdots + a_k = n$.

Let $V_{a_i} = \{v \in K_{a_i} : v$ is an end vertex of the cut edges of $G\}$. Choose $G \in \mathcal{G}^k_n$ such that the spectral radius of $G$ is as large as possible. In the following, we will prove some facts.

**Fact 1.** $|V_{a_i}| = 1$ for $0 \leq i \leq k$. 

Proof of Fact 1. Suppose that $|V_{a_i}| > 1$ for some $i$, $0 \leq i \leq k$. Let $u, u' \in V_{a_i}$ and assume that $x_u > x_{u'}$. Denote $N(u') \setminus N(u) = \{w_1, w_2, \ldots, w_s\}$. Then $s \geq 1$ by $u' \in V_{a_i}$. Let $G^* = G - \{u'w_1, \ldots, u'w_s\} + \{uw_1, \ldots, uw_s\}$. Then $G^* \in \mathcal{E}_n^k$. By Lemma 2, $\rho(G^*) > \rho(G)$, a contradiction. Therefore $|V_{a_i}| = 1$. □

Fact 2. $G \in \mathcal{E}_n^k$.

Proof of Fact 2. Assume that $G \notin \mathcal{E}_n^k$. Then there exist $v \in V(K_{a_i})$ and $v' \in V(K_{a_j}), 0 \leq i, j \leq k, i \neq j$ such that $|N(v) \setminus V(K_{a_i})| \geq 2$ and $|N(v') \setminus V(K_{a_j})| \geq 2$. Obviously, $[v] = V_{a_i}$ and $[v'] = V_{a_j}$ by Fact 1. Assume, without loss of generality, that $x_v \geq x_{v'}$. Denote $N(v') \setminus (V(K_{a_i}) \cup \{v\}) = \{z_1, \ldots, z_t\}$. Then $t \geq 1$ by $|N(v') \setminus V(K_{a_j})| \geq 2$. Let $G^* = G - \{v'z_1, \ldots, v'u_1\} + \{vz_1, \ldots, vz_t\}$. Then $G^* \in \mathcal{E}_n^k$. By Lemma 2, $\rho(G^*) > \rho(G)$, a contradiction. □

By Fact 1, we can assume that $V_{a_i} = \{v_i\}, 0 \leq i \leq k$. By Fact 2, we can assume that $v_0v_j \in E(G), 1 \leq j \leq k$. Assume, without loss of generality, that $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 1$.

Fact 3. $G \cong K\left(a_0, \{1, \ldots, 1, n - a_0 - k + 1\}\right)$.

Proof of Fact 3. Suppose that $a_i > 1$ for some $i$, $1 \leq i \leq k - 1$. Then $a_k > 1$. Without loss of generality, we assume that $x_{v_1} \geq x_{v_0}$. Denote $N(v_1) \setminus \{v_0\} = \{w_1, \ldots, w_{a_1-1}\}$. Let $G^* = G - \{v_1w_1, \ldots, v_1w_{a_1-1}\} + \{v_kw_1, \ldots, v_kw_{a_1-1}\}$. Then $G^* \in \mathcal{E}_n^k$. By Lemma 2, $\rho(G^*) > \rho(G)$, a contradiction. □

Fact 4. $a_0 = n - k$.

Proof of Fact 4. Obviously, $a_0 = n - (a_1 + a_2 + \cdots + a_k) \leq n - k$. Suppose that $a_0 < n - k$. Then $a_1 > 1$ by Fact 3. Denote $N(v_1) \setminus \{v_0\} = \{w_1, \ldots, w_{a_1-1}\}$ and $N(v_k) \setminus \{v_0\} = \{z_1, \ldots, z_{a_k-1}\}$. If $x_{v_1} \geq x_{v_0}$, we let
\[
G^* = G - \{v_0v_1, \ldots, v_0v_{k-1}, v_0z_1, \ldots, v_0z_{a_k-1}\} + \{v_kw_1, \ldots, v_kw_{k-1}, v_kz_1, \ldots, v_kz_{a_k-1}\}.
\]
If $x_{v_1} \leq x_{v_0}$, we let
\[
G^* = G - \{v_kw_1, \ldots, v_kw_{a_k-1}\} + \{v_0w_1, \ldots, v_0w_{a_k-1}\}.
\]
Then in either case, $G^* \in \mathcal{E}_n^k$. By Lemma 2, $\rho(G^*) > \rho(G)$, a contradiction. □

By Fact 4, Theorem 3 holds. □
Theorem 4. The characteristic polynomial of $K_k^\ast n$ is 
$$
\lambda^{k-1}(\lambda + 1)^{n-k-2}(\lambda^3 - (n - k - 2)\lambda^2 - (n - 1)\lambda + (n - k - 2)k).
$$

Proof. By a repeated using of Lemma 1, we have 
$$
p(K_k^\ast n; \lambda) = \lambda p(K_{n-1}^\ast; \lambda) - p(K_{n-k-1} \cup (k-1)K_1; \lambda) 
= \lambda^2 p(K_{n-2}^\ast; \lambda) - \lambda p(K_{n-k-1} \cup (k-2)K_1; \lambda) 
- p(K_{n-k-1} \cup (k-1)K_1; \lambda) 
= \cdots 
= \lambda^k p(K_{n-k}; \lambda) - \lambda^{k-1} p(K_{n-k-1}; \lambda) - \lambda^{k-2} p(K_{n-k-1} \cup K_1; \lambda) 
- \cdots - \lambda p(K_{n-k-1} \cup (k-2)K_1; \lambda) 
- p(K_{n-k-1} \cup (k-1)K_1; \lambda) 
= \lambda^k (\lambda - n + k + 1)(\lambda + 1)^{n-k-1} 
- k\lambda^{k-1}(\lambda - n + k + 2)(\lambda + 1)^{n-k-2} 
= \lambda^{k-1}(\lambda + 1)^{n-k-2}(\lambda^3 - (n - k - 2)\lambda^2 
- (n - 1)\lambda + (n - k - 2)k). \quad \Box
$$

Corollary 5. The spectral radius $\rho$ of the graph $K_k^\ast n$ satisfies the equation 
$$
\rho^3 - (n - k - 2)\rho^2 - (n - 1)\rho + (n - k - 2)k = 0.
$$

By Theorem 3 and Corollary 5, we have the following corollaries.

Corollary 6 [6]. Let $T$ be a tree on $n$ vertices. Then 
$$
\rho(T) \leq \sqrt{n - 1},
$$
and the equality holds if and only if $T \cong K_{1,n-1}$, the star with $n$ vertices.

Corollary 7. Let $\rho$ be the spectral radius of the graph $K_k^\ast n$. If $1 \leq k \leq n - 1 - \sqrt{n - 1}$, then 
$$
\rho < n - k - 1 + \frac{k}{(n-k)^2 - n}.
$$
Moreover, if $k$ is fixed, then 
$$
\lim_{n \to \infty} \left\{ \rho - \left( n - k - 1 + \frac{k}{(n-k)^2 - n} \right) \right\} = 0.
$$

Proof. Since $K_k^\ast n$ contain a complete subgraph of order $n - k$ and a star $K_{1,n-1}$, 
$$
\rho(K_k^\ast n) > \max\{n - k - 1, \sqrt{n - 1}\}. \text{ Since } 1 \leq k \leq n - 1 - \sqrt{n - 1}, \text{ we have } n \geq 3 \text{ and } n - k - 1 \geq \sqrt{n - 1} > 0. \text{ Hence } \rho(K_k^\ast n) > n - k - 1 \text{ and } (n-k)^2 - n \geq 2\sqrt{n - 1} > 0. \text{ Denote } \rho = n - k - 1 + x, \text{ where } x > 0. \text{ By Corollary 5, we have }$$
\[ x^3 + (2n - 2k - 1)x^2 + ((n - k)^2 - n)x - k = 0. \]
Thus \( x < \frac{k}{(n - k)^2 - n} \) and the results hold. \( \square \)

**Corollary 8.** Let \( \rho \) be the spectral radius of the graph \( K_n^k \). If \( k \) is fixed and \( n - 1 - \sqrt{n - 1} < k \leq n - 3 \), then\[ \rho < \sqrt{n - 1} + \frac{n - k - 2}{2(\sqrt{n - 1} - (n - k - 2))}. \]

**Proof.** Since \( K_n^k \) contain a complete subgraph of order \( n - k \) and a star \( K_1, n - 1 \), \( \rho(K_n^k) > \max\{n - k - 1, \sqrt{n - 1}\} \). Since \( k > n - 1 - \sqrt{n - 1} \), we have \( n - k - 1 < \sqrt{n - 1} \). Thus \( \rho(K_n^k) > \sqrt{n - 1} \). Denote \( \rho = \sqrt{n - 1} + y \), where \( y > 0 \). By Corollary 5, we have\[ y^3 + \left(3\sqrt{n - 1} - (n - k - 2)\right)y^2 + 2\left((n - 1) - \sqrt{n - 1}(n - k - 2)\right)y - (n - k - 1)(n - k - 2) = 0. \]
Thus\[ y < \frac{(n - k - 2)(n - k - 1)}{2((n - 1) - \sqrt{n - 1}(n - k - 2))} < \frac{n - k - 2}{2(\sqrt{n - 1} - (n - k - 2))} \]
as \( \sqrt{n - 1} > n - k - 1 \), and hence \( \rho < \sqrt{n - 1} + \frac{n - k - 2}{2(\sqrt{n - 1} - (n - k - 2))}. \) \( \square \)

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**References**