Edge fault tolerance of super edge connectivity for three families of interconnection networks

Dongye Wang, Mei Lu *

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

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ABSTRACT

Let \( G = (V, E) \) be a connected graph. \( G \) is said to be super edge connected (or super-\( \kappa \) for short) if every minimum edge cut of \( G \) isolates one of the vertex of \( G \). A graph \( G \) is called \( m \)-super-\( \kappa \) if for any edge set \( S \subset E(G) \) with \( |S| < m \), \( G - S \) is still super-\( \kappa \). The maximum cardinality of \( m \)-super-\( \kappa \) is called the edge fault tolerance of super edge connectivity of \( G \). In this paper, we discuss the edge fault tolerance of super edge connectivity of three families of interconnection networks.

1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and only consider finite simple undirected graphs. Let \( G = (V, E) \) be a connected graph. For \( v \in V(G) \), the degree of \( v \), written by \( d(v) \), is the number of edges incident with \( v \). Let \( \delta(G) = \min\{d(v) | v \in V(G)\} \) and it is called the minimum degree of \( G \). For a subset \( S \subset V(G) \), \( G[S] \) is the subgraph of \( G \) induced by \( S \). An edge subset \( T \subset E(G) \) is an edge cut if \( G - T \) is disconnected. The edge connectivity, denoted by \( \kappa \), is the minimum cardinality of the set of all edge cuts of \( G \).

It is well known that the edge connectivity \( \kappa \) is an important measurement for the fault tolerance of networks. In general, the larger \( \kappa \) is, the more reliable a network is. Obviously, \( \kappa \leq \delta(G) \). In [2], Bauer et al.*** defined the so-called super-\( \kappa \) graphs. A graph \( G \) is said to be super edge-connected (in short, super-\( \kappa \)) if every minimum edge cut is the set of edges incident with some vertex of \( G \). There are much research on super-\( \kappa \), the reader is referred to [5,11,13,16,18] and the references therein.

In [8,9], Esfahanian and Hakimi proposed the concept of restricted edge connectivity of graphs which generalized the concept of super-\( \kappa \). Then Fábrega and Fiol [10] introduced the \( k \)-restricted edge connectivity of interconnection networks. Let \( G \) be a graph. An edge set \( S \subset E \) is said to be a \( k \)-restricted edge cut if \( G - S \) is disconnected and there are no components whose cardinalities are smaller than \( k \) in \( G - S \). The minimum cardinality of \( k \)-restricted edge cut of \( G \) is called \( k \)-restricted edge connectivity of \( G \), denoted by \( \lambda_k(G) \). \( k \)-restricted edge connectivity is another important parameter in measuring the reliability and fault tolerance of large interconnection networks. In particular, estimating the bound for \( \lambda_k(G) \) is of great interest, and many results have been obtained in [1,6,12,17,19–26].

In [14], Hong and Meng defined another index to measure the reliability of networks.

* Corresponding author.

E-mail addresses: wangdy04@mails.tsinghua.edu.cn (D. Wang), mlu@math.tsinghua.edu.cn (M. Lu).
Definition 1.1 ([14]). A graph $G$ is said to be $m$-super edge connected ($m$-super-$\lambda$ for short) if $G - S$ is super-$\lambda$ for any $S \subseteq E(G)$ with $|S| \leq m$.

From the definition, we know that $G$ is 0-super-$\lambda$ is equivalent to that $G$ is super-$\lambda$. Furthermore, if $G$ is $a$-super-$\lambda$, then $G$ is also $b$-super-$\lambda$, for any $0 \leq b \leq a$. So $m$-super-$\lambda$ is a generalization of super-$\lambda$.

The edge fault tolerance of super edge connectivity of $G$ is an integer $m$ such that $G$ is $m$-super-$\lambda$ but not $(m + 1)$-super-$\lambda$, denoted by $S_k(G)$.

In [14], Hong and Meng gave an upper and lower bound for $S_k(G)$. Moreover, more refined bounds for $S_k(G)$ of Cartesian product graphs, edge transitive graphs and regular graphs are given.

In this paper, we will give some bounds of $S_k(G)$ for three families of interconnection networks.

Before proceeding, we introduce some notions which will be used in the discussions in the next sections. Let $G = (V, E)$ be a graph. For two disjoint vertex sets $U_1, U_2 \subseteq V(G)$, we use $[U_1, U_2]_G$ to denote the edge set of $G$ with one end in $U_1$ and the other end in $U_2$. For any vertex set $A \subseteq V(G)$, denote \( \omega_G(A) = \begin{bmatrix} A \setminus \overline{A} \end{bmatrix}_G \), where $\overline{A} = V(G) - A$ is the complement of $A$. The subscription $G$ is omitted when the graph under consideration is obvious. Next we cite two lemmas which will be used in the following proofs.

Lemma 1.2 ([14]). A graph $G$ is super-$\lambda$ if and only if $\omega(A) > \delta(G)$ for any $A \subseteq V(G)$ with $2 \leq |A| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and $G[A]$ and $G[\overline{A}]$ being connected.

Lemma 1.3 ([14]). Let $G$ be a connected graph with minimum degree $\delta(G)$. Then $S_k(G) \leq \delta(G) - 1$.

2. Three families of interconnection networks

The following three families of interconnection networks which we will discuss in the next sections were introduced in [4].
2.1. The first family $G(G_0, G_1; M)$ of networks

Let $G_0$ and $G_1$ be two graphs with the same number of vertices. Then $G(G_0, G_1; M)$ is a new graph with vertex set $V(G) = V(G_0) \cup V(G_1)$ and edge set $E(G) = E(G_0) \cup E(G_1) \cup M$, where $M$ is an arbitrary perfect matching between the vertices of $G_0$ and $G_1$ (see Fig. 1).

2.2. The second family $G(G_0, G_1, \ldots, G_{r-1}; M_{r})$ of networks

Let $r$ and $t$ be positive integers with $r \geq 3$. Let $G_0, G_1, \ldots, G_{r-1}$ be graphs with $|V(G_i)| = t$ for $i = 0, 1, \ldots, r-1$. Then $G = G(G_0, G_1, \ldots, G_{r-1}; M_{r})$ is a graph with vertex set $V(G) = V(G_0) \cup V(G_1) \cup \cdots \cup V(G_{r-1})$ and edge set $E(G) = E(G_0) \cup E(G_1) \cup \cdots \cup E(G_{r-1}) \cup M_{r}$, where $M_{r} = \bigcup_{i=0}^{r-1} M_{i,i+1(\text{mod} r)}$ and $M_{i,i+1(\text{mod} r)}$ is an arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1(\text{mod} r)})$ (see Fig. 2). Recursive circulant graphs [15] and $k$-ary $n$-cubes [7] are special cases of this family.

2.3. The third family $SP_n$ of networks

We define the graph $SP_n$ for $n \geq 3$. $SP_3$ is a cycle of length 6. For $n \geq 4$, $SP_n$ consists of $n$ disjoint $SP_{n-1}$'s, say $SP_1^{(1)}, SP_2^{(1)}, \ldots, SP_{n-1}^{(1)}$. The vertex set of each $SP_{n-1}^{(i)}$ for $1 \leq i \leq n$ is divided arbitrarily into $n-1$ disjoint vertex sets equally, say $S_1^{(i)}, S_2^{(i)}, \ldots, S_{n-1}^{(i)}$. For every $SP_{n-1}^{(i)}$ and $SP_{n-1}^{(j)}$, $i \neq j$, there exists a perfect matching between $S_x^{(i)}$ and $S_y^{(j)}$ for some $x$ and $y$, so that $SP_n$ is $(n-1)$-regular. Examples of $SP_3, SP_4$ and $SP_5$ are shown in Fig. 3.

![Fig. 3](image-url)

(a) $SP_3$, (b) $SP_4$, (c) $SP_5$. 

3. The first family $G(G_0,G_1;M)$ of networks

In this section, we will give lower bound of $S_i(G)$ for $G = G(G_0,G_1;M)$.

**Theorem 3.1.** Let $G_i = (V_i,E_i)$ be a connected graph of order $n$ with $\delta_i = \delta(G_i) = \lambda(G_i) = \lambda_i \geq 2$, $i = 0,1$. Let $G = G(G_0,G_1;M)$ and $\delta = \min\{\delta_0,\delta_1\}$. Then

$$S_i(G) \geq \begin{cases} \delta - 2 & \text{if } \delta \leq \frac{n}{2}, \\ n - \delta - 2 & \text{if } \frac{n}{2} \leq \delta \leq n - 2. \end{cases}$$

**Proof.** Set $m = \min \{\delta - 2, n - \delta - 2\}$. Then $m \geq 0$. We will show that $G$ is $m$-super-$\delta$, that is, for any $S \subseteq E(G)$ with $|S| \leq m$, $G - S$ is super-$\delta$. Let $S \subseteq E(G)$ with $|S| \leq m$, $G' = G - S$ and $A$ a vertex set of $V(G)$ with $2 \leq |A| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor = n$ and $G[A]$ and $G'[\overline{A}]$ being connected.

If $G_0$ and $G_1$ are disconnected in $G - \overline{A}$, then $\omega(A) \geq \lambda_0 + \lambda_1 = \delta_0 + \delta_1$. Thus we have

$$\omega_A(G) \geq \omega(A) \geq |S| \geq n - (\delta - 2) \geq \delta + 2 > \delta(G').$$

By Lemma 1.2, $G - S$ is super-$\delta$. If $G_0$ and $G_1$ are connected in $G - \overline{A}$, then $\omega(A) = n$ by the definition of $G$. Thus

$$\omega_A(G) \geq \omega(A) \geq n - (n - \delta - 2) = \delta + 2 > \delta(G').$$

and $G - S$ is super-$\delta$ by Lemma 1.2.

Now we assume, without loss of generality, that $G_0$ is disconnected and $G_1$ is connected in $G - \overline{A}$. Then $A \subset V(G_0)$ from $|A| \leq n$. Let $A = a$. Then $a \geq 2$. We suppose, to the contrary, that $\omega_A(G) \leq \delta(G')$.

Since $\delta(G') \geq \omega_A(G) \geq \omega(A) - |S| \geq \omega(A) - (\delta - 2)$ and $\delta(G') \leq \delta(G) = \delta + 1$, we have

$$2\delta \geq \omega(A) + 1 = \omega_{G_0}(A) + ||A, G_1|| + 1 = \omega_{G_0}(A) + a + 1. \quad (1)$$

Since $G_0$ is disconnected in $G - \overline{A}$, $\omega_{G_0}(A) \geq \lambda_0 = \delta_0 \geq \delta$. By (1), we have $2\delta \geq \delta + a + 1$. Thus $\delta \geq a + 1$.

On the other hand, for any $z \in A, d_{G_i,\overline{A}}(z) = d_{G_i}(z) - d_A(z) \geq \delta - (a - 1)$. Thus

$$\omega_{G_0}(A) = \sum_{z \in A} d_{G_0,\overline{A}}(z) \geq a(\delta - a + 1).$$

From (1) and $\delta \geq a + 1$, we have

$$(2 - a)(a + 1) \geq -a^2 + 2a + 1,$$

which implies $a \leq 1$, a contradiction. Thus $\omega_A(G) > \delta(G')$ and then $G - S$ is super-$\delta$ by Lemma 1.2. \qed

If $G_0$ and $G_1$ are regular graphs, we will have the following result which is stronger than Theorem 3.1.

**Theorem 3.2.** Let $G_i = (V_i,E_i)$ be a connected $\delta_i$-regular graph of order $n$ with $\delta_i = \delta(G_i) = \lambda(G_i) = \lambda_i \geq 2$, $i = 0,1$. Let $G = G(G_0,G_1;M)$ and $\delta = \min\{\delta_0,\delta_1\}$. Then

$$S_i(G) \geq \begin{cases} \delta - 1 & \text{if } \delta \leq \frac{n}{2}, \\ n - \delta - 1 & \text{if } \frac{n}{2} \leq \delta \leq n - 2. \end{cases}$$

**Proof.** Let $m = \min \{\delta - 1, n - \delta - 1\}, S \subseteq E(G)$ with $|S| \leq m$ and $G' = G - S$. We will show that $G - S$ is super-$\delta$.

Let $A$ be a vertex set of $V(G)$ with $2 \leq |A| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor = n$ and $G[A]$ and $G'[\overline{A}]$ being connected.

If $G_0$ and $G_1$ are disconnected in $G - \overline{A}$ and $|S| = 0$, then $\omega_A(G) = \omega_A(G) \geq \delta_0 + \delta_1 \geq 2\delta > \delta + 1 = \delta(G')$. If $G_0$ and $G_1$ are disconnected and $|S| > 0$, then $\omega_A(G) \geq \lambda_0 + \lambda_1 = \delta_0 + \delta_1$. Thus

$$\omega_A(G) \geq \omega(A) - |S| \geq \delta_0 + \delta_1 - (\delta - 1) \geq \delta + 1.$$
We will show that for any proof.

Theorem 4.1. Assume, without loss of generality, that \( G_0 \) is disconnected and \( G_1 \) is connected in \( G = [A, \overline{A}] \). Then \( A \subset V(G_0) \). Let \(|A| = a\). Then \( a \geq 2 \). If \(|S| = 0\), then \( \omega_C(A) = \omega(A) + a \geq \delta + 2 > \delta(G') \) and \( G - S \) is super-\( \lambda \) by Lemma 1.2. Hence we can assume \(|S| \geq 1\) in the following discussion. Thus \( \delta(G') \leq \delta(G) - 1 = \delta \).

We suppose, to the contrary, that \( \omega_C(A) \leq \delta(G') \). Since \( \omega_C(A) \geq \omega(A) - |S| \geq \omega(A) - (\delta - 1) \) and \( \delta(G') \leq \delta(G) - 1 = \delta \), we have

\[
2\delta \geq \omega(A) + 1.
\]

Then by the same argument as that of Theorem 3.1, we can get that \( a \leq 1 \), a contradiction. Hence \( \omega_C(A) > \delta(G') \) and then \( G - S \) is super-\( \lambda \) by Lemma 1.2. \( \square \)

Note. By the definition of \( S_i(G) \), the graphs \( G \) in Theorems 3.1 and 3.2 are super-\( \lambda \). Now we use a graph to show that the condition \( \delta \leq n - 2 \) in Theorem 3.2 is necessary. Let \( G_0 = G_1 = K_3 \). Then \( \delta = n - 1 \). But the graph \( G = G(G_0; G_1; M) \) is not super-\( \lambda \).

4. The second family \( G(G_0, G_1, \ldots, G_{r - 1}; M) \) of networks

In this section, we will give the lower bound of \( S_i(G) \) for \( G = G(G_0, G_1, \ldots, G_{r - 1}; M) \).

Theorem 4.1. Let \( G = (V, E) \) be a connected graph of order \( n \) with \( \delta_i = \delta(G_i) = \lambda_i = 2i \geq 0 \), \( i = 0, 1, \ldots, r - 1 \) and \( r \geq 3 \). Let \( G = G(G_0, G_1, \ldots, G_{r - 1}; M) \) and \( \delta = \min \{ \delta_0, \delta_1, \ldots, \delta_{r - 1} \} \). Then

\[
S_i(G) \geq \delta - 1.
\]

Proof. We will show that for any \( S \subset E(G) \) with \(| |S| \leq \delta - 1 \), \( G' = G - S \) is super-\( \lambda \).

Let \( A \) be a vertex set of \( V(G) \) with \( 2 \leq |A| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor \) and \( G'[A] \) and \( G' \overline{A} \) being connected. We will complete the proof by considering the following three cases.

Case 1. \( G_i \) is connected in \( G = [A, \overline{A}] \) for \( 0 \leq i \leq r - 1 \).

In this case, \( \omega(A) = 2n \). Thus

\[
\omega_C(A) \geq \omega(A) - |A| \geq 2n - (\delta - 1) > \delta + 2 \geq \delta(G'),
\]
and \( G' = G - S \) is super-\( \lambda \) by Lemma 1.2.

Case 2. There are \( k \) subgraphs in \( G_0, G_1, \ldots, G_{r - 1} \), say \( G_i, \ldots, G_k \), such that they are disconnected in \( G = [A, \overline{A}] \), where \( k \geq 2 \). If \( k \geq 3 \), then \( \omega(A) \geq \lambda_k + \ldots + \lambda_i \geq k\delta \). Thus

\[
\omega_C(A) \geq \omega(A) - |A| \geq k\delta - (\delta - 1) = (k - 1)\delta + 1 \geq \delta + 3 \geq \delta(G'),
\]
and \( G' = G - S \) is super-\( \lambda \) by Lemma 1.2. If \( k = 2 \), then \( \omega(A) \geq 2 + \lambda_i + \lambda_0 \geq 2 + 2\delta \) by the definition of \( G \) and \( |A| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor \). Thus

\[
\omega_C(A) \geq \omega(A) - |A| \geq 2 + 2\delta - (\delta - 1) > \delta + 2 \geq \delta(G'),
\]
and \( G' = G - S \) is super-\( \lambda \) by Lemma 1.2.

Case 3. Assume, without loss of generality, that \( G_0 \) is disconnected and \( G_i (1 \leq i \leq r - 1) \) is connected in \( G = [A, \overline{A}] \).

If there exists \( G_i \) with \( i \neq 0 \) such that \( A \cap V(G_i) \neq \emptyset \), then \( V(G_i) \subset A \). Since \( G_0 \) is disconnected in \( G = [A, \overline{A}] \) and \( V(G_i) \subset A \), we have \( \omega(A) \geq \lambda_0 + n + 1 \geq \delta + n + 1 \). Note that \( n \geq \delta + 1 \), we have

\[
\omega_C(A) \geq \omega(A) - |A| \geq \delta + n + 1 - (\delta - 1) \geq \delta + 3 \geq \delta(G'),
\]
and \( G' = G - S \) is super-\( \lambda \) by Lemma 1.2.

Now we consider the case \( A \subset V(G_0) \). We suppose, to the contrary, that \( \omega_C(A) \leq \delta(G') \). Let \(|A| = a\). Then \( a \geq 2 \). Since \( \delta(G') \geq \omega_C(A) \geq \omega(A) - |A| \geq \omega(A) - (\delta - 1) \) and \( \delta(G') \leq \delta + 2 \), we have

\[
2\delta \geq \omega(A) - 1 = \omega(C_0(A)) + 2a - 1.
\]

(2)

Since \( G_0 \) is disconnected in \( G = [A, \overline{A}] \), \( \omega(C_0(A)) \geq \lambda_0 \geq \delta \). Thus \( \delta \geq 2a - 1 \) by (2).

On the other hand, for any \( z \in A, d_{C_0(A)}(z) = d_{C_0}(z) - d_A(z) \geq \delta - (a - 1) = \delta - a + 1 \). Thus
\[ \omega_G(A) = \sum_{z \in A} d_G(z) \geq a(\delta - a + 1). \] (3)

From (2), (3) and \( \delta \geq 2a - 1 \), we have \( (2a - 1)(a - 2) \leq a^2 - 3a + 1 \), which implies \( a \leq 1 \), a contradiction. Thus \( \omega_G(A) > \delta(G) \) and \( G = G - S \) is super-\( \lambda \) by Lemma 1.2. \( \square \)

5. The third family \( S_{\lambda}(G) \) of networks

Next we consider the value of \( S_{\lambda}(G) \) for \( G = SP_n \).

**Lemma 5.1.** \( G \) is an \((n - 1)\)-regular graph with \(|V(G)| = n! \) and \( \lambda = n - 1 \).

**Proof.** By the definition of \( G \), \( G \) is \((n - 1)\)-regular and \(|V(G)| = n! \). We will show, by induction on \( n \), that \( \lambda = n - 1 \). If \( n = 3 \) or \( n = 4 \), then it is easy to check that \( \lambda = \delta(G) = n - 1 \) (see Fig. 3(a) and (b)).

Let \( n \geq 5 \). Note that \( \lambda \leq \delta(G) = n - 1 \). To show that \( \lambda \geq n - 1 \), we just need to prove that for any edge set \( S \subseteq E(G) \) with \(|S| \leq n - 2 \), \( G - S \) is connected. By the definition of \( SP_n \), for any \( i \neq j \) \((1 \leq i, j \leq n) \), there is an edge set \( E_{ij} = [V(SP^i_{n-1}), V(SP^j_{n-1})] \) such that \(|E_{ij}| = (n - 2)! \).

If there exists \( i \), such that \( |S \cap E(SP^i_{n-1})| = n - 2 \), say \( i = 1 \), then \( S \subseteq E(SP^1_{n-1}) \). So \( SP^1_{n-1} \) is connected in \( G - S \) for \( 2 \leq i \leq n \). By the definition of \( SP_n \), \( G - V(SP^1_{n-1}) \) is connected. On the other hand, for any vertex \( v \in V(SP^1_{n-1}) \), there exists \( u \in V(SP^j_{n-1}) \) \((j \neq 1) \) such that \( uv \in E(G - S) \). Thus \( G - S \) is connected.

Now we assume that for any \( i = 1, 2, \ldots, n \), \( |S \cap E(SP^i_{n-1})| \leq n - 3 \). Then \( SP^1_{n-1} - S \) is connected from induction for \( 1 \leq i \leq n \). Since \((n - 2)! > (n - 2) \) by \( n \geq 5 \), \( E_{i,j} - S = \emptyset \) for any \( i \neq j \). Thus \( G - S \) is connected. Hence \( \lambda \geq n - 1 \) which implies \( \lambda = n - 1 \). \( \square \)

It is obvious that \( SP_3 \) is not super-\( \lambda \). By Lemma 5.1, we have the following result for \( SP_n \) when \( n \geq 4 \).

**Lemma 5.2.** Let \( n \geq 4 \). Then \( n - 3 < S_{\lambda}(G) < n - 2 \). In particular, \( G = SP_n \) is super-\( \lambda \).

**Proof.** By Lemmas 1.3 and 5.1, we just need to show \( S_{\lambda}(G) \geq n - 3 \). That is, we will show that for any \( S \subseteq E(G) \) with \(|S| \leq n - 3 \), \( G = G - S \) is super-\( \lambda \).

Let \( A \) be a vertex set of \( V(G) \) with \( 2 \leq |A| \leq \left\lceil \frac{|V(G)|}{2} \right\rceil \) and \( G[A] \) and \( G[\overline{A}] \) being connected. Let \(|A| = a \). Then \( a \geq 2 \). We will complete the proof by considering the following three cases.

**Case 1.** \( SP^k_{n-1} \) is connected in \( G - \left[ A, \overline{A} \right] \) for \( k = 1, 2, \ldots, n \).

Let \( l = \left| \left[ SP^k_{n-1}, V(SP^k_{n-1}) \right] \subseteq A, 1 \leq k \leq n \right| \), then \( 1 \leq l \leq \left\lceil \frac{n}{2} \right\rceil \). Thus \( \omega_G(A) \geq l((n - l)(n - 2)! - (n - 3)! - (n - 2)! - n + 3 > n - 1 > \delta(G) \) and \( G = G - S \) is super-\( \lambda \) by Lemma 1.2.

**Case 2.** There exist \( SP^i_{n-1}, SP^j_{n-1}, \ldots, SP^k_{n-1} \) with \( k \geq 2 \), such that they are disconnected in \( G - \left[ A, \overline{A} \right] \).

By Lemma 5.1, \( \left| SP^i_{n-1} \right| = n - 2 \) for \( i = 1, 2, \ldots, n \). If \(|S| = 0 \), then \( \delta(G) = \delta(G) \) and \( \omega_G(A) = \omega_G(A) \geq k(n - 2)! - 2(n - 2)! - n + 3 \geq n - 1 = \delta(G) = \delta(G) \).

If \(|S| > 0 \), then \( \omega_G(A) \geq \omega_G(A) - |S| \geq k(n - 2)! - 2(n - 2)! - n + 3 = n - 1 \).

On the other hand, since \( G \) is regular and \( S \neq \emptyset \), \( \delta(G) < \delta(G) \), which implies \( \omega_G(A) > \delta(G) \). In these two cases, \( G = G - S \) is super-\( \lambda \) by Lemma 1.2.

**Case 3.** Assume, without loss of generality, that \( SP^i_{n-1} \) is disconnected and \( SP^i_{n-1} \) is connected for \( 2 \leq i \leq n \) in \( G - \left[ A, \overline{A} \right] \).

If there exists an \( SP^i_{n-1} \) such that \( A \cap V(SP^i_{n-1}) \neq \emptyset \), where \( i \neq 1 \), then \( V(SP^i_{n-1}) \subseteq A \). Since \( SP^1_{n-1} \) is disconnected in \( G - \left[ A, \overline{A} \right] \) and \( V(SP^i_{n-1}) \subseteq A \), we have \( \omega_G(A) \geq \lambda + 2(n - 2)! - n + 2 + 2(n - 2)! \) Then we have \( \omega_G(A) \geq \omega_G(A) - |S| \geq n - 2 + 2(n - 2)! - (n - 3) > n - 1 > \delta(G) \) and \( G = G - S \) is super-\( \lambda \) by Lemma 1.2.
Now we consider the case $A \subset V\left(\text{SP}_{n-1}\right)$. If $|S| = 0$, then $\omega_{G}(A) = \omega(A) > \lambda_1 + 2 = n - 2 + 2 > \delta(G)$ and $G - S$ is super-$\lambda$ by Lemma 1.2. So we can assume $|S| > 0$. Then $\delta(G) \leq \delta(G) - 1 = n - 2$.

We suppose, to the contrary, that $\omega_{G}(A) \leq \delta(G)$.

Since $n - 2 \geq \delta(G)$, we have $\omega_{G}(A) \geq \omega(A) - |S| \geq \omega(A) - (n - 3)$, we have

$$2n \geq \omega(A) + 5 = \omega_{\text{sp}_{1}}(A) + a + 5.$$  \hspace{1cm} (4)

Since $\text{SP}_{i-1}$ is disconnected in $G - \left[A, \overline{A}\right]$, we have $\omega_{\text{sp}_{i-1}}(A) > \lambda\left(\text{SP}_{n-1}\right) = n - 2$. Thus we have $n \geq a + 3$ from (4).

On the other hand, for any $z \in A$, $d_{\text{sp}_{i-1}}(z) = d_{G}(z) - d(z) \geq n - a - 1$. Thus

$$\omega_{\text{sp}_{i-1}}(A) = \sum_{z \in A} d_{\text{sp}_{i-1}}(z) \geq a(n - a - 1).$$  \hspace{1cm} (5)

From (4), (5) and $n \geq a + 3$, we have

$$(a - 2)(a + 3) \leq a^2 - 5,$$

which implies $a \leq 1$, a contradiction. Thus $\omega_{G}(A) > \delta(G)$ and $G - S$ is super-$\lambda$ by Lemma 1.2.

Furthermore, since $S_{i}(G) > n - 3 > 0$, $G$ is super-$\lambda$. \hspace{1cm} \square

By Lemmas 5.1 and 5.2, we have the following result.

**Theorem 5.3.** Let $G = \text{SP}_{n}$ and $n \geq 5$. Then $S_{i}(G) = n - 2$.

**Proof.** By Lemma 5.2, we only need to show that $S_{i}(G) \geq n - 2$.

Let $S \subseteq E(G)$ with $|S| 

\leq n - 2$ and $G' = G - S$. We will show that $G'$ is super-$\lambda$. If $|S| = 0$, then $G' = G$ is super-$\lambda$ by Lemma 5.2. So we will assume that $|S| \geq 1$ in the following discussion. Then $\delta(G') \leq \delta(G) - 1 = n - 2$.

Let $A$ be a vertex set of $V(G)$ with $2 \leq |A| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and $G[A]$ and $G'[\overline{A}]$ being connected. Let $|A| = a$. Then $a \geq 2$. We will complete the proof by considering the following three cases.

**Case 1.** $\text{SP}_{i-1}$ is connected in $G - \left[A, \overline{A}\right]$ for $1 \leq i \leq n$.

Then by the same argument in Lemma 5.2, $\omega_{G}(A) \geq (n - 1)! - (n - 2) > n - 2 \geq \delta(G)$ and $G'$ is super-$\lambda$ by Lemma 1.2.

**Case 2.** Suppose $\text{SP}_{i-1}^{1}, \text{SP}_{i-1}^{2}, \ldots, \text{SP}_{i-1}^{k}$ with $k \geq 2$ are disconnected in $G - \left[A, \overline{A}\right]$.

If $k \geq 3$, then $\omega_{G}(A) \geq k(n - 2) - n + 2 \geq 3(n - 2) - n + 2 = 2n - 4 > n - 1 > \delta(G)$ by $n \geq 5$ and $G'$ is super-$\lambda$ by Lemma 1.2.

Now we consider the case $k = 2$. Without loss of generality, we assume that $\text{SP}_{n-1}^{1}$ and $\text{SP}_{n-1}^{2}$ are disconnected in $G - \left[A, \overline{A}\right]$. If $|A \cap V\left(\text{SP}_{n-1}^{1}\right)| \geq 2$ or $|A \cap V\left(\text{SP}_{n-1}^{2}\right)| \geq 2$, then $\delta(G) \geq n - 1 > \delta(G)$ and $G'$ is super-$\lambda$. Suppose $|A \cap V\left(\text{SP}_{n-1}^{1}\right)| = |A \cap V\left(\text{SP}_{n-1}^{2}\right)| = 1$. Set $A = \{u, v\}$, where $u \in V\left(\text{SP}_{n-1}^{1}\right)$ and $v \in V\left(\text{SP}_{n-1}^{2}\right)$. Since $G[A]$ is connected, $uv \in E(G)$ and then $\omega_{G}(A) = 2(n - 2)$. If $|S \cap \left[A, \overline{A}\right]| < n - 2$, then $\omega_{G}(A) > 2(n - 2) - (n - 2) = n - 2 > \delta(G)$ and $G'$ is super-$\lambda$ by Lemma 1.2.

If $|S \cap \left[A, \overline{A}\right]| = n - 2$, then $\omega_{G}(A) = \omega(G) - |S \cap \left[A, \overline{A}\right]| = 2(n - 2) - (n - 2) = n - 2$. Since $S \subseteq \left[A, \overline{A}\right]$ and $|S \cap \left[A, \overline{A}\right]| = n - 2 \geq 3$, there is a vertex of $A$, say $u$, is adjacent to at least two edges of $S$. Then $\delta(G) \geq d_{G}(u) \geq d(u) - 2 = n - 3 < n - 2 = \omega_{G}(A)$ and $G'$ is super-$\lambda$ by Lemma 1.2.

**Case 3.** Assume, without loss of generality, that $\text{SP}_{i-1}^{1}$ is disconnected and $\text{SP}_{i-1}^{2} (2 \leq i \leq n)$ is connected in $G - \left[A, \overline{A}\right]$.

By the same argument as that of Lemma 5.2, we can assume $A \subset V\left(\text{SP}_{i-1}^{1}\right)$. Let $|A| = a$. Then $a \geq 2$. For any $z \in A, d_{\text{sp}_{i-1}}(z) = d_{G}(z) - d(z) \geq n - a - 1$. Thus

$$\omega_{\text{sp}_{i-1}^{1}}(A) = \sum_{z \in A} d_{\text{sp}_{i-1}^{1}}(z) \geq a(n - a - 1).$$  \hspace{1cm} (6)

We suppose, to the contrary, that $\omega_{G}(A) \leq \delta(G)$.

**Subcase 3.1.** $a \geq 3$.

Since $n - 2 \geq \delta(G)$, we have $\omega_{G}(A) > \omega(G) - |S| \geq \omega(G) - (n - 2)$. we have
2n ≥ ω(A) + 4 = ω_{SP^{+1}_{1}}(A) + a + 4.  \hspace{1cm} (7)

Since $SP^{+1}_{1}$ is super-$\lambda$ and $SP^{+1}_{1}$ is disconnected in $G - [A, \overline{\lambda}]$, $\omega_{SP^{+1}_{1}}(A) ≥ i(SP^{+1}_{n+1}) + 1 = n - 1$. Then $n ≥ a + 3$ from (7). From (6), (7) and $n ≥ a + 3$, we have

$$(a - 2)(a + 3) ≤ a^2 - 4,$$

which implies $a ≤ 2$, a contradiction. Thus $\omega_{C}(A) > \delta(G)$ and $G$ is super-$\lambda$ by Lemma 1.2.

**Subcase 3.2.** $a = 2$.

Let $A = [u, v]$. Since $G[A]$ is connected, $uv \in E(G)$. If $|S \cap [A, \overline{\lambda}]| ≤ n - 3$, then $n - 2 ≥ \delta(G) ≥ \omega_{C}(A) = ω(A) - |S ∩ [A, \overline{\lambda}]| ≥ ω(A) - (n - 3)$. So

$$2n ≥ ω(A) + 5 = ω_{SP^{+1}_{1}}(A) + a + 5.$$  \hspace{1cm} (8)

Since $ω_{SP^{+1}_{1}}(A) ≥ i(SP^{+1}_{n+1}) + 1 = n - 1$, we have $n ≥ a + 4$ from (8). From (6), (8) and $n ≥ a + 4$, we have $a^2 - 5 ≤ a^2 + 2a - 8$, which implies $2a ≤ 3$, a contradiction. Thus $ω_{C}(A) > \delta(G)$ and $G$ is super-$\lambda$ by Lemma 1.2.

Now we consider the case $|S ∩ [A, \overline{\lambda}]| = n - 2$.

Since $|S ∩ [A, \overline{\lambda}]| = n - 2 ≥ 3$ and $|A| = 2$, there is a vertex of A, say $u$, which is adjacent to at least two edges in $S$. Then $\delta(G) ≤ d_{C}(u) ≤ d(u) - 2 = n - 3$. On the other hand, $n - 3 ≥ \delta(G) ≥ ω_{C}(A) = ω(A) - n + 2$, which implies

$$2n ≥ ω(A) + 5 = ω_{SP^{+1}_{1}}(A) + a + 5.$$  \hspace{1cm} (8)

By the same argument as above, we have the same contradiction. Thus $ω_{C}(A) > \delta(G)$ and $G$ is super-$\lambda$ by Lemma 1.2. \hfill $\square$

For $n = 4$, we have the following conclusion.

**Theorem 5.4.** Let $G = SP_{4}$. Then $S_{\lambda}(G) = 1$.

**Proof.** By Lemma 5.2, $1 ≤ S_{\lambda}(G) ≤ 2$. So we just need to show that there exists an edge set $S ⊆ E(G)$ with $|S| = 2$ such that $G - S$ is not super-$\lambda$.

Let $uv \in E(G)$, $e_{j} = ui_{j}$ $(j = 1, 2)$ and $e_{j} = ui_{j}v$ $(j = 3, 4)$. Then there are two nonadjacent edges in $e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{6}$. Let $A = [u, v]$ and $S = \{e_{1}, e_{2}\}$. Since $G = SP_{4}$ is 3-regular, $ω_{C, S}(A) = 2 = \delta(G - S)$. By Lemma 1.2, $G - S$ is not super-$\lambda$. Thus $S_{\lambda}(G) = 1$. \hfill $\square$

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**References**


