# Contents

Preface xi

0 Introduction 1
  0.1 Distribution Function 1
  0.2 How to Handle Frequency 2
  0.3 How to Handle Belief Degree 3
  0.4 Ellsberg Urn: A Benchmark Test 4
  0.5 How to Choose Your Mathematical Tool 7

1 Uncertain Measure 9
  1.1 Measurable Space 9
  1.2 Uncertain Measure 11
  1.3 Uncertainty Space 16
  1.4 Product Uncertain Measure 17
  1.5 Independence 23
  1.6 Polyrectangular Theorem 25
  1.7 Conditional Uncertain Measure 27
  1.8 Bibliographic Notes 30

2 Uncertain Variable 31
  2.1 Uncertain Variable 31
  2.2 Uncertainty Distribution 34
  2.3 Inverse Uncertainty Distribution 43
  2.4 Independence 47
  2.5 Operational Law: Inverse Distribution 49
  2.6 Operational Law: Distribution 59
  2.7 Operational Law: Boolean System 65
  2.8 Expected Value 70
  2.9 Variance 81
  2.10 Moment 84
  2.11 Distance 87
  2.12 Entropy 89
  2.13 Conditional Uncertainty Distribution 95
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.14 Uncertain Sequence</td>
<td>98</td>
</tr>
<tr>
<td>2.15 Uncertain Vector</td>
<td>103</td>
</tr>
<tr>
<td>2.16 Bibliographic Notes</td>
<td>105</td>
</tr>
<tr>
<td>3 Uncertain Programming</td>
<td>107</td>
</tr>
<tr>
<td>3.1 Uncertain Programming</td>
<td>107</td>
</tr>
<tr>
<td>3.2 Numerical Method</td>
<td>110</td>
</tr>
<tr>
<td>3.3 Machine Scheduling Problem</td>
<td>112</td>
</tr>
<tr>
<td>3.4 Vehicle Routing Problem</td>
<td>115</td>
</tr>
<tr>
<td>3.5 Project Scheduling Problem</td>
<td>119</td>
</tr>
<tr>
<td>3.6 Uncertain Multiobjective Programming</td>
<td>122</td>
</tr>
<tr>
<td>3.7 Uncertain Goal Programming</td>
<td>124</td>
</tr>
<tr>
<td>3.8 Uncertain Multilevel Programming</td>
<td>125</td>
</tr>
<tr>
<td>3.9 Bibliographic Notes</td>
<td>126</td>
</tr>
<tr>
<td>4 Uncertain Risk Analysis</td>
<td>127</td>
</tr>
<tr>
<td>4.1 Loss Function</td>
<td>127</td>
</tr>
<tr>
<td>4.2 Risk Index</td>
<td>129</td>
</tr>
<tr>
<td>4.3 Series System</td>
<td>130</td>
</tr>
<tr>
<td>4.4 Parallel System</td>
<td>131</td>
</tr>
<tr>
<td>4.5 $k$-out-of-$n$ System</td>
<td>131</td>
</tr>
<tr>
<td>4.6 Standby System</td>
<td>132</td>
</tr>
<tr>
<td>4.7 Structural Risk Analysis</td>
<td>132</td>
</tr>
<tr>
<td>4.8 Value-at-Risk</td>
<td>136</td>
</tr>
<tr>
<td>4.9 Expected Loss</td>
<td>137</td>
</tr>
<tr>
<td>4.10 Investment Risk Analysis</td>
<td>138</td>
</tr>
<tr>
<td>4.11 Bibliographic Notes</td>
<td>138</td>
</tr>
<tr>
<td>5 Uncertain Reliability Analysis</td>
<td>141</td>
</tr>
<tr>
<td>5.1 Structure Function</td>
<td>141</td>
</tr>
<tr>
<td>5.2 Reliability Index</td>
<td>142</td>
</tr>
<tr>
<td>5.3 Series System</td>
<td>143</td>
</tr>
<tr>
<td>5.4 Parallel System</td>
<td>143</td>
</tr>
<tr>
<td>5.5 $k$-out-of-$n$ System</td>
<td>144</td>
</tr>
<tr>
<td>5.6 General System</td>
<td>144</td>
</tr>
<tr>
<td>5.7 Bibliographic Notes</td>
<td>145</td>
</tr>
<tr>
<td>6 Uncertain Propositional Logic</td>
<td>147</td>
</tr>
<tr>
<td>6.1 Uncertain Proposition</td>
<td>147</td>
</tr>
<tr>
<td>6.2 Truth Value</td>
<td>149</td>
</tr>
<tr>
<td>6.3 Chen-Ralescu Theorem</td>
<td>151</td>
</tr>
<tr>
<td>6.4 Uncertain Predicate Logic</td>
<td>154</td>
</tr>
<tr>
<td>6.5 Bibliographic Notes</td>
<td>157</td>
</tr>
</tbody>
</table>
7 Uncertain Entailment
7.1 Uncertain Entailment Model
7.2 Uncertain Modus Ponens
7.3 Uncertain Modus Tollens
7.4 Uncertain Hypothetical Syllogism
7.5 Bibliographic Notes

8 Uncertain Set
8.1 Uncertain Set
8.2 Membership Function
8.3 Inverse Membership Function
8.4 Independence
8.5 Set Operational Law
8.6 Arithmetic Operational Law
8.7 Inclusion Relation
8.8 Expected Value
8.9 Variance
8.10 Distance
8.11 Entropy
8.12 Conditional Membership Function
8.13 Bibliographic Notes

9 Uncertain Logic
9.1 Individual Feature Data
9.2 Uncertain Quantifier
9.3 Uncertain Subject
9.4 Uncertain Predicate
9.5 Uncertain Proposition
9.6 Truth Value
9.7 Linguistic Summarizer
9.8 Bibliographic Notes

10 Uncertain Inference
10.1 Uncertain Inference Rule
10.2 Uncertain System
10.3 Uncertain Control
10.4 Inverted Pendulum
10.5 Bibliographic Notes

11 Uncertain Process
11.1 Uncertain Process
11.2 Uncertainty Distribution
11.3 Independence and Operational Law
11.4 Independent Increment Process
11.5 Extreme Value Theorem
11.6 First Hitting Time .............................. 271  
11.7 Time Integral ................................. 273  
11.8 Stationary Increment Process ................. 276  
11.9 Bibliographic Notes ............................ 281

12 Uncertain Renewal Process 283  
12.1 Uncertain Renewal Process ...................... 283  
12.2 Block Replacement Policy ...................... 287  
12.3 Renewal Reward Process ....................... 288  
12.4 Uncertain Insurance Model .................... 291  
12.5 Age Replacement Policy ....................... 295  
12.6 Alternating Renewal Process ................... 300  
12.7 Bibliographic Notes ............................ 304

13 Uncertain Calculus 305  
13.1 Liu Process ..................................... 305  
13.2 Liu Integral .................................... 310  
13.3 Fundamental Theorem .......................... 315  
13.4 Chain Rule ..................................... 317  
13.5 Change of Variables ............................ 317  
13.6 Integration by Parts ............................ 318  
13.7 Bibliographic Notes ............................ 320

14 Uncertain Differential Equation 321  
14.1 Uncertain Differential Equation ............... 321  
14.2 Analytic Methods .............................. 324  
14.3 Existence and Uniqueness ...................... 329  
14.4 Stability ....................................... 331  
14.5 α-Path ......................................... 333  
14.6 Yao-Chen Formula .............................. 334  
14.7 Numerical Methods ............................. 342  
14.8 Bibliographic Notes ............................ 344

15 Uncertain Finance 345  
15.1 Uncertain Stock Model .......................... 345  
15.2 European Options ............................... 345  
15.3 American Options ............................... 349  
15.4 Asian Options ................................. 351  
15.5 General Stock Model ............................ 355  
15.6 Multifactor Stock Model ....................... 358  
15.7 Uncertain Interest Rate Model ................. 361  
15.8 Uncertain Currency Model ..................... 366  
15.9 Bibliographic Notes ............................ 371
16 Uncertain Statistics

16.1 Expert’s Experimental Data ........................................ 373
16.2 Empirical Uncertainty Distribution ................................. 375
16.3 Principle of Least Squares ........................................... 376
16.4 Method of Moments .................................................. 378
16.5 Multiple Domain Experts ............................................. 379
16.6 Delphi Method .......................................................... 380
16.7 Uncertain Regression Analysis ...................................... 381
16.8 Uncertain Time Series Analysis ..................................... 390
16.9 Bibliographic Notes .................................................... 396

A Chance Theory

A.1 Chance Measure .......................................................... 399
A.2 Uncertain Random Variable .......................................... 403
A.3 Chance Distribution .................................................... 404
A.4 Operational Law ........................................................ 406
A.5 Expected Value .......................................................... 413
A.6 Variance ................................................................. 417
A.7 Law of Large Numbers ................................................ 421
A.8 Ellsberg Experiment ..................................................... 422
A.9 Uncertain Random Programming .................................... 428
A.10 Uncertain Random Risk Analysis ................................... 431
A.11 Uncertain Random Reliability Analysis ........................... 435
A.12 Uncertain Random Graph ............................................ 437
A.13 Uncertain Random Network ........................................ 440
A.14 Uncertain Random Process ......................................... 442
A.15 Bibliographic Notes .................................................... 454

B Frequently Asked Questions

B.1 What is the meaning that an object follows the laws of probability theory? ........................................... 457
B.2 Why does frequency follow the laws of probability theory? .......................................................... 458
B.3 Why does Dutch book argument fail to prove that belief degree follows the laws of probability theory? ........................................................................................................... 459
B.4 Why does Cox’s theorem fail to prove that belief degree follows the laws of probability theory? ........................................................................................................... 461
B.5 What is the difference between probability theory and uncertainty theory? ........................................... 461
B.6 How do we distinguish between randomness and uncertainty in practice? ........................................... 462
B.7 Why is stochastic differential equation not suitable for modelling stock price? ...................................... 462
B.8 In what situations should we use uncertainty theory? .......................................................... 465
B.9 Why do I think fuzzy set theory is bad mathematics? .......................................................... 465
B.10 Why is fuzzy variable not suitable for modelling indeterminate quantity? ........................................... 467
B.11 What is the difference between uncertainty theory and possibility theory? ................................. 469
B.12 How do we handle interval numbers by uncertainty theory? ..................................................... 469
B.13 Why do I think none of interval analysis, rough set theory and grey system are consistent? ........ 472
B.14 How did “uncertainty” evolve over the past 100 years? ................................................................. 474

Bibliography ........................................................................................................................................ 477

List of Frequently Used Symbols ........................................................................................................ 490

Index .................................................................................................................................................... 491
Preface

When no samples are available to estimate distribution functions, or some emergency (e.g., war, flood, earthquake, accident, and even rumour) arises, we have to invite some domain experts to evaluate belief degree that each event will happen. Perhaps some people think that belief degree should be modeled by subjective probability or fuzzy set theory. However, it is usually inappropriate because both of them may lead to counterintuitive results in this case. In order to rationally deal with personal belief degrees, uncertainty theory was founded in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of mathematics.

Uncertain Measure

The most fundamental concept is uncertain measure that is a type of set function satisfying the axioms of uncertainty theory. It is used to indicate the belief degree that an uncertain event may happen. Chapter 1 will introduce normality, duality, subadditivity and product axioms. From those four axioms, this chapter will also present uncertain measure, product uncertain measure, and conditional uncertain measure.

Uncertain Variable

Uncertain variable is a measurable function from an uncertainty space to the set of real numbers. It is used to represent quantities with uncertainty. Chapter 2 is devoted to uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, distance, entropy, conditional uncertainty distribution, uncertain sequence, and uncertain vector.

Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Chapter 3 will provide the tool of uncertain programming model with applications to machine scheduling problem, vehicle routing problem, and project scheduling problem. In addition, uncertain multiobjective programming, uncertain goal programming and uncertain multilevel programming are also documented.
Uncertain Risk Analysis

The term risk has been used in different ways in literature. In this book the risk is defined as the accidental loss plus the uncertain measure of such loss, and a risk index is defined as the uncertain measure that some specified loss occurs. Chapter 4 will introduce uncertain risk analysis that is a tool to quantify risk via uncertainty theory. As applications of uncertain risk analysis, Chapter 4 will also discuss structural risk analysis and investment risk analysis.

Uncertain Reliability Analysis

Reliability index is defined as the uncertain measure that some system is working. Chapter 5 will introduce uncertain reliability analysis that is a tool to deal with system reliability via uncertainty theory.

Uncertain Propositional Logic

Uncertain propositional logic is a generalization of propositional logic in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. Chapter 6 will present uncertain propositional logic and uncertain predicate logic. In addition, uncertain entailment is a methodology for determining the truth value of an uncertain proposition via the maximum uncertainty principle when the truth values of other uncertain propositions are given. Chapter 7 will discuss an uncertain entailment model from which uncertain modus ponens, uncertain modus tollens and uncertain hypothetical syllogism are deduced.

Uncertain Set

Uncertain set is a set-valued function on an uncertainty space, and attempts to model unsharp concepts like “young”, “tall”, “warm”, and “most”. The main difference between uncertain set and uncertain variable is that the former takes values of set and the latter takes values of point. Uncertain set theory will be introduced in Chapter 8.

Uncertain Logic

Some knowledge in human brain is actually an uncertain set. This fact encourages us to design an uncertain logic that is a methodology for calculating the truth values of uncertain propositions via uncertain set theory. Uncertain logic may provide a flexible means for extracting linguistic summary from a collection of raw data. Chapter 9 will be devoted to uncertain logic and linguistic summarizer.
Uncertain Inference

Uncertain inference is a process of deriving consequences from human knowledge via uncertain set theory. Chapter 10 will present a set of uncertain inference rules, uncertain system, and uncertain control with application to an inverted pendulum system.

Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time. Thus an uncertain process is usually used to model uncertain phenomena that vary with time. Chapter 11 is devoted to basic concepts of uncertain process and uncertainty distribution. In addition, extreme value theorem, first hitting time and time integral of uncertain processes are also introduced. Chapter 12 deals with uncertain renewal process, renewal reward process, and alternating renewal process. Chapter 12 also provides block replacement policy, age replacement policy, and uncertain insurance model.

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes. Chapter 13 will introduce Liu process that is a stationary independent increment process whose increments are normal uncertain variables, and discuss Liu integral that is a type of uncertain integral with respect to Liu process. In addition, the fundamental theorem of uncertain calculus will be proved in this chapter from which the techniques of chain rule, change of variables, and integration by parts are also derived.

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation involving uncertain processes. Chapter 14 will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and will introduce Yao-Chen formula that represents the solution of an uncertain differential equation by a family of solutions of ordinary differential equations. On the basis of this formula, some formulas to calculate extreme value, first hitting time, and time integral of solution are provided. Furthermore, some numerical methods for solving general uncertain differential equations are designed.

Uncertain Finance

As applications of uncertain differential equation, Chapter 15 will introduce uncertain stock model, uncertain interest rate model, and uncertain currency model. Based on the fair price principle, Chapter 15 will also price European options, American options, Asian options, zero-coupon bond, interest rate ceiling, and interest rate floor.
Uncertain Statistics

Uncertain statistics is a methodology for collecting and interpreting expert’s experimental data by uncertainty theory. Chapter 16 will present a questionnaire survey for collecting expert’s experimental data. In order to determine uncertainty distributions from those expert’s experimental data, Chapter 16 will also introduce linear interpolation method, principle of least squares, method of moments, and Delphi method. In addition, uncertain regression analysis and uncertain time series analysis are also introduced when the imprecise observations are characterized in terms of uncertain variables.

Law of Truth Conservation

The law of excluded middle tells us that a proposition is either true or false, and the law of contradiction tells us that a proposition cannot be both true and false. In the state of indeterminacy, some people said, the law of excluded middle and the law of contradiction are no longer valid because the truth degree of a proposition is no longer 0 or 1. I cannot gainsay this viewpoint to a certain extent. But it does not mean that you might “go as you please”. The truth values of a proposition and its negation should sum to unity. This is the law of truth conservation that is weaker than the law of excluded middle and the law of contradiction. Furthermore, the law of truth conservation agrees with the law of excluded middle and the law of contradiction when the uncertainty vanishes.

Maximum Uncertainty Principle

An event has no uncertainty if its uncertain measure is 1 because we may believe that the event happens. An event has no uncertainty too if its uncertain measure is 0 because we may believe that the event does not happen. An event is the most uncertain if its uncertain measure is 0.5 because the event and its complement may be regarded as “equally likely”. In practice, if there is no information about the uncertain measure of an event, we should assign 0.5 to it. Sometimes, only partial information is available. In this case, the value of uncertain measure may be specified in some range. What value does the uncertain measure take? For any event, if there are multiple reasonable values that an uncertain measure may take, then the value as close to 0.5 as possible is assigned to the event. This is the maximum uncertainty principle.

Lecture Slides

If you need lecture slides for uncertainty theory, please download them from the website at http://orsc.edu.cn/liu/resources.htm.
Uncertainty Theory Online

If you want to read more books, dissertations and papers related to uncer-
tainty theory, please visit the website at http://orsc.edu.cn/online.

Purpose

The purpose is to equip the readers with a branch of mathematics to deal
with belief degrees. The textbook is suitable for researchers, engineers, and
students in the field of mathematics, information science, operations research,
industrial engineering, computer science, artificial intelligence, automation,
economics, and management science.

A Guide for the Readers

The readers are not required to read the book from cover to cover. The logic
dependence of chapters is illustrated by the figure below.

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To most people, uncertainty means something is unknown. However, uncertainty theory is concerned with to what extent something can be known.
Chapter 0

Introduction

Real decisions are usually made in the state of indeterminacy. To rationally deal with indeterminacy, there exist two mathematical systems, one is probability theory (Kolmogorov, 1933) and the other is uncertainty theory (Liu, 2007). Probability theory is a branch of mathematics for modelling frequencies, while uncertainty theory is a branch of mathematics for modelling belief degrees.

0.1 Distribution Function

By indeterminacy we mean the phenomena whose outcomes cannot be exactly predicted in advance. For example, we cannot exactly predict which face will appear before we toss dice. Thus “tossing dice” is a type of indeterminacy. As another example, we cannot exactly predict tomorrow’s stock price. That is, “stock price” is also a type of indeterminacy. Some other instances of indeterminacy include “roulette wheel”, “product lifetime”, “market demand”, “travel time”, “construction cost”, etc.

Indeterminacy is absolute, while determinacy is relative. This is the reason why we say real decisions are usually made in the state of indeterminacy. How to model indeterminacy is thus an important research subject in not only mathematics but also science and engineering.

In order to deal with an indeterminate quantity (e.g. stock price), the first action we take is to produce a distribution function representing the “degree” $\alpha$ that the quantity falls into the left side of the current point $x$. See Figure 1. Such a function will always have bigger values as the current point moves from the left to right. If the distribution function takes value 0, then it is completely impossible that the quantity falls into the left side of the current point; if the distribution function takes value 1, then it is completely impossible that the quantity falls into the right side; if the distribution function takes value 0.6, then we are 60% sure that the quantity falls into the left side and 40% sure that the quantity falls into the right side.
In order to obtain the distribution function for some indeterminate quantity, there exist only two ways, one is frequency generated by samples (i.e., historical data), and the other is belief degree evaluated by domain experts. Could you imagine a third way?

0.2 How to Handle Frequency

Assume we have collected a set of samples for some indeterminate quantity (e.g. stock price). By cumulative frequency we mean a function representing the percentage of all samples that fall into the left side of the current point. It is clear that the cumulative frequency looks like a step function in Figure 2.

Frequency is a factual property of indeterminate quantity, and does not change with our state of knowledge and preference. When the sample size is large enough and no emergency (e.g., war, flood, earthquake, accident, and even rumour) arises, it is possible for us to find a distribution function that is close enough to the frequency. In this case, there is no doubt that probability theory is the only legitimate approach to deal with our problem.

However, in many cases, no samples are available or some emergency
arises. Thus the estimated distribution function may deviate far from the frequency because the latter is unknown to us at the moment. If this distribution function is used, then probability theory may lead to counterintuitive results.

0.3 How to Handle Belief Degree

Belief degrees are familiar to all of us. The object of belief is an event (i.e., a proposition). For example, “the sun will rise tomorrow”, “it will be sunny next week”, and “John is a young man” are all instances of object of belief. A belief degree represents the strength with which you believe the event will happen. If you completely believe the event will happen, then your belief degree is 1 (complete belief). If you think it is completely impossible, then your belief degree is 0 (complete disbelief). Generally, you will assign a number between 0 and 1 to the belief degree for each event because you can be neither in more belief than “complete belief” nor in more disbelief than “complete disbelief”. The higher the belief degree is, the more strongly you believe the event will happen.

The belief degree of an event may also be interpreted as the betting ratio (price/stake) for the event. Assume a bet offers $1 if the event happens and nothing otherwise. What price of this bet do you think is reasonable? If you think the bet is worth $1, then your belief degree of this event is 100%; if you think the bet is worth nothing, then your belief degree is 0%; and if you think the bet is worth 60¢, then your belief degree is 60%.

The belief degree depends heavily on the personal knowledge and preference concerning the event. When the personal knowledge and preference change, the belief degree changes too. For example, let us consider my birthdate. Someone who does not know me would be only 8% (i.e., 1/12) sure that I was born in February. Some friends of mine might be 80% sure for it since they shared my birthday cake last year. However, my mother can be 100% sure that I was born in February. Different people hold different belief degrees due to their different knowledge and preference.

Perhaps some readers may ask which belief degree is correct. I have to say that all belief degrees are wrong, but some are useful. Through a lot of surveys, Kahneman and Tversky [75] showed that human beings usually overweight unlikely events. From another side, Liu [106] showed that human beings usually estimate a much wider range of values than the object actually takes. This conservatism of human beings makes the belief degrees deviate far from the frequency. Thus all belief degrees are wrong compared with its frequency. However, it cannot be denied that those belief degrees are indeed helpful for decision making. A belief degree becomes “correct” only when it coincides with the frequency. However, usually we cannot make it to that.

---

1In contrast, frequency does not change with the personal knowledge and preference.
In order to describe an indeterminate quantity, what we need is a belief degree function that represents the degree with which we believe the quantity falls into the left side of the current point. Generally, a belief degree function takes values between 0 and 1, and has bigger values as the current point moves from the left to right.

Belief degree function is certainly a type of distribution function for the indeterminate quantity. Since it usually deviates far from the frequency, using probability theory may lead to counterintuitive results. In this case, we should use uncertainty theory.

### 0.4 Ellsberg Urn: A Benchmark Test

Assume I have filled 100 urns each with 100 balls that are either red or black. For me, they are Pólya urns\(^2\). However, for you, they are Ellsberg urns. You are only told that the compositions in those urns are iid, but the distribution function is completely unknown to you. Consider the following three urn problems:

(i) How many balls do you think are red in the first urn?

(ii) How many balls do you think are red in the 100 urns?

(iii) How likely do you think the number of red balls is 10,000?

#### How do we solve urn problems by probability theory?

Since you do not know the number of red balls completely, Laplace criterion makes you assign equal probabilities to the possible numbers of red balls 0, 1, 2, \cdots, 100. Thus, for each \( i \) with \( 1 \leq i \leq 100 \), the number of red balls in the \( i \)th urn is a random variable,

\[
\xi_i = k \text{ with probability } \frac{1}{101}, \quad k = 0, 1, 2, \cdots, 100.
\]

Note that \( \xi_1, \xi_2, \cdots, \xi_n \) are iid random variables according to my promise. The total number of red balls in the 100 urns is the sum

\[
\xi = \xi_1 + \xi_2 + \cdots + \xi_{100}
\]

that can take any integer between 0 and 10,000. Since the total number of red balls is 10,000 if and only if the 100 urns each contain 100 red balls, the

---

\(^2\)Assume an urn contains a finite number of balls of different colors. It is called a Pólya urn (named after George Pólya) if its composition is perfectly known, and an Ellsberg urn (named after Daniel Ellsberg) if its composition is completely or partially unknown.
probability of the total number of red balls being 10,000 is
\[ \Pr\{\xi = 10,000\} = \Pr\{\xi_i = 100, i = 1, 2, \cdots, 100\} \]
\[ = \prod_{i=1}^{100} \Pr\{\xi_i = 100\} = \prod_{i=1}^{100} \frac{1}{101} \]
\[ \approx 3.6 \times 10^{-201} \]

where \(\Pr\{\cdot\}\) represents the probability measure.

**How do we solve urn problems by uncertainty theory?**

Since you do not know the number of red balls completely, you have to assign equal belief degrees to the possible numbers of red balls 0, 1, 2, \cdots, 100. Thus, for each \(i\) with \(1 \leq i \leq 100\), the number of red balls in the \(i\)th urn is an uncertain variable,

\[ \eta_i = k \text{ with belief degree } \frac{1}{101}, \quad k = 0, 1, 2, \cdots, 100. \]

Note that \(\eta_1, \eta_2, \cdots, \eta_n\) are iid uncertain variables according to my promise. The total number of red balls in the 100 urns is the sum

\[ \eta = \eta_1 + \eta_2 + \cdots + \eta_{100} \]

that can also take any integer between 0 and 10,000. Since the total number of red balls is 10,000 if and only if the 100 urns each contain 100 red balls, the belief degree of the total number of red balls being 10,000 is

\[ \mathcal{M}\{\eta = 10,000\} = \mathcal{M}\{\eta_i = 100, i = 1, 2, \cdots, 100\} \]
\[ = \bigwedge_{i=1}^{100} \mathcal{M}\{\eta_i = 100\} = \bigwedge_{i=1}^{100} \frac{1}{101} \]
\[ = \frac{1}{101} \]

where \(\mathcal{M}\{\cdot\}\) represents the belief degree (i.e., uncertain measure).

**Which result is more reasonable?**

Probability theory tells you that the probability of the total number of red balls being 10,000 is \(3.6 \times 10^{-201}\), while uncertainty theory tells you that the belief degree is \(1/101\). Which result is more reasonable? In order to answer this question, let us introduce the following two options:

A: You lose \$1,000,000 if the total number of red balls is 10,000, and receive \$1 otherwise;
B: Don’t bet.

What is your choice between A and B? If probability theory is used, then the probability of the total number of red balls being 10,000 is $3.6 \times 10^{-201}$, and the expected income of A is

$$A = 1 \times (1 - 3.6 \times 10^{-201}) - 1000000 \times 3.6 \times 10^{-201} \approx 1.$$ 

Since the income of B is always 0, we have $A > B$.

That is, probability theory makes you choose A. If uncertainty theory is used, then the belief degree of the total number of red balls being 10,000 is $1/101$, and the expected income of A is

$$A = 1 \times \left(1 - \frac{1}{101}\right) - 1000000 \times \frac{1}{101} \approx -9900.$$ 

Since the income of B is always 0, we have $A < B$.

That is, uncertainty theory makes you choose B. Probability theory and uncertainty theory give you two diametrically opposed choices. Which choice do you think is better?

**How did I fill the 100 urns?**

In order to compare the decisions produced by probability theory and uncertainty theory, I would like to show you how I filled the 100 urns. First I took a distribution function,

$$\Upsilon(x) = \begin{cases} 
0, & \text{if } x < 100 \\
1, & \text{if } x \geq 100
\end{cases}$$

that is just the constant 100 (please recognize that I have the option to choose my preferred distribution function). Next I generated a random number $k$ from the distribution function $\Upsilon$, and filled the first urn with $k$ red balls and $100 - k$ black balls. Then I generated a new random number $k$ from $\Upsilon$, and filled the second urn with $k$ red balls and $100 - k$ black balls. Repeated this process until 100 urns were filled. Note that 100, 100, · · · , 100 are indeed iid, and the total number of red balls happens to be 10,000.

You would lose $1,000,000 if you used probability theory (i.e., you chose A). If this experiment is repeated, then you have to choose A again and continue to lose $1,000,000 as long as you use probability theory.
Why does probability theory fail?

The root cause is that your uniform distribution function (approximatively) of number of red balls in each urn,

\[ \Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{x}{100}, & \text{if } 0 \leq x \leq 100 \\
1, & \text{if } x > 100 
\end{cases} \]

is not close to the real frequency,

\[ \Upsilon(x) = \begin{cases} 
0, & \text{if } x < 100 \\
1, & \text{if } x \geq 100 
\end{cases} \]

In this case, probability theory led to wrong results. However, uncertainty theory was proven successful to deal with those urn problems.

0.5 How to Choose Your Mathematical Tool

To rationally deal with indeterminacy, there exist two mathematical systems, one is probability theory and the other is uncertainty theory. Liu [97] declared that probability theory is applicable to modelling frequencies, and uncertainty theory is applicable to modelling belief degrees. In other words, frequency is the empirical basis of probability theory, while belief degree is the empirical basis of uncertainty theory. Using uncertainty theory to model frequency may produce a crude result, while using probability theory to model belief degree may produce a big disaster.

Perhaps some people may complain that they cannot distinguish between frequency and belief degree in practice. It doesn’t matter because you can quickly distinguish between them in this way: For some quantity, if you believe your distribution function (no matter how you get it) is close enough to the real frequency, then you should treat the quantity as a random variable. Otherwise, you have to regard the quantity as an uncertain variable. Thus you may choose your mathematical tool by the following criterion:

_If your distribution function is close enough to the frequency, then you should use probability theory. Otherwise, you have to use uncertainty theory._

Should you be satisfied with this answer? Perhaps some people may continue to ask how to verify whether the distribution function is close enough to the frequency or not. I have no idea. But, to my knowledge, the distribution function obtained in most practical problems is, unfortunately, not close enough to the frequency. Thus you have to use uncertainty theory in most practical problems. Are you willing to learn uncertainty theory?
Figure 3: When your distribution function (left curve) is close enough to the frequency (left histogram), you should use probability theory. When your distribution function (right curve) deviates far from the frequency (right histogram but unknown), you have to use uncertainty theory.
Chapter 1

Uncertain Measure

Uncertainty theory was founded by Liu [88] in 2007 and subsequently studied by many researchers. Nowadays uncertainty theory has become a branch of mathematics for modelling belief degrees. This chapter will provide normality, duality, subadditivity and product axioms of uncertainty theory. From those four axioms, this chapter will also introduce an uncertain measure that is a fundamental concept in uncertainty theory. In addition, product uncertain measure and conditional uncertain measure will be explored at the end of this chapter.

1.1 Measurable Space

From the mathematical viewpoint, uncertainty theory is essentially an alternative theory of measure. Thus uncertainty theory should begin with a measurable space. In order to learn it, let us introduce algebra, $\sigma$-algebra, measurable set, Borel algebra, Borel set, and measurable function. The main results in this section are well-known. For this reason the credit references are not provided. You may skip this section if you are familiar with them.

**Definition 1.1** Let $\Gamma$ be a nonempty set (sometimes called universal set). A collection $\mathcal{L}$ consisting of subsets of $\Gamma$ is called an algebra over $\Gamma$ if the following three conditions hold: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, then

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}. \quad (1.1)$$

The collection $\mathcal{L}$ is called a $\sigma$-algebra over $\Gamma$ if the condition (c) is replaced with closure under countable union, i.e., when $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, we have

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}. \quad (1.2)$$
Example 1.1: The collection \( \{\emptyset, \Gamma\} \) is the smallest \( \sigma \)-algebra over \( \Gamma \), and the power set (i.e., all subsets of \( \Gamma \)) is the largest \( \sigma \)-algebra.

Example 1.2: Let \( \Lambda \) be a proper nonempty subset of \( \Gamma \). Then \( \{\emptyset, \Lambda, \Lambda^c, \Gamma\} \) is a \( \sigma \)-algebra over \( \Gamma \).

Example 1.3: Let \( \mathcal{L} \) be the collection of all finite disjoint unions of all intervals of the form
\[
(-\infty, a], \quad (a, b], \quad (b, \infty), \quad \emptyset. \tag{1.3}
\]
Then \( \mathcal{L} \) is an algebra over \( \mathbb{R} \) (the set of real numbers), but not a \( \sigma \)-algebra because \( \Lambda_i = (0, (i - 1)/i] \in \mathcal{L} \) for all \( i \) but
\[
\bigcup_{i=1}^{\infty} \Lambda_i = (0,1) \notin \mathcal{L}. \tag{1.4}
\]

Example 1.4: A \( \sigma \)-algebra \( \mathcal{L} \) is closed under countable union, countable intersection, difference, and limit. That is, if \( \Lambda_1, \Lambda_2, \cdots \in \mathcal{L} \), then
\[
\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \bigcap_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \Lambda_1 \setminus \Lambda_2 \in \mathcal{L}; \quad \lim_{i \to \infty} \Lambda_i \in \mathcal{L}. \tag{1.5}
\]

Definition 1.2 Let \( \Gamma \) be a nonempty set, and let \( \mathcal{L} \) be a \( \sigma \)-algebra over \( \Gamma \). Then \((\Gamma, \mathcal{L})\) is called a measurable space, and any element in \( \mathcal{L} \) is called a measurable set.

Example 1.5: Let \( \mathbb{R} \) be the set of real numbers. Then \( \mathcal{L} = \{\emptyset, \mathbb{R}\} \) is a \( \sigma \)-algebra over \( \mathbb{R} \). Thus \((\mathbb{R}, \mathcal{L})\) is a measurable space. Note that there exist only two measurable sets in this space, one is \( \emptyset \) and another is \( \mathbb{R} \). However, the intervals \([0, 1]\) and \((0, +\infty)\) are not measurable in this space.

Example 1.6: Let \( \Gamma = \{a, b, c\} \). Then \( \mathcal{L} = \{\emptyset, \{a\}, \{b, c\}, \Gamma\} \) is a \( \sigma \)-algebra over \( \Gamma \). Thus \((\Gamma, \mathcal{L})\) is a measurable space. Furthermore, \( \{a\} \) and \( \{b, c\} \) are measurable sets in this space, but \( \{b\}, \{c\}, \{a, b\}, \{a, c\} \) are not.

Definition 1.3 The smallest \( \sigma \)-algebra \( \mathcal{B} \) containing all open intervals is called the Borel algebra over the set of real numbers, and any element in \( \mathcal{B} \) is called a Borel set.

Example 1.7: It has been proved that intervals, open sets, closed sets, rational numbers, and irrational numbers are all Borel sets.

Example 1.8: There exists a non-Borel set over \( \mathbb{R} \). Let \([a]\) represent the set of all rational numbers plus \( a \). Note that if \( a_1 - a_2 \) is not a rational number, then \([a_1]\) and \([a_2]\) are disjoint sets. Thus \( \mathbb{R} \) is divided into an infinite number of those disjoint sets. Let \( A \) be a new set containing precisely one element from them. Then \( A \) is not a Borel set.
Definition 1.4 A function $\xi$ from a measurable space $(\Gamma, \mathcal{L})$ to the set of real numbers is said to be measurable if

$$\xi^{-1}(B) = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \in \mathcal{L}$$

(1.6)

for any Borel set $B$ of real numbers.

Continuous function and monotone function are instances of measurable function. Let $\xi_1, \xi_2, \cdots$ be a sequence of measurable functions. Then the following functions are also measurable:

$$\sup_{1 \leq i < \infty} \xi_i(\gamma); \quad \inf_{1 \leq i < \infty} \xi_i(\gamma); \quad \limsup_{i \to \infty} \xi_i(\gamma); \quad \liminf_{i \to \infty} \xi_i(\gamma).$$

(1.7)

Especially, if $\lim_{i \to \infty} \xi_i(\gamma)$ exists for each $\gamma$, then the limit is also a measurable function.

1.2 Uncertain Measure

Let $(\Gamma, \mathcal{L})$ be a measurable space. Recall that each element $\Lambda$ in $\mathcal{L}$ is called a measurable set. The first action we take is to rename measurable set as event in uncertainty theory. The second action is to define an uncertain measure $\mathcal{M}$ on the $\sigma$-algebra $\mathcal{L}$. That is, a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event $\Lambda$ to indicate the belief degree with which we believe $\Lambda$ will happen. There is no doubt that the assignment is not arbitrary, and the uncertain measure $\mathcal{M}$ must have certain mathematical properties. In order to rationally deal with belief degrees, Liu [88] suggested the following three axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set $\Gamma$.

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda$.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M} \left\{ \bigcup_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \quad (1.8)$$

Remark 1.1: Uncertain measure is interpreted as the belief degree of an uncertain event that may happen\(^1\). Thus uncertain measure and belief degree are synonymous, and will be used interchangeably in this book.

Remark 1.2: Uncertain measure (i.e., belief degree) depends on the personal knowledge concerning the event. It will change if the state of knowledge changes.

\(^1\)In contrast, probability measure is interpreted as the frequency of a random event that may happen.
Remark 1.3: Since “1” means “complete belief” and we cannot be in more belief than “complete belief”, the belief degree of any event cannot exceed 1. Furthermore, the belief degree of the universal set takes value 1 because it is completely believable. Thus the belief degree meets the normality axiom.

Remark 1.4: Duality axiom is in fact an application of the law of truth conservation in uncertainty theory. The property ensures that the uncertainty theory is consistent with the law of excluded middle and the law of contradiction. In addition, the human thinking is always dominated by the duality. For example, if someone tells us that a proposition is true with belief degree 0.6, then all of us will think that the proposition is false with belief degree 0.4.

Remark 1.5: Given two events with known belief degrees, it is frequently asked that how the belief degree for their union is generated from the individuals. Personally, I do not think there exists any rule to make it. A lot of surveys showed that, generally speaking, the belief degree of a union of events is neither the sum of belief degrees of the individual events (e.g. probability measure) nor the maximum (e.g. possibility measure). It seems that there is no explicit relation between the union and individuals except for the subadditivity axiom.

Remark 1.6: Pathology occurs if subadditivity axiom is not assumed. For example, suppose that a universal set contains 3 elements. We define a set function that takes value 0 for each singleton, and 1 for each event with at least 2 elements. Then such a set function satisfies all axioms but subadditivity. Do you think it is strange if such a set function serves as a measure?

Remark 1.7: Although probability measure satisfies the above three axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fourth axiom, namely the product axiom on Page 17.

Definition 1.5 (Liu [88]) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.

Exercise 1.1: Let $\Gamma$ be a nonempty set. For each subset $\Lambda$ of $\Gamma$, we define

$$
\mathcal{M}\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
0.5, & \text{otherwise.}
\end{cases}
$$

(1.9)

Show that $\mathcal{M}$ is an uncertain measure. (Hint: Verify $\mathcal{M}$ meets the three axioms.)

Exercise 1.2: Let $\Gamma = \{\gamma_1, \gamma_2\}$. It is clear that there exist 4 events in the power set,

$$
\mathcal{L} = \{\emptyset, \{\gamma_1\}, \{\gamma_2\}, \Gamma\}.
$$

(1.10)
Assume \( c \) is a real number with \( 0 < c < 1 \), and define
\[
M\{\emptyset\} = 0, \quad M\{\gamma_1\} = c, \quad M\{\gamma_2\} = 1 - c, \quad M\{\Gamma\} = 1.
\]
Show that \( M \) is an uncertain measure.

**Exercise 1.3:** Let \( \Gamma = \{\gamma_1, \gamma_2, \gamma_3\} \). It is clear that there exist 8 events in the power set,
\[
\mathcal{L} = \{\emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_1, \gamma_2\}, \{\gamma_1, \gamma_3\}, \{\gamma_2, \gamma_3\}, \Gamma\}. \quad (1.11)
\]
Assume \( c_1, c_2, c_3 \) are nonnegative numbers satisfying the consistency condition
\[
c_i + c_j \leq 1 \leq c_1 + c_2 + c_3, \quad \forall i \neq j.
\]
Define
\[
M\{\gamma_1\} = c_1, \quad M\{\gamma_2\} = c_2, \quad M\{\gamma_3\} = c_3,
\]
\[
M\{\gamma_1, \gamma_2\} = 1 - c_3, \quad M\{\gamma_1, \gamma_3\} = 1 - c_2, \quad M\{\gamma_2, \gamma_3\} = 1 - c_1,
\]
\[
M\{\emptyset\} = 0, \quad M\{\Gamma\} = 1.
\]
Show that \( M \) is an uncertain measure.

**Exercise 1.4:** Let \( \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \), and let \( c \) be a real number with \( 0.5 \leq c < 1 \). It is clear that there exist 16 events in the power set. For each subset \( \Lambda \), define
\[
M\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
c, & \text{if } \gamma_1 \in \Lambda \neq \Gamma \\
1 - c, & \text{if } \gamma_1 \not\subset \Lambda \neq \emptyset.
\end{cases} \quad (1.13)
\]
Show that \( M \) is an uncertain measure.

**Exercise 1.5:** Let \( \Gamma = \{\gamma_1, \gamma_2, \cdots\} \), and let \( c_1, c_2, \cdots \) be nonnegative numbers such that \( c_1 + c_2 + \cdots = 1 \). For each subset \( \Lambda \), define
\[
M\{\Lambda\} = \sum_{\gamma_1 \in \Lambda} c_i. \quad (1.14)
\]
Show that \( M \) is an uncertain measure.

**Exercise 1.6:** Lebesgue measure, named after French mathematician Henri Lebesgue, is the standard way of assigning a length, area or volume to subsets of Euclidean space. For example, the Lebesgue measure of the interval \([a, b]\) of real numbers is the length \( b - a \). Let \( \Gamma = [0, 1] \), and let \( M \) be the Lebesgue measure. Show that \( M \) is an uncertain measure.
Exercise 1.7: Let $\Gamma$ be the set of real numbers, and let $c$ be a real number with $0 < c \leq 0.5$. For each subset $\Lambda$, define

$$
M\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
c, & \text{if } \Lambda \text{ is upper bounded and } \Lambda \neq \emptyset \\
0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\
1 - c, & \text{if } \Lambda^c \text{ is upper bounded and } \Lambda \neq \Gamma \\
1, & \text{if } \Lambda = \Gamma.
\end{cases} \tag{1.15}
$$

Show that $M$ is an uncertain measure.

Exercise 1.8: Suppose $\rho(x)$ is a nonnegative and integrable function on $(-\infty, +\infty)$ such that

$$
\int_{-\infty}^{+\infty} \rho(x)dx \geq 1. \tag{1.16}
$$

Define a set function

$$
M\{\Lambda\} = \begin{cases} 
\int_{\Lambda} \rho(x)dx, & \text{if } \int_{\Lambda} \rho(x)dx < 0.5 \\
1 - \int_{\Lambda^c} \rho(x)dx, & \text{if } \int_{\Lambda^c} \rho(x)dx < 0.5 \\
0.5, & \text{otherwise}
\end{cases} \tag{1.17}
$$

for each Borel set $\Lambda$ of real numbers. Show that $M$ is an uncertain measure.

Theorem 1.1 (Monotonicity Theorem) The uncertain measure is a monotone increasing set function. That is, for any events $\Lambda_1$ and $\Lambda_2$ with $\Lambda_1 \subset \Lambda_2$, we have

$$
M\{\Lambda_1\} \leq M\{\Lambda_2\}. \tag{1.18}
$$

Proof: The normality axiom says $M\{\Gamma\} = 1$, and the duality axiom says $M\{\Lambda_1^c\} = 1 - M\{\Lambda_1\}$. Since $\Lambda_1 \subset \Lambda_2$, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$. By using the subadditivity axiom, we obtain

$$
1 = M\{\Gamma\} \leq M\{\Lambda_1^c\} + M\{\Lambda_2\} = 1 - M\{\Lambda_1\} + M\{\Lambda_2\}.
$$

Thus $M\{\Lambda_1\} \leq M\{\Lambda_2\}$.

Theorem 1.2 The empty set $\emptyset$ always has an uncertain measure zero. That is,

$$
M\{\emptyset\} = 0. \tag{1.19}
$$

Proof: Since $\emptyset = \Gamma^c$ and $M\{\Gamma\} = 1$, it follows from the duality axiom that

$$
M\{\emptyset\} = 1 - M\{\Gamma\} = 1 - 1 = 0.
$$
Theorem 1.3  The uncertain measure takes values between 0 and 1. That is, for any event \( \Lambda \), we have

\[ 0 \leq \mathcal{M}\{\Lambda\} \leq 1. \]  \hspace{1cm} (1.20)

Proof: It follows from the monotonicity theorem that \( 0 \leq \mathcal{M}\{\Lambda\} \leq 1 \) because \( \emptyset \subset \Lambda \subset \Gamma \) and \( \mathcal{M}\{\emptyset\} = 0, \mathcal{M}\{\Gamma\} = 1 \).

Theorem 1.4  Let \( \Lambda_1, \Lambda_2, \cdots \) be a sequence of events with \( \mathcal{M}\{\Lambda_i\} \to 0 \) as \( i \to \infty \). Then for any event \( \Lambda \), we have

\[ \lim_{i \to \infty} \mathcal{M}\{\Lambda \cup \Lambda_i\} = \lim_{i \to \infty} \mathcal{M}\{\Lambda \setminus \Lambda_i\} = \mathcal{M}\{\Lambda\}. \]  \hspace{1cm} (1.21)

Especially, an uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

Proof: It follows from the monotonicity theorem and subadditivity axiom that

\[ \mathcal{M}\{\Lambda\} \leq \mathcal{M}\{\Lambda \cup \Lambda_i\} \leq \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda_i\} \]

for each \( i \). Thus we get \( \mathcal{M}\{\Lambda \cup \Lambda_i\} \to \mathcal{M}\{\Lambda\} \) by using \( \mathcal{M}\{\Lambda_i\} \to 0 \). Since \( (\Lambda \setminus \Lambda_i) \subset \Lambda \subset ((\Lambda \setminus \Lambda_i) \cup \Lambda_i) \), we have

\[ \mathcal{M}\{\Lambda \setminus \Lambda_i\} \leq \mathcal{M}\{\Lambda\} \leq \mathcal{M}\{\Lambda \setminus \Lambda_i\} + \mathcal{M}\{\Lambda_i\} . \]

Hence \( \mathcal{M}\{\Lambda \setminus \Lambda_i\} \to \mathcal{M}\{\Lambda\} \) by using \( \mathcal{M}\{\Lambda_i\} \to 0 \).

Theorem 1.5  (Asymptotic Theorem) For any events \( \Lambda_1, \Lambda_2, \cdots \), we have

\[ \lim_{i \to \infty} \mathcal{M}\{\Lambda_i\} > 0, \text{ if } \Lambda_i \uparrow \Gamma, \]  \hspace{1cm} (1.22)

\[ \lim_{i \to \infty} \mathcal{M}\{\Lambda_i\} < 1, \text{ if } \Lambda_i \downarrow \emptyset. \]  \hspace{1cm} (1.23)

Proof: Assume \( \Lambda_i \uparrow \Gamma \). Since \( \Gamma = \cup_i \Lambda_i \), it follows from the subadditivity axiom that

\[ 1 = \mathcal{M}\{\Gamma\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \]

Since \( \mathcal{M}\{\Lambda_i\} \) is increasing with respect to \( i \), we have \( \lim_{i \to \infty} \mathcal{M}\{\Lambda_i\} > 0 \). If \( \Lambda_i \downarrow \emptyset \), then \( \Lambda_i^c \uparrow \Gamma \). It follows from the first inequality and the duality axiom that

\[ \lim_{i \to \infty} \mathcal{M}\{\Lambda_i\} = 1 - \lim_{i \to \infty} \mathcal{M}\{\Lambda_i^c\} < 1. \]

The theorem is proved.
Example 1.9: Assume \( \Gamma \) is the set of real numbers. Let \( \alpha \) be a number with \( 0 < \alpha \leq 0.5 \). Define an uncertain measure as follows,

\[
M\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
\alpha, & \text{if } \Lambda \text{ is upper bounded and } \Lambda \neq \emptyset \\
0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\
1 - \alpha, & \text{if } \Lambda^c \text{ is upper bounded and } \Lambda \neq \Gamma \\
1, & \text{if } \Lambda = \Gamma.
\end{cases}
\] (1.24)

(i) Write \( \Lambda_i = (-\infty, i] \) for \( i = 1, 2, \cdots \). Then \( \Lambda_i \uparrow \Gamma \) and \( \lim_{i \to \infty} M\{\Lambda_i\} = \alpha \).

(ii) Write \( \Lambda_i = [i, +\infty) \) for \( i = 1, 2, \cdots \). Then \( \Lambda_i \downarrow \emptyset \) and \( \lim_{i \to \infty} M\{\Lambda_i\} = 1 - \alpha \).

1.3 Uncertainty Space

Definition 1.6 (Liu [88]) Let \( \Gamma \) be a nonempty set, let \( \mathcal{L} \) be a \( \sigma \)-algebra over \( \Gamma \), and let \( M \) be an uncertain measure. Then the triplet \((\Gamma, \mathcal{L}, M)\) is called an uncertainty space.

Example 1.10: Let \( \Gamma \) be a two-point set \( \{\gamma_1, \gamma_2\} \), let \( \mathcal{L} \) be the power set of \( \{\gamma_1, \gamma_2\} \), and let \( M \) be an uncertain measure determined by \( M\{\gamma_1\} = 0.6 \) and \( M\{\gamma_2\} = 0.4 \). Then \((\Gamma, \mathcal{L}, M)\) is an uncertainty space.

Example 1.11: Let \( \Gamma \) be a three-point set \( \{\gamma_1, \gamma_2, \gamma_3\} \), let \( \mathcal{L} \) be the power set of \( \{\gamma_1, \gamma_2, \gamma_3\} \), and let \( M \) be an uncertain measure determined by \( M\{\gamma_1\} = 0.6 \), \( M\{\gamma_2\} = 0.3 \) and \( M\{\gamma_3\} = 0.2 \). Then \((\Gamma, \mathcal{L}, M)\) is an uncertainty space.

Example 1.12: Let \( \Gamma \) be the interval \([0, 1]\), let \( \mathcal{L} \) be the Borel algebra over \([0, 1]\), and let \( M \) be the Lebesgue measure. Then \((\Gamma, \mathcal{L}, M)\) is an uncertainty space.

For practical purposes, the study of uncertainty spaces is sometimes restricted to complete uncertainty spaces.

Definition 1.7 (Liu [106]) An uncertainty space \((\Gamma, \mathcal{L}, M)\) is called complete if for any \( \Lambda_1, \Lambda_2 \in \mathcal{L} \) with \( M\{\Lambda_1\} = M\{\Lambda_2\} \) and any subset \( A \) with \( \Lambda_1 \subset A \subset \Lambda_2 \), one has \( A \in \mathcal{L} \). In this case, we also have

\[
M\{A\} = M\{\Lambda_1\} = M\{\Lambda_2\}.
\] (1.25)

Exercise 1.9: Let \((\Gamma, \mathcal{L}, M)\) be a complete uncertainty space, and let \( \Lambda \) be an event with \( M\{\Lambda\} = 0 \). Show that \( A \) is an event and \( M\{A\} = 0 \) whenever \( A \subset \Lambda \).

Exercise 1.10: Let \((\Gamma, \mathcal{L}, M)\) be a complete uncertainty space, and let \( \Lambda \) be an event with \( M\{\Lambda\} = 1 \). Show that \( A \) is an event and \( M\{A\} = 1 \) whenever \( A \supset \Lambda \).
Definition 1.8 (Gao [46]) An uncertainty space $(\Gamma, \mathcal{L}, M)$ is called continuous if for any events $\Lambda_1, \Lambda_2, \cdots$, we have
\[
M \left\{ \lim_{i \to \infty} \Lambda_i \right\} = \lim_{i \to \infty} M \{\Lambda_i\}
\] (1.26)
provided that $\lim_{i \to \infty} \Lambda_i$ exists.

Exercise 1.11: Show that an uncertainty space $(\Gamma, \mathcal{L}, M)$ is always continuous if $\Gamma$ consists of a finite number of points.

Exercise 1.12: Let $\Gamma = [0, 1]$, let $\mathcal{L}$ be the Borel algebra over $\Gamma$, and let $M$ be the Lebesgue measure. Show that $(\Gamma, \mathcal{L}, M)$ is a continuous uncertainty space.

Exercise 1.13: Let $\Gamma = [0, 1]$, and let $\mathcal{L}$ be the power set over $\Gamma$. For each subset $\Lambda$ of $\Gamma$, define
\[
M \{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
0.5, & \text{otherwise} 
\end{cases}
\] (1.27)
Show that $(\Gamma, \mathcal{L}, M)$ is a discontinuous uncertainty space.

1.4 Product Uncertain Measure

Product uncertain measure was defined by Liu [91] in 2009, thus producing the fourth axiom of uncertainty theory. Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Write
\[
\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots
\] (1.28)
that is the set of all ordered tuples of the form $(\gamma_1, \gamma_2, \cdots)$, where $\gamma_k \in \Gamma_k$ for $k = 1, 2, \cdots$. A measurable rectangle in $\Gamma$ is a set
\[
\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots
\] (1.29)
where $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots$. The smallest $\sigma$-algebra containing all measurable rectangles of $\Gamma$ is called the product $\sigma$-algebra, denoted by
\[
\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots
\] (1.30)
Then the product uncertain measure $M$ on the product $\sigma$-algebra $\mathcal{L}$ is defined by the following product axiom (Liu [91]).

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $M$ is an uncertain measure satisfying
\[
M \left\{ \prod_{k=1}^{\infty} \Lambda_k \right\} = \bigwedge_{k=1}^{\infty} M_k \{\Lambda_k\}
\] (1.31)
where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \ldots \), respectively.

**Remark 1.8:** Note that (1.31) defines a product uncertain measure only for rectangles. How do we extend the uncertain measure \( M \) from the class of rectangles to the product \( \sigma \)-algebra \( \mathcal{L} \)? In fact, for each event \( \Lambda \in \mathcal{L} \), we may set

\[
M\{\Lambda\} = \begin{cases} 
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\}, & \text{if } \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} > 0.5 \\
1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k\{\Lambda_k\}, & \text{if } \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} > 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]  

(1.32)

![Figure 1.1](image.png)

Figure 1.1: Extension from Rectangles to Product \( \sigma \)-Algebra. The uncertain measure of \( \Lambda \) (the disk) is essentially the acreage of its inscribed rectangle \( \Lambda_1 \times \Lambda_2 \) if it is greater than 0.5. Otherwise, we have to examine its complement \( \Lambda^c \). If the inscribed rectangle of \( \Lambda^c \) is greater than 0.5, then \( M\{\Lambda^c\} \) is just its inscribed rectangle and \( M\{\Lambda\} = 1 - M\{\Lambda^c\} \). If there does not exist an inscribed rectangle of \( \Lambda \) or \( \Lambda^c \) greater than 0.5, then we set \( M\{\Lambda\} = 0.5 \).

**Remark 1.9:** The sum of the uncertain measures of the maximum rectangles in \( \Lambda \) and \( \Lambda^c \) is always less than or equal to 1, i.e.,

\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} \leq 1.
\]
This means that at most one of
\[ \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} \quad \text{and} \quad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} \]
is greater than 0.5. Thus the expression (1.32) is reasonable.

Remark 1.10: It is clear that for each \( \Lambda \in \mathcal{L} \), the uncertain measure \( M\{\Lambda\} \) defined by (1.32) takes possible values on the interval
\[ \left[ \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\}, 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} \right]. \]
Thus (1.32) coincides with the maximum uncertainty principle (Liu [88]), that is, \( M\{\Lambda\} \) takes the value as close to 0.5 as possible within the above interval.

Remark 1.11: If the sum of the uncertain measures of the maximum rectangles in \( \Lambda \) and \( \Lambda^c \) is just 1, i.e.,
\[ \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k\{\Lambda_k\} = 1, \]
then the product uncertain measure (1.32) is simplified as
\[ M\{\Lambda\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k\{\Lambda_k\}. \quad (1.33) \]

Remark 1.12: The product uncertain measure \( M \) defined by (1.31) will be denoted as
\[ M = M_1 \land M_2 \land \cdots \quad (1.34) \]

Exercise 1.14: Let \((\Gamma_1, \mathcal{L}_1, M_1)\) be the interval \([0, 1]\) with Borel algebra and Lebesgue measure, and let \((\Gamma_2, \mathcal{L}_2, M_2)\) be also the interval \([0, 1]\) with Borel algebra and Lebesgue measure. Then
\[ \Lambda = \{ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \gamma_1 + \gamma_2 \leq 1 \} \quad (1.35) \]
is an event on the product uncertainty space \((\Gamma_1, \mathcal{L}_1, M_1) \times (\Gamma_2, \mathcal{L}_2, M_2)\). Show that
\[ M\{\Lambda\} = \frac{1}{2}. \quad (1.36) \]

Exercise 1.15: Let \((\Gamma_1, \mathcal{L}_1, M_1)\) be the interval \([0, 1]\) with Borel algebra and Lebesgue measure, and let \((\Gamma_2, \mathcal{L}_2, M_2)\) be also the interval \([0, 1]\) with Borel algebra and Lebesgue measure. Then
\[ \Lambda = \{ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid (\gamma_1 - 0.5)^2 + (\gamma_2 - 0.5)^2 \leq 0.5^2 \} \quad (1.37) \]
is an event on the product uncertainty space \((\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)\). (i) Show that
\[
\mathcal{M}\{\Lambda\} = \frac{1}{\sqrt{2}}. \tag{1.38}
\]
(ii) From the above result we derive \(\mathcal{M}\{\Lambda^c\} = 1 - 1/\sqrt{2}\). Please find a rectangle \(\Lambda_1 \times \Lambda_2\) in \(\Lambda^c\) such that \(\mathcal{M}\{\Lambda_1 \times \Lambda_2\} = 1 - 1/\sqrt{2}\).

**Theorem 1.6 (Peng-Iwamura [138])** The product uncertain measure defined by (1.32) is an uncertain measure.

**Proof:** In order to prove that the product uncertain measure (1.32) is indeed an uncertain measure, we should verify that the product uncertain measure satisfies the normality, duality and subadditivity axioms.

**STEP 1:** The product uncertain measure is clearly normal, i.e., \(\mathcal{M}\{\Gamma\} = 1\).

**STEP 2:** We prove the duality, i.e., \(\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1\). The argument breaks down into three cases. Case 1: Assume
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5.
\]
Then we immediately have
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\} < 0.5.
\]
It follows from (1.32) that
\[
\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\},
\]
\[
\mathcal{M}\{\Lambda^c\} = 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset (\Lambda^c)^c} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\} = 1 - \mathcal{M}\{\Lambda\}.
\]
The duality is proved. Case 2: Assume
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5.
\]
This case can be proved by a similar process. Case 3: Assume
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\} \leq 0.5
\]
and
\[
\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} \mathcal{M}_k\{\Lambda_k\} \leq 0.5.
\]
It follows from (1.32) that \(\mathcal{M}\{\Lambda\} = \mathcal{M}\{\Lambda^c\} = 0.5\) which proves the duality.
Step 3: Let us prove that $M$ is an increasing set function. Suppose $\Lambda$ and $\Delta$ are two events in $\mathcal{L}$ with $\Lambda \subset \Delta$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k \{\Lambda_k\} > 0.5.$$ 

Then

$$\sup_{\Delta_1 \times \Delta_2 \times \cdots \subset \Delta} \min_{1 \leq k < \infty} M_k \{\Delta_k\} \geq \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k \{\Lambda_k\} > 0.5.$$ 

It follows from (1.32) that

$$M\{\Lambda\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k \{\Lambda_k\},$$

$$M\{\Delta\} = \sup_{\Delta_1 \times \Delta_2 \times \cdots \subset \Delta} \min_{1 \leq k < \infty} M_k \{\Delta_k\}.$$ 

Thus $M\{\Lambda\} \leq M\{\Delta\}$. Case 2: Assume

$$\sup_{\Delta_1 \times \Delta_2 \times \cdots \subset \Delta^c} \min_{1 \leq k < \infty} M_k \{\Delta_k\} > 0.5.$$ 

Then

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k \{\Lambda_k\} \geq \sup_{\Delta_1 \times \Delta_2 \times \cdots \subset \Delta^c} \min_{1 \leq k < \infty} M_k \{\Delta_k\} > 0.5.$$ 

Thus

$$M\{\Lambda\} = 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \leq k < \infty} M_k \{\Lambda_k\}$$

$$
\leq 1 - \sup_{\Delta_1 \times \Delta_2 \times \cdots \subset \Delta^c} \min_{1 \leq k < \infty} M_k \{\Delta_k\} = M\{\Delta\}.$$

Case 3: Assume

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \leq k < \infty} M_k \{\Lambda_k\} \leq 0.5$$

and

$$\sup_{\Delta_1 \times \Delta_2 \times \cdots \subset \Delta^c} \min_{1 \leq k < \infty} M_k \{\Delta_k\} \leq 0.5.$$ 

It follows from (1.32) that $M\{\Lambda\} \leq 0.5$ and $M\{\Delta\} \geq 0.5$. Thus we obtain $M\{\Lambda\} \leq M\{\Delta\}$.

Step 4: Finally, we prove the subadditivity of $M$. For simplicity, we only prove the case of two events $\Lambda$ and $\Delta$. The argument breaks down into three cases. Case 1: Assume $M\{\Lambda\} < 0.5$ and $M\{\Delta\} < 0.5$. For any given $\varepsilon > 0$, there are two rectangles

$$\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c, \quad \Delta_1 \times \Delta_2 \times \cdots \subset \Delta^c.$$
such that
\[ 1 - \min_{1 \leq k < \infty} M_k \{\Lambda_k\} \leq M\{\Lambda\} + \varepsilon/2, \]
\[ 1 - \min_{1 \leq k < \infty} M_k \{\Delta_k\} \leq M\{\Delta\} + \varepsilon/2. \]

Note that
\[(\Lambda_1 \cap \Delta_1) \times (\Lambda_2 \cap \Delta_2) \times \cdots \subset (\Lambda \cup \Delta)^c.\]

It follows from the duality and subadditivity axioms of \(M_k\) that
\[
M_k \{\Lambda_k \cap \Delta_k\} = 1 - M_k \{(\Lambda_k \cap \Delta_k)^c\} \\
= 1 - M_k \{\Lambda_k^c \cup \Delta_k^c\} \\
\geq 1 - (M_k \{\Lambda_k^c\} + M_k \{\Delta_k^c\}) \\
= 1 - (1 - M_k \{\Lambda_k\}) - (1 - M_k \{\Delta_k\}) \\
= M_k \{\Lambda_k\} + M_k \{\Delta_k\} - 1
\]

for any \(k\). Since \(M\) has been proved to be dual and increasing, we have
\[
M\{\Lambda \cup \Delta\} = 1 - M\{(\Lambda \cup \Delta)^c\} \\
\leq 1 - M\{(\Lambda_1 \cap \Delta_1) \times (\Lambda_2 \cap \Delta_2) \times \cdots\} \\
= 1 - \min_{1 \leq k < \infty} M_k \{\Lambda_k \cap \Delta_k\} \\
\leq 1 - \min_{1 \leq k < \infty} M_k \{\Lambda_k\} + 1 - \min_{1 \leq k < \infty} M_k \{\Delta_k\} \\
\leq M\{\Lambda\} + M\{\Delta\} + \varepsilon.
\]

Letting \(\varepsilon \to 0\), we obtain
\[
M\{\Lambda \cup \Delta\} \leq M\{\Lambda\} + M\{\Delta\}.
\]

Case 2: Assume \(M\{\Lambda\} \geq 0.5\) and \(M\{\Delta\} < 0.5\). When \(M\{\Lambda \cup \Delta\} = 0.5\), the subadditivity is obvious. Now we consider the case \(M\{\Lambda \cup \Delta\} > 0.5\), i.e., \(M\{\Lambda^c \cap \Delta^c\} < 0.5\). By using \(\Lambda^c \cup \Delta = (\Lambda^c \cap \Delta^c) \cup \Delta\) and Case 1, we get
\[
M\{\Lambda^c \cup \Delta\} \leq M\{\Lambda^c \cap \Delta^c\} + M\{\Delta\}.
\]

Thus
\[
M\{\Lambda \cup \Delta\} = 1 - M\{\Lambda^c \cap \Delta^c\} \leq 1 - M\{\Lambda^c \cup \Delta\} + M\{\Delta\} \\
\leq 1 - M\{\Lambda^c\} + M\{\Delta\} = M\{\Lambda\} + M\{\Delta\}.
\]

Case 3: If both \(M\{\Lambda\} \geq 0.5\) and \(M\{\Delta\} \geq 0.5\), then the subadditivity is obvious because \(M\{\Lambda\} + M\{\Delta\} \geq 1 \geq M\{\Lambda \cup \Delta\}\). The theorem is proved.
**Definition 1.9** Assume \((\Gamma_k, \mathcal{L}_k, M_k)\) are uncertainty spaces for \(k = 1, 2, \cdots\) and
\[
\begin{align*}
\Gamma &= \Gamma_1 \times \Gamma_2 \times \cdots \quad (1.39) \\
\mathcal{L} &= \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \quad (1.40) \\
M &= M_1 \land M_2 \land \cdots \quad (1.41)
\end{align*}
\]
Then the triplet \((\Gamma, \mathcal{L}, M)\) is called a product uncertainty space.

### 1.5 Independence

Note that an event is essentially a measurable set. The independence of two sets means that knowing the occurrence of one does not change our estimation of another. What events meet this condition? A typical case is that they belong to different uncertainty spaces. For example, let \(\Lambda_1\) and \(\Lambda_2\) be events on the uncertainty spaces \((\Gamma_1, \mathcal{L}_1, M_1)\) and \((\Gamma_2, \mathcal{L}_2, M_2)\), respectively. Then \(\Lambda_1\) and \(\Lambda_2\) can be understood as \(\Lambda_1 \times \Gamma_2\) and \(\Gamma_1 \times \Lambda_2\) on the product uncertainty space \((\Gamma_1, \mathcal{L}_1, M_1) \times (\Gamma_2, \mathcal{L}_2, M_2)\), respectively. See Figure 1.2.

\[
\begin{align*}
\Gamma_2 \\
&\Lambda_2 \\
&\Lambda_1 \times \Lambda_2 \\
\end{align*}
\]

![Figure 1.2](image_url)

**Figure 1.2:** \((\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2) = \Lambda_1 \times \Lambda_2\) (abbreviated as \(\Lambda_1 \cap \Lambda_2\))

It follows from the product axiom that the product uncertain measure of the intersection is
\[
M\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = M\{\Lambda_1 \times \Lambda_2\} = M_1\{\Lambda_1\} \land M_2\{\Lambda_2\}.
\]
By using \(M\{\Lambda_1 \times \Gamma_2\} = M_1\{\Lambda_1\}\) and \(M\{\Gamma_1 \times \Lambda_2\} = M_2\{\Lambda_2\}\), we obtain
\[
M\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = M\{\Lambda_1 \times \Gamma_2\} \land M\{\Gamma_1 \times \Lambda_2\}.
\]
Similarly, we may prove other three equations as follows,

\[
\begin{align*}
\mathcal{M}((\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)) &= \mathcal{M}((\Lambda_1 \times \Gamma_2)^c) \land \mathcal{M}\{\Gamma_1 \times \Lambda_2\}, \\
\mathcal{M}((\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)^c) &= \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \land \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\}, \\
\mathcal{M}((\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)^c) &= \mathcal{M}((\Lambda_1 \times \Gamma_2)^c) \land \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\}.
\end{align*}
\]

For simplicity, we denote \(\Lambda_1 \times \Gamma_2\) and \(\Gamma_1 \times \Lambda_2\) by \(\Lambda_1\) and \(\Lambda_2\), respectively. Then the above four equations become

\[
\begin{align*}
\mathcal{M}\{\Lambda_1 \cap \Lambda_2\} &= \mathcal{M}\{\Lambda_1\} \land \mathcal{M}\{\Lambda_2\}, \\
\mathcal{M}\{\Lambda_1^c \cap \Lambda_2\} &= \mathcal{M}\{\Lambda_1^c\} \land \mathcal{M}\{\Lambda_2\}, \\
\mathcal{M}\{\Lambda_1 \cap \Lambda_2^c\} &= \mathcal{M}\{\Lambda_1\} \land \mathcal{M}\{\Lambda_2^c\}, \\
\mathcal{M}\{\Lambda_1^c \cap \Lambda_2^c\} &= \mathcal{M}\{\Lambda_1^c\} \land \mathcal{M}\{\Lambda_2^c\}.
\end{align*}
\]

Thus we say two events \(\Lambda_1\) and \(\Lambda_2\) are independent if and only if those four equations hold. Generally, we may define independence of events in the following form.

**Definition 1.10** (Liu [95]) The events \(\Lambda_1, \Lambda_2, \cdots, \Lambda_n\) are said to be independent if

\[
\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^c\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^c\}
\]

(1.42)

where \(\Lambda_i^c\) are arbitrarily chosen from \(\{\Lambda_i, \Lambda_i^c, \Gamma\}\), \(i = 1, 2, \cdots, n\), respectively, and \(\Gamma\) is the universal set.

**Example 1.13:** The impossible event \(\emptyset\) is independent of any event \(\Lambda\) because the following four equations hold:

\[
\begin{align*}
\mathcal{M}\{\emptyset \cap \Lambda\} &= \mathcal{M}\{\emptyset\} = \mathcal{M}\{\emptyset\} \land \mathcal{M}\{\Lambda\}, \\
\mathcal{M}\{\emptyset^c \cap \Lambda\} &= \mathcal{M}\{\Lambda\} = \mathcal{M}\{\emptyset^c\} \land \mathcal{M}\{\Lambda\}, \\
\mathcal{M}\{\emptyset \cap \Lambda^c\} &= \mathcal{M}\{\emptyset\} = \mathcal{M}\{\emptyset\} \land \mathcal{M}\{\Lambda^c\}, \\
\mathcal{M}\{\emptyset^c \cap \Lambda^c\} &= \mathcal{M}\{\Lambda^c\} = \mathcal{M}\{\emptyset^c\} \land \mathcal{M}\{\Lambda^c\}.
\end{align*}
\]

**Example 1.14:** The sure event \(\Gamma\) is independent of any event \(\Lambda\) because the following four equations hold:

\[
\begin{align*}
\mathcal{M}\{\Gamma \cap \Lambda\} &= \mathcal{M}\{\Lambda\} = \mathcal{M}\{\Gamma\} \land \mathcal{M}\{\Lambda\}, \\
\mathcal{M}\{\Gamma^c \cap \Lambda\} &= \mathcal{M}\{\Lambda\} = \mathcal{M}\{\Gamma^c\} \land \mathcal{M}\{\Lambda\}, \\
\mathcal{M}\{\Gamma \cap \Lambda^c\} &= \mathcal{M}\{\Lambda^c\} = \mathcal{M}\{\Gamma\} \land \mathcal{M}\{\Lambda^c\}, \\
\mathcal{M}\{\Gamma^c \cap \Lambda^c\} &= \mathcal{M}\{\Lambda^c\} = \mathcal{M}\{\Gamma^c\} \land \mathcal{M}\{\Lambda^c\}.
\end{align*}
\]

**Exercise 1.16:** Let \(\Lambda_1, \Lambda_2, \cdots, \Lambda_n\) be independent events. Show that \(\Lambda_i\) and \(\Lambda_j\) are independent for any indexes \(i\) and \(j\) with \(1 \leq i < j \leq n\).
Exercise 1.17: Let $\Lambda$ be an event. Are $\Lambda$ and $\Lambda^c$ independent? Please justify your answer.

Exercise 1.18: Construct $n$ independent events. (Hint: Define them on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, M_1) \times (\Gamma_2, \mathcal{L}_2, M_2) \times \cdots \times (\Gamma_n, \mathcal{L}_n, M_n)$.)

Theorem 1.7 (Liu [95]) The events $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are independent if and only if

$$M\left\{ \bigcup_{i=1}^{n} \Lambda_i^* \right\} = \bigvee_{i=1}^{n} M\{\Lambda_i^*\}$$

(1.43)

where $\Lambda_i^*$ are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \ldots, n$, respectively, and $\emptyset$ is the impossible event.

Proof: Assume $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are independent events. It follows from the duality axiom of uncertain measure that

$$M\left\{ \bigcup_{i=1}^{n} \Lambda_i^* \right\} = 1 - M\left\{ \bigcap_{i=1}^{n} \Lambda_i^{*c} \right\} = 1 - \bigwedge_{i=1}^{n} M\{\Lambda_i^{*c}\} = \bigvee_{i=1}^{n} M\{\Lambda_i^*\}$$

where $\Lambda_i^*$ are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \ldots, n$, respectively. The equation (1.43) is proved. Conversely, if the equation (1.43) holds, then

$$M\left\{ \bigcap_{i=1}^{n} \Lambda_i^* \right\} = 1 - M\left\{ \bigcup_{i=1}^{n} \Lambda_i^{*c} \right\} = 1 - \bigvee_{i=1}^{n} M\{\Lambda_i^{*c}\} = \bigwedge_{i=1}^{n} M\{\Lambda_i^*\}.$$  

where $\Lambda_i^*$ are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \ldots, n$, respectively. The equation (1.42) is true. The theorem is proved.

1.6 Polyrectangular Theorem

Definition 1.11 (Liu [103]) Let $(\Gamma_1, \mathcal{L}_1, M_1)$ and $(\Gamma_2, \mathcal{L}_2, M_2)$ be two uncertainty spaces. A set on $\Gamma_1 \times \Gamma_2$ is called a polyrectangle if it has the form

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i})$$

(1.44)

where $\Lambda_{1i} \in \mathcal{L}_1$ and $\Lambda_{2i} \in \mathcal{L}_2$ for $i = 1, 2, \ldots, m$, and

$$\Lambda_{11} \subset \Lambda_{12} \subset \cdots \subset \Lambda_{1m},$$

(1.45)

$$\Lambda_{21} \supset \Lambda_{22} \supset \cdots \supset \Lambda_{2m}.$$  

(1.46)

A rectangle $\Lambda_1 \times \Lambda_2$ is clearly a polyrectangle. In addition, a “cross”-like set is also a polyrectangle. See Figure 1.3.
Figure 1.3: Three Polyrectangles

**Theorem 1.8** (Liu [103], Polyrectangular Theorem) Let \((\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)\) and \((\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)\) be two uncertainty spaces. Then the polyrectangle
\[
\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i})
\]  
(1.47)
on the product uncertainty space \((\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)\) has an uncertain measure
\[
\mathcal{M}\{\Lambda\} = \bigvee_{i=1}^{m} \mathcal{M}\{\Lambda_{1i} \times \Lambda_{2i}\}.
\]  
(1.48)

**Proof:** It is clear that the maximum rectangle in the polyrectangle \(\Lambda\) is one of \(\Lambda_{1i} \times \Lambda_{2i}, \ i = 1, 2, \ldots, n\). Denote the maximum rectangle by \(\Lambda_{1k} \times \Lambda_{2k}\).

Case I: If
\[
\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_1\{\Lambda_{1k}\},
\]
then the maximum rectangle in \(\Lambda^c\) is \(\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\), and
\[
\mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = \mathcal{M}_1\{\Lambda_{1k}^c\} = 1 - \mathcal{M}_1\{\Lambda_{1k}\}.
\]
Thus
\[
\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = 1.
\]

Case II: If
\[
\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_2\{\Lambda_{2k}\},
\]
then the maximum rectangle in \(\Lambda^c\) is \(\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\), and
\[
\mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = \mathcal{M}_2\{\Lambda_{2k}^c\} = 1 - \mathcal{M}_2\{\Lambda_{2k}\}.
\]
Thus
\[
\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = 1.
\]
No matter what case happens, the sum of the uncertain measures of the maximum rectangles in \(\Lambda\) and \(\Lambda^c\) is always 1. It follows from the product axiom that (1.48) holds.
Remark 1.13: Since \( M_\{A_{1i} \times A_{2i}\} = M_1 \{A_{1i}\} \wedge M_2 \{A_{2i}\} \) for each index \( i \), we also have
\[
M_\{\Lambda\} = \bigvee_{i=1}^{m} M_1 \{A_{1i}\} \wedge M_2 \{A_{2i}\}.
\] (1.49)

Remark 1.14: Note that the polyrectangular theorem is also applicable to the polyrectangles that are unions of infinitely many rectangles. In this case, the polyrectangles may become the shapes in Figure 1.4.

1.7 Conditional Uncertain Measure

We consider the uncertain measure of an event \( \Lambda \) after it has been learned that some other event \( A \) has occurred. This new uncertain measure of \( \Lambda \) is called the \textit{conditional uncertain measure} of \( \Lambda \) given \( A \).

In order to define a conditional uncertain measure \( M_\{\Lambda | A\} \), at first we have to enlarge \( M_\{\Lambda \cap A\} \) because \( M_\{\Lambda \cap A\} < 1 \) for all events whenever \( M_\{A\} < 1 \). It seems that we have no alternative but to divide \( M_\{\Lambda \cap A\} \) by \( M_\{A\} \). Unfortunately, \( M_\{\Lambda \cap A\} / M_\{A\} \) is not always an uncertain measure. However, the value \( M_\{\Lambda | A\} \) should not be greater than \( M_\{\Lambda \cap A\} / M_\{A\} \) (otherwise the normality will be lost), i.e.,
\[
M_\{\Lambda | A\} \leq \frac{M_\{\Lambda \cap A\}}{M_\{A\}}.
\] (1.50)

On the other hand, in order to preserve the duality, we should have
\[
M_\{\Lambda | A\} = 1 - M_\{\Lambda^c | A\} \geq 1 - \frac{M_\{\Lambda^c \cap A\}}{M_\{A\}}.
\] (1.51)

Furthermore, since \( (\Lambda \cap A) \cup (\Lambda^c \cap A) = A \), we have \( M_\{A\} \leq M_\{\Lambda \cap A\} + M_\{\Lambda^c \cap A\} \) by using the subadditivity axiom. Thus
\[
0 \leq 1 - \frac{M_\{\Lambda^c \cap A\}}{M_\{A\}} \leq \frac{M_\{\Lambda \cap A\}}{M_\{A\}} \leq 1.
\] (1.52)
Hence any numbers between \(1 - \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}}\) and \(\frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}}\) are reasonable values that the conditional uncertain measure may take. Based on the maximum uncertainty principle (Liu [88]), we have the following conditional uncertain measure.

**Definition 1.12 (Liu [88])** Let \((\Gamma, \mathcal{L}, \mathcal{M})\) be an uncertainty space, and \(\Lambda, A \in \mathcal{L}\). Then the conditional uncertain measure of \(\Lambda\) given \(A\) is defined by

\[
\mathcal{M}\{\Lambda | A\} = \begin{cases} 
\frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}}, & \text{if } \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\
1 - \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}}, & \text{if } \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\
0.5, & \text{otherwise}
\end{cases}
\quad (1.53)
\]

provided that \(\mathcal{M}\{A\} > 0\).

**Remark 1.15:** It follows immediately from the definition of conditional uncertain measure that

\[
1 - \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}} \leq \mathcal{M}\{\Lambda | A\} \leq \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}}. \quad (1.54)
\]

**Remark 1.16:** The conditional uncertain measure \(\mathcal{M}\{\Lambda | A\}\) yields the posterior uncertain measure of \(\Lambda\) after the occurrence of event \(A\).

**Theorem 1.9 (Liu [88])** Let \((\Gamma, \mathcal{L}, \mathcal{M})\) be an uncertainty space, and let \(A\) be an event with \(\mathcal{M}\{A\} > 0\). Then \(\mathcal{M}\{\cdot | A\}\) defined by (1.53) is an uncertain measure, and \((\Gamma, \mathcal{L}, \mathcal{M}\{\cdot | A\})\) is an uncertainty space.

**Proof:** It is sufficient to prove that \(\mathcal{M}\{\cdot | A\}\) satisfies the normality, duality and subadditivity axioms. At first, it satisfies the normality axiom, i.e.,

\[
\mathcal{M}\{\Gamma | A\} = 1 - \frac{\mathcal{M}\{\Gamma^c \cap A\}}{\mathcal{M}\{A\}} = 1 - \frac{\mathcal{M}\{\emptyset\}}{\mathcal{M}\{A\}} = 1.
\]

For any event \(\Lambda\), if

\[
\frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}} \geq 0.5, \quad \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}} \geq 0.5,
\]

then we have \(\mathcal{M}\{\Lambda | A\} + \mathcal{M}\{\Lambda^c | A\} = 0.5 + 0.5 = 1\) immediately. Otherwise, without loss of generality, suppose

\[
\frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}} < 0.5 < \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}},
\]
then we have
\[ M\{\Lambda|A\} + M\{\Lambda^c|A\} = \frac{M\{\Lambda \cap A\}}{M\{A\}} + \left(1 - \frac{M\{\Lambda \cap A\}}{M\{A\}}\right) = 1. \]

That is, \(M\{\cdot|A\}\) satisfies the duality axiom. Finally, for any countable sequence \(\{\Lambda_i\}\) of events, if \(M\{\Lambda_i|A\} < 0.5\) for all \(i\), it follows from (1.54) and the subadditivity axiom that
\[
M\left\{\bigcup_{i=1}^{\infty} \Lambda_i | A\right\} \leq \frac{M\left\{\bigcap_{i=1}^{\infty} \Lambda_i \cap A\right\}}{M\{A\}} \leq \sum_{i=1}^{\infty} \frac{M\{\Lambda_i \cap A\}}{M\{A\}} = \sum_{i=1}^{\infty} M\{\Lambda_i|A\}.
\]

Suppose there is one term greater than 0.5, say \(M\{\Lambda_1|A\} \geq 0.5, \ M\{\Lambda_i|A\} < 0.5, \ i = 2, 3, \cdots\)

If \(M\{\cup_i \Lambda_i|A\} = 0.5\), then we immediately have
\[
M\left\{\bigcup_{i=1}^{\infty} \Lambda_i | A\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i|A\}.
\]

If \(M\{\cup_i \Lambda_i|A\} > 0.5\), we may prove the above inequality by the following facts:
\[
\Lambda_1^c \cap A \subset \bigcup_{i=2}^{\infty} (\Lambda_i \cap A) \cup \left(\bigcap_{i=1}^{\infty} \Lambda_i^c \cap A\right),
\]
\[
M\{\Lambda_1^c \cap A\} \leq \sum_{i=2}^{\infty} M\{\Lambda_i \cap A\} + M\left\{\bigcap_{i=1}^{\infty} \Lambda_i^c \cap A\right\},
\]
\[
M\left\{\bigcup_{i=1}^{\infty} \Lambda_i | A\right\} = 1 - \frac{M\left\{\bigcap_{i=1}^{\infty} \Lambda_i^c \cap A\right\}}{M\{A\}},
\]
\[
\sum_{i=1}^{\infty} M\{\Lambda_i|A\} \geq 1 - \frac{M\{\Lambda_1^c \cap A\}}{M\{A\}} + \sum_{i=2}^{\infty} \frac{M\{\Lambda_i \cap A\}}{M\{A\}}.
\]

If there are at least two terms greater than 0.5, then the subadditivity is clearly true. Thus \(M\{\cdot|A\}\) satisfies the subadditivity axiom. Hence \(M\{\cdot|A\}\) is an uncertain measure. Furthermore, \((\Gamma, \mathcal{L}, M\{\cdot|A\})\) is an uncertainty space.
1.8 Bibliographic Notes

When no samples are available to estimate distribution functions or some emergency (e.g., war, flood, earthquake, accident, and even rumour) arises, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree is subjective probability or fuzzy concept. However, Liu [97] declared that it is usually inappropriate because both probability theory and fuzzy set theory may lead to counterintuitive results in this case.

In order to rationally deal with belief degrees, uncertainty theory was founded by Liu [88] in 2007 and perfected by Liu [91] in 2009. The core of uncertainty theory is uncertain measure defined by the normality axiom, duality axiom, subadditivity axiom, and product axiom. In practice, uncertain measure is interpreted as the personal belief degree of an uncertain event that may happen.

Uncertain measure was also actively investigated by Gao [46], Liu [95], Zhang [222], Peng-Iwamura [138], and Liu [103], among others. Since then, the tool of uncertain measure was well developed and became a rigorous footstone of uncertainty theory.
Uncertain variable is a fundamental concept in uncertainty theory. It is used to represent quantities with uncertainty. The emphasis in this chapter is mainly on uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, distance, entropy, conditional uncertainty distribution, uncertain sequence, and uncertain vector.

2.1 Uncertain Variable

Roughly speaking, an uncertain variable is a measurable function on an uncertainty space. A formal definition is given as follows.

**Definition 2.1 (Liu [88])** An uncertain variable is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set $B$ of real numbers.

![Figure 2.1: An Uncertain Variable](image-url)

Remark 2.1: Note that the event $\{\xi \in B\}$ is a subset of the universal set
Chapter 2 - Uncertain Variable

Γ, i.e.,
\[ \{ \xi \in B \} = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B \}. \]  

(2.1)

**Example 2.1:** Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{ \gamma_1, \gamma_2 \}\) with power set and \(\mathcal{M}\{\gamma_1\} = 0.6, \mathcal{M}\{\gamma_2\} = 0.4\). Then

\[ \xi(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2 
\end{cases} \]  

(2.2)

is an uncertain variable. Furthermore, we have

\[ \mathcal{M}\{\xi = 0\} = \mathcal{M}\{\gamma \mid \xi(\gamma) = 0\} = \mathcal{M}\{\gamma_1\} = 0.6, \]  

(2.3)

\[ \mathcal{M}\{\xi = 1\} = \mathcal{M}\{\gamma \mid \xi(\gamma) = 1\} = \mathcal{M}\{\gamma_2\} = 0.4. \]  

(2.4)

**Example 2.2:** Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then

\[ \xi(\gamma) = 3\gamma, \quad \forall \gamma \in \Gamma \]  

(2.5)

is an uncertain variable. Furthermore, we have

\[ \mathcal{M}\{\xi \in [0, 2]\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \in [0, 2]\} = \mathcal{M}\{[0, 2/3]\} = 2/3, \]  

(2.6)

\[ \mathcal{M}\{\xi > 2\} = \mathcal{M}\{\gamma \mid \xi(\gamma) > 2\} = \mathcal{M}\{(2/3, 1]\} = 1/3. \]  

(2.7)

**Example 2.3:** A real number \(c\) may be regarded as a special uncertain variable. In fact, it is the constant function

\[ \xi(\gamma) \equiv c \]  

(2.9)

on the uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\). Furthermore, for any Borel set \(B\) of real numbers, we have

\[ \mathcal{M}\{\xi \in B\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \in B\} = \mathcal{M}\{\Gamma\} = 1, \quad \text{if } c \in B; \]  

(2.10)

\[ \mathcal{M}\{\xi \in B\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \in B\} = \mathcal{M}\{\emptyset\} = 0, \quad \text{if } c \notin B. \]  

(2.11)

**Example 2.4:** Let \(\xi\) be an uncertain variable and let \(b\) be a real number. Then

\[ \{ \xi = b \}^c = \{ \gamma \mid \xi(\gamma) = b \}^c = \{ \gamma \mid \xi(\gamma) \neq b \} = \{ \xi \neq b \}. \]

Thus \(\{ \xi = b \}\) and \(\{ \xi \neq b \}\) are opposite events. Furthermore, by the duality axiom, we obtain

\[ \mathcal{M}\{\xi = b\} + \mathcal{M}\{\xi \neq b\} = 1. \]  

(2.12)
**Exercise 2.1:** Let $\xi$ be an uncertain variable and let $B$ be a Borel set of real numbers. Show that $\{\xi \in B\}$ and $\{\xi \in B^c\}$ are opposite events, and
\[
M\{\xi \in B\} + M\{\xi \in B^c\} = 1.
\] (2.13)

**Exercise 2.2:** Let $\xi$ and $\eta$ be two uncertain variables. Show that $\{\xi \geq \eta\}$ and $\{\xi < \eta\}$ are opposite events, and
\[
M\{\xi \geq \eta\} + M\{\xi < \eta\} = 1.
\] (2.14)

**Definition 2.2** An uncertain variable $\xi$ on the uncertainty space $(\Gamma, L, M)$ is said to be (a) nonnegative if $M\{\xi < 0\} = 0$; and (b) positive if $M\{\xi \leq 0\} = 0$.

**Definition 2.3** Let $\xi$ and $\eta$ be uncertain variables defined on the uncertainty space $(\Gamma, L, M)$. We say $\xi = \eta$ if $\xi(\gamma) = \eta(\gamma)$ for almost all $\gamma \in \Gamma$.

**Definition 2.4** Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables, and let $f$ be a real-valued measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable defined by
\[
\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma.
\] (2.15)

**Example 2.5:** Let $\xi_1$ and $\xi_2$ be two uncertain variables. Then the sum $\xi = \xi_1 + \xi_2$ is an uncertain variable defined by
\[
\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma), \quad \forall \gamma \in \Gamma.
\]
The multiplication $\xi = \xi_1 \xi_2$ is also an uncertain variable defined by
\[
\xi(\gamma) = \xi_1(\gamma) \cdot \xi_2(\gamma), \quad \forall \gamma \in \Gamma.
\]

The reader may wonder whether $\xi(\gamma)$ defined by (2.15) is an uncertain variable. The following theorem answers this question.

**Theorem 2.1** Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables, and let $f$ be a real-valued measurable function. Then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.

**Proof:** Since $\xi_1, \xi_2, \ldots, \xi_n$ are uncertain variables, they are measurable functions from an uncertainty space $(\Gamma, L, M)$ to the set of real numbers. Thus $f(\xi_1, \xi_2, \ldots, \xi_n)$ is also a measurable function from the uncertainty space $(\Gamma, L, M)$ to the set of real numbers. Hence $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.
2.2 Uncertainty Distribution

This section introduces a concept of uncertainty distribution in order to describe uncertain variables. Mention that uncertainty distribution is a carrier of incomplete information of uncertain variable. However, in many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself.

Definition 2.5 (Liu [88]) The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$  \hspace{1cm} (2.16)

for any real number $x$.

![Figure 2.2: An Uncertainty Distribution](image)

Exercise 2.3: A real number $c$ is a special uncertain variable $\xi(\gamma) \equiv c$. Show that such an uncertain variable has an uncertainty distribution

$$\Phi(x) = \begin{cases} 
0, & \text{if } x < c \\
1, & \text{if } x \geq c.
\end{cases}$$

Exercise 2.4: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.7$, $\mathcal{M}\{\gamma_2\} = 0.3$. Show that the uncertain variable

$$\xi(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2
\end{cases}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
0.7, & \text{if } 0 \leq x < 1 \\
1, & \text{if } x \geq 1.
\end{cases}$$
Exercise 2.5: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2, \gamma_3\}\) with power set and \(\mathcal{M}\{\gamma_1\} = 0.6, \mathcal{M}\{\gamma_2\} = 0.3, \mathcal{M}\{\gamma_3\} = 0.2\). Show that the uncertain variable
\[
\xi(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
2, & \text{if } \gamma = \gamma_2 \\
3, & \text{if } \gamma = \gamma_3 
\end{cases}
\]
has an uncertainty distribution
\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 1 \\
0.6, & \text{if } 1 \leq x < 2 \\
0.8, & \text{if } 2 \leq x < 3 \\
1, & \text{if } x \geq 3.
\end{cases}
\]

Exercise 2.6: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. (i) Show that the uncertain variable
\[
\xi(\gamma) = \gamma, \quad \forall \gamma \in [0, 1]
\]
has an uncertainty distribution
\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
x, & \text{if } 0 < x \leq 1 \\
1, & \text{if } x > 1.
\end{cases}
\]
(ii) What is the uncertainty distribution of \(\xi(\gamma) = 1 - \gamma\)? (iii) What do those two uncertain variables make you think about? (iv) Design a third uncertain variable whose uncertainty distribution is also (2.18).

Exercise 2.7: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. (i) Show that the uncertain variable \(\xi(\gamma) = \gamma^2\) has an uncertainty distribution
\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\sqrt{x}, & \text{if } 0 \leq x \leq 1 \\
1, & \text{if } x > 1.
\end{cases}
\]

Exercise 2.8: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. What is the uncertainty distribution of \(\xi(\gamma) = 1/\gamma\)?

Exercise 2.9: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. What is the uncertainty distribution of \(\xi(\gamma) = \ln \gamma\)?
Exercise 2.10: Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \), and let \( a \) and \( b \) be real numbers with \( a > 0 \). Show that \( a\xi + b \) has an uncertainty distribution
\[
\Psi(x) = \Phi\left(\frac{x - b}{a}\right), \quad \forall x \in \mathbb{R}. \tag{2.20}
\]

Exercise 2.11: Let \( \xi \) be an uncertain variable with continuous uncertainty distribution \( \Phi \), and let \( a \) and \( b \) be real numbers with \( a < 0 \). Show that \( a\xi + b \) has an uncertainty distribution
\[
\Psi(x) = 1 - \Phi\left(\frac{x - b}{a}\right), \quad \forall x \in \mathbb{R}. \tag{2.21}
\]

Exercise 2.12: Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). Show that \( \exp(\xi) \) has an uncertainty distribution
\[
\Psi(x) = \Phi(\ln(x)), \quad \forall x > 0. \tag{2.22}
\]

Exercise 2.13: Let \( \xi \) be a positive uncertain variable with continuous uncertainty distribution \( \Phi \). Show that \( \frac{1}{\xi} \) has an uncertainty distribution
\[
\Psi(x) = 1 - \Phi\left(\frac{1}{x}\right), \quad \forall x > 0. \tag{2.23}
\]

Exercise 2.14: Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \), and let \( f \) be a continuous and strictly increasing function. Show that \( f(\xi) \) has an uncertainty distribution
\[
\Psi(x) = \Phi(f^{-1}(x)), \quad \forall x \in \mathbb{R}. \tag{2.24}
\]

Exercise 2.15: Let \( \xi \) be an uncertain variable with continuous uncertainty distribution \( \Phi \), and let \( f \) be a continuous and strictly decreasing function. Show that \( f(\xi) \) has an uncertainty distribution
\[
\Psi(x) = 1 - \Phi(f^{-1}(x)), \quad \forall x \in \mathbb{R}. \tag{2.25}
\]

Definition 2.6 Uncertain variables are said to be identically distributed if they have the same uncertainty distribution.

It is clear that uncertain variables \( \xi \) and \( \eta \) are identically distributed if \( \xi = \eta \). However, identical distribution does not imply \( \xi = \eta \). For example, let \( (\Gamma, \mathcal{L}, \mathcal{M}) \) be \( \{\gamma_1, \gamma_2\} \) with power set and \( \mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5 \). Define
\[
\xi(\gamma) = \begin{cases} 
  1, & \text{if } \gamma = \gamma_1 \\
  -1, & \text{if } \gamma = \gamma_2
\end{cases}, \quad \eta(\gamma) = \begin{cases} 
  -1, & \text{if } \gamma = \gamma_1 \\
  1, & \text{if } \gamma = \gamma_2
\end{cases}.
\]
Then $\xi$ and $\eta$ have the same uncertainty distribution,

$$\Phi(x) = \begin{cases} 
0, & \text{if } x < -1 \\
0.5, & \text{if } -1 \leq x < 1 \\
1, & \text{if } x \geq 1.
\end{cases}$$

Thus the two uncertain variables $\xi$ and $\eta$ are identically distributed but $\xi \neq \eta$.

**Sufficient and Necessary Condition**

**Theorem 2.2** (Peng-Iwamura Theorem [137]) A function $\Phi(x) : \mathbb{R} \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.

**Proof:** It is obvious that an uncertainty distribution $\Phi$ is a monotone increasing function. In addition, both $\Phi(x) \neq 0$ and $\Phi(x) \neq 1$ follow from the asymptotic theorem immediately. Conversely, suppose that $\Phi$ is a monotone increasing function but $\Phi(x) \neq 0$ and $\Phi(x) \neq 1$. We will prove that there is an uncertain variable whose uncertainty distribution is just $\Phi$. Let $\mathcal{C}$ be a collection of all intervals of the form $(-\infty, a]$, $(b, \infty)$, $\emptyset$ and $\mathbb{R}$. We define a set function on $\mathbb{R}$ as follows,

$$
\mathcal{M}\{(\infty, a]\} = \Phi(a), \\
\mathcal{M}\{(b, +\infty]\} = 1 - \Phi(b), \\
\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\mathbb{R}\} = 1.
$$

For an arbitrary Borel set $B$ of real numbers, there exists a sequence $\{A_i\}$ in $\mathcal{C}$ such that

$$B \subset \bigcup_{i=1}^{\infty} A_i.$$ 

Note that such a sequence is not unique. We define a set function $\mathcal{M}\{B\}$ by

$$
\mathcal{M}\{B\} = \begin{cases} 
\inf_{B \subset \bigcup_{i=1}^{\infty} A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf_{B \subset \bigcup_{i=1}^{\infty} A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\
1 - \inf_{B^c \subset \bigcup_{i=1}^{\infty} A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf_{B^c \subset \bigcup_{i=1}^{\infty} A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\
0.5, & \text{otherwise}.
\end{cases}
$$

Then the set function $\mathcal{M}$ is indeed an uncertain measure on $\mathbb{R}$, and the uncertain variable defined by the identity function $\xi(\gamma) = \gamma$ has the uncertainty distribution $\Phi$. 
Example 2.6: A “completely unknown number” may be regarded as an uncertain variable whose uncertainty distribution is

\[ \Phi(x) \equiv 0.5. \]  
(2.26)

It follows from the sufficient and necessary condition that \( \Phi(x) \equiv 0.5 \) is indeed an uncertainty distribution. Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \( \mathbb{R} \) with power set and

\[ \mathcal{M}\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \mathbb{R} \\
0.5, & \text{otherwise.} 
\end{cases} \]  
(2.27)

Then the uncertain variable \( \xi(\gamma) = \gamma \) has the uncertainty distribution (2.26).

Exercise 2.16: (i) Design an uncertain variable whose uncertainty distribution is

\[ \Phi(x) = 0.4 \]  
(2.28)

for any real number \( x \). (ii) Design an uncertain variable whose uncertainty distribution is

\[ \Phi(x) = 0.6 \]  
(2.29)

for any real number \( x \).

Exercise 2.17: Design an uncertain variable whose uncertainty distribution is

\[ \Phi(x) = (1 + \exp(-x))^{-1} \]  
(2.30)

for any real number \( x \).

Some Special Uncertainty Distributions

Definition 2.7 An uncertain variable \( \xi \) is called linear if it has a linear uncertainty distribution

\[ \Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
\frac{x - a}{b - a}, & \text{if } a < x \leq b \\
1, & \text{if } b < x 
\end{cases} \]  
(2.31)

denoted by \( \mathcal{L}(a, b) \) where \( a \) and \( b \) are real numbers with \( a < b \).

Example 2.7: In practice, some quantities are sometimes only given by lower and upper bounds. For example, someone thinks John is neither
younger than 24 nor older than 28. Then John’s age is a linear uncertain variable \( L(24, 28) \) whose uncertainty distribution is

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 24 \\
\frac{x - 24}{4}, & \text{if } 24 < x \leq 28 \\
1, & \text{if } 28 < x.
\end{cases}
\]  

(2.32)

Example 2.8: Someone thinks James’ height is between 180 and 190 centimeters. Then James’ height is a linear uncertain variable \( L(180, 190) \) whose uncertainty distribution is

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 180 \\
\frac{(x - 180)}{10}, & \text{if } 180 < x \leq 190 \\
1, & \text{if } 190 < x.
\end{cases}
\]  

(2.33)

Definition 2.8 An uncertain variable \( \xi \) is called zigzag if it has a zigzag uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
\frac{x - a}{2(b - a)}, & \text{if } a < x \leq b \\
\frac{x + c - 2b}{2(c - b)}, & \text{if } b < x \leq c \\
1, & \text{if } c < x
\end{cases}
\]  

(2.34)

denoted by \( Z(a, b, c) \) where \( a, b, c \) are real numbers with \( a < b < c \).

Example 2.9: If a quantity is only given by median\(^1\), lower and upper

\(^1\)Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi(x) \). The median of \( \xi \) is a point \( x \) at which \( \Phi(x) = 0.5 \). Thus the median may be thought of as the “middle” point. That is, we are 50% sure that the quantity falls into the left side and 50% sure that the quantity falls into the right side of the median.
bounds, then it is of zigzag form. For example, someone thinks James’ height is between 180 and 190 centimeters, and the median is 187 centimeters. Then James’ height is a zigzag uncertain variable \( Z(180, 187, 190) \) whose uncertainty distribution is

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 180 \\
(x - 180)/14, & \text{if } 180 < x \leq 187 \\
(x - 184)/6, & \text{if } 187 < x \leq 190 \\
1, & \text{if } 190 < x. 
\end{cases}
\] (2.35)

**Definition 2.9** An uncertain variable \( \xi \) is called normal if it has a normal uncertainty distribution

\[
\Phi(x) = \left( 1 + \exp \left( \frac{\pi(e - x)}{\sqrt{3}\sigma} \right) \right)^{-1}, \quad x \in \mathbb{R}
\] (2.36)
denoted by \( \mathcal{N}(e, \sigma) \) where \( e \) and \( \sigma \) are real numbers with \( \sigma > 0 \).
**Definition 2.10** An uncertain variable $\xi$ is called lognormal if $\ln \xi$ is a normal uncertain variable $N(e, \sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution

$$
\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \geq 0
$$

denoted by $LOGN(e, \sigma)$, where $e$ and $\sigma$ are real numbers with $\sigma > 0$.

![Lognormal Uncertainty Distribution](image)

**Definition 2.11** An uncertain variable $\xi$ is called empirical if it has an empirical uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq x_1 \\
\alpha_i + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i < x \leq x_{i+1}, 1 \leq i < n \\
1, & \text{if } x_n < x
\end{cases}
$$

where $x_1 < x_2 < \cdots < x_n$ and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1$.

**Measure Inversion Theorem**

**Theorem 2.3** (Liu [95], Measure Inversion Theorem) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then for any real number $x$, we have

$$
M\{\xi \leq x\} = \Phi(x), \quad M\{\xi > x\} = 1 - \Phi(x).
$$

**Proof:** The equation $M\{\xi \leq x\} = \Phi(x)$ follows from the definition of uncertainty distribution immediately. By using the duality of uncertain measure, we get

$$
M\{\xi > x\} = 1 - M\{\xi \leq x\} = 1 - \Phi(x).
$$
Theorem is verified.

**Remark 2.2:** When the uncertainty distribution $\Phi$ is a continuous function, we also have
\[ M\{\xi < x\} = \Phi(x), \quad M\{\xi \geq x\} = 1 - \Phi(x). \quad (2.40) \]

**Remark 2.3:** Generally speaking, it is impossible to get the exact value of $M\{a < \xi \leq b\}$ (except $a = -\infty$ or $b = +\infty$) if only an uncertainty distribution is available. However, the lower and upper bounds are given by the following theorem.

**Theorem 2.4** Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then for any real numbers $a$ and $b$ with $a < b$, we have
\[ \Phi(b) - \Phi(a) \leq M\{a < \xi \leq b\} \leq (1 - \Phi(a)) \wedge \Phi(b). \quad (2.41) \]

**Proof:** Since \( \{\xi \leq b\} = \{\xi \leq a\} \cup \{a < \xi \leq b\} \), it follows from the subadditivity axiom that
\[ M\{\xi \leq b\} \leq M\{\xi \leq a\} + M\{a < \xi \leq b\}. \]
That is,
\[ \Phi(b) \leq \Phi(a) + M\{a < \xi \leq b\}. \]
Thus the left inequality is proved. By using the monotonicity theorem and duality axiom, we have
\[ M\{a < \xi \leq b\} \leq M\{a < \xi\} = 1 - M\{\xi \leq a\} = 1 - \Phi(a) \]
and

\[ M\{a < \xi \leq b\} \leq M\{\xi \leq b\} = \Phi(b). \]

Thus the right inequality is proved.

**Remark 2.4:** It is inappropriate to regard the derivative \( \Phi'(x) \) as an uncertainty density function because uncertain measure is not additive, i.e., generally speaking,

\[ M\{a < \xi \leq b\} \neq \int_a^b \Phi'(x)dx. \quad (2.42) \]

**Regular Uncertainty Distribution**

**Definition 2.12** (Liu [95]) An uncertainty distribution \( \Phi(x) \) is said to be regular if it is a continuous and strictly increasing function with respect to \( x \) at which \( 0 < \Phi(x) < 1 \), and

\[
\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1. \quad (2.43)
\]

For example, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution, and lognormal uncertainty distribution are all regular. However, the uncertainty distribution \( \Phi(x) \equiv 0.5 \) is not regular.

**2.3 Inverse Uncertainty Distribution**

It is clear that a regular uncertainty distribution \( \Phi(x) \) has an inverse function on the range of \( x \) with \( 0 < \Phi(x) < 1 \), and the inverse function \( \Phi^{-1}(\alpha) \) exists on the open interval \((0, 1)\).

**Definition 2.13** (Liu [95]) Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi(x) \). Then the inverse function \( \Phi^{-1}(\alpha) \) is called the inverse uncertainty distribution of \( \xi \).

Note that the inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) is well defined on the open interval \((0, 1)\). If needed, we may extend the domain to \([0, 1]\) via

\[
\Phi^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha). \quad (2.44)
\]

**Example 2.10:** The inverse uncertainty distribution of linear uncertain variable \( L(a, b) \) is

\[
\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b. \quad (2.45)
\]
Example 2.11: The inverse uncertainty distribution of zigzag uncertain variable $Z(a, b, c)$ is
\[
\Phi^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \geq 0.5.
\end{cases}
\] (2.46)

Example 2.12: The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e, \sigma)$ is
\[
\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\] (2.47)

Example 2.13: The inverse uncertainty distribution of lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is
\[
\Phi^{-1}(\alpha) = \exp \left( e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).
\] (2.48)
Theorem 2.5 A continuous and strictly increasing function $\Phi^{-1} : (0, 1) \to \mathbb{R}$ is the inverse uncertainty distribution of an uncertain variable $\xi$ if and only if

$$M\{\xi \leq \Phi^{-1}(\alpha)\} = \alpha \quad (2.49)$$

for all $\alpha \in (0, 1)$.

Proof: Since $\Phi^{-1}$ is a continuous and strictly increasing function, its inverse function $\Phi$ exists. Suppose $\Phi^{-1}$ is the inverse uncertainty distribution of $\xi$. Then for any $\alpha$, we have

$$M\{\xi \leq \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

Conversely, suppose $\Phi^{-1}$ meets (2.49). Write $x = \Phi^{-1}(\alpha)$. Then $\alpha = \Phi(x)$ and

$$M\{\xi \leq x\} = \alpha = \Phi(x).$$

That is, $\Phi$ is the uncertainty distribution of $\xi$ and $\Phi^{-1}$ is its inverse uncertainty distribution. The theorem is verified.
**Exercise 2.18:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $a$ and $b$ be real numbers with $a > 0$. Show that $a\xi + b$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = a\Phi^{-1}(\alpha) + b.$$  \hspace{1cm} (2.50)

**Exercise 2.19:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $a$ and $b$ be real numbers with $a < 0$. Show that $a\xi + b$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = a\Phi^{-1}(1 - \alpha) + b.$$  \hspace{1cm} (2.51)

**Exercise 2.20:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Show that $\exp(\xi)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \exp(\Phi^{-1}(\alpha)).$$  \hspace{1cm} (2.52)

**Exercise 2.21:** Let $\xi$ be a positive uncertain variable with regular uncertainty distribution $\Phi$. Show that the reciprocal $1/\xi$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1 - \alpha)}.$$  \hspace{1cm} (2.53)

**Exercise 2.22:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $f$ be a continuous and strictly increasing function. Show that $f(\xi)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha)).$$  \hspace{1cm} (2.54)

**Exercise 2.23:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $f$ be a continuous and strictly decreasing function. Show that $f(\xi)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(1 - \alpha)).$$  \hspace{1cm} (2.55)

**Theorem 2.6** Let $\xi$ be an uncertain variable with inverse uncertainty distribution $\Phi^{-1}(\alpha)$. Then

$$\mathbb{M}\{\xi \leq c\} \geq \alpha$$

if and only if

$$\Phi^{-1}(\alpha) \leq c$$

where $\alpha$ and $c$ are constants with $0 < \alpha < 1$.  \hspace{1cm} (2.56)
**Proof:** It follows from $M\{\xi \leq c\} = \Phi(c)$ that $M\{\xi \leq c\} \geq \alpha$ if and only if $\Phi(c) \geq \alpha$, i.e., $\Phi^{-1}(\alpha) \leq c$. The theorem is thus proved.

**Exercise 2.24:** Let $\xi$ be an uncertain variable with inverse uncertainty distribution $\Phi^{-1}(\alpha)$. Show that

$$M\{\xi \geq c\} \geq \alpha$$  \hspace{1cm} (2.58)  

if and only if  

$$\Phi^{-1}(1 - \alpha) \geq c$$  \hspace{1cm} (2.59)  

where $\alpha$ and $c$ are constants with $0 < \alpha < 1$.

**Theorem 2.7 (Liu [100], Sufficient and Necessary Condition)** A function $\Phi^{-1}(\alpha) : (0,1) \to \mathbb{R}$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to $\alpha$.

**Proof:** Suppose $\Phi^{-1}(\alpha)$ is an inverse uncertainty distribution. It follows from the definition of inverse uncertainty distribution that $\Phi^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha \in (0,1)$.

Conversely, suppose $\Phi^{-1}(\alpha)$ is a continuous and strictly increasing function on $(0,1)$. Define

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha) \\ \alpha, & \text{if } x = \Phi^{-1}(\alpha) \\ 1, & \text{if } x \geq \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha). \end{cases}$$

It follows from Peng-Iwamura theorem that $\Phi(x)$ is an uncertainty distribution of some uncertain variable $\xi$. Then for each $\alpha \in (0,1)$, we have

$$M\{\xi \leq \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

Thus $\Phi^{-1}(\alpha)$ is just the inverse uncertainty distribution of the uncertain variable $\xi$. The theorem is verified.

### 2.4 Independence

Note that an uncertain variable is a measurable function from an uncertainty space to the set of real numbers. The independence of two functions means that knowing the value of one does not change our estimation of the value of another. What uncertain variables meet this condition? A typical case is that they are defined on different uncertainty spaces. For example, let $\xi_1(\gamma_1)$

---

2For example, in a rectangular coordinate system $(x, y, z)$, it is clear that $z = f(x)$ and $z = g(y)$ are always independent for any univariate functions $f$ and $g$. However, $z = x + 1$ and $z = x - y$ are not.
and $\xi_2(\gamma_2)$ be uncertain variables on the uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively. It is clear that they are also uncertain variables on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$. Then for any Borel sets $B_1$ and $B_2$ of real numbers, we have

$$M\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\}$$

$$= M\{(\gamma_1, \gamma_2) | \xi_1(\gamma_1) \in B_1, \xi_2(\gamma_2) \in B_2\}$$

$$= M\{(\gamma_1 | \xi_1(\gamma_1) \in B_1) \times (\gamma_2 | \xi_2(\gamma_2) \in B_2)\}$$

$$= M_1\{\gamma_1 | \xi_1(\gamma_1) \in B_1\} \wedge M_2\{\gamma_2 | \xi_2(\gamma_2) \in B_2\}$$

That is,

$$M\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\} = M\{\xi_1 \in B_1\} \wedge M\{\xi_2 \in B_2\}. \quad (2.60)$$

Thus we say two uncertain variables are independent if the equation (2.60) holds. Generally, we may define independence in the following form.

**Definition 2.14** (Liu [91]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\} \quad (2.61)$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

**Exercise 2.25:** Show that a constant (a special uncertain variable) is always independent of any uncertain variable.

**Exercise 2.26:** John gives Tom 2 dollars. Thus John gets “−2 dollars” and Tom “+2 dollars”. Are John’s “−2 dollars” and Tom’s “+2 dollars” independent? Why?

**Exercise 2.27:** Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables. Show that $\xi_i$ and $\xi_j$ are independent for any indexes $i$ and $j$ with $1 \leq i < j \leq n$.

**Exercise 2.28:** Let $\xi$ be an uncertain variable. Are $\xi$ and $1-\xi$ independent? Please justify your answer.

**Exercise 2.29:** Construct $n$ independent uncertain variables. (Hint: Define them on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2) \times \cdots \times (\Gamma_n, \mathcal{L}_n, \mathcal{M}_n)$.)

**Theorem 2.8** (Liu [91]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if

$$M\left\{\bigcup_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \bigvee_{i=1}^{n} M\{\xi_i \in B_i\} \quad (2.62)$$
for any Borel sets \( B_1, B_2, \cdots, B_n \) of real numbers.

**Proof:** It follows from the duality of uncertain measure that \( \xi_1, \xi_2, \cdots, \xi_n \) are independent if and only if

\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} (\xi_i \in B_i) \right\} = 1 - \mathcal{M}\left\{ \bigcup_{i=1}^{n} (\xi_i \in B_i^c) \right\}
\]

\[
= 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}.
\]

Thus the proof is complete.

**Theorem 2.9** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables, and let \( f_1, f_2, \cdots, f_n \) be measurable functions. Then \( f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n) \) are independent uncertain variables.

**Proof:** For any Borel sets \( B_1, B_2, \cdots, B_n \) of real numbers, it follows from the definition of independence that

\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} (f_i(\xi_i) \in B_i) \right\} = \mathcal{M}\left\{ \bigcap_{i=1}^{n} (\xi_i \in f_i^{-1}(B_i)) \right\}
\]

\[
= \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in f_i^{-1}(B_i)\} = \bigvee_{i=1}^{n} \mathcal{M}\{f_i(\xi_i) \in B_i\}.
\]

Thus \( f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n) \) are independent uncertain variables.

### 2.5 Operational Law: Inverse Distribution

This section provides some operational laws for calculating the inverse uncertainty distributions of strictly increasing function, strictly decreasing function, and strictly monotone function of uncertain variables.

**Strictly Increasing Function of Uncertain Variables**

A real-valued function \( f(x_1, x_2, \cdots, x_n) \) is said to be strictly increasing if

\[
f(x_1, x_2, \cdots, x_n) \leq f(y_1, y_2, \cdots, y_n)
\]

whenever \( x_i \leq y_i \) for \( i = 1, 2, \cdots, n \), and

\[
f(x_1, x_2, \cdots, x_n) < f(y_1, y_2, \cdots, y_n)
\]
whenever \( x_i < y_i \) for \( i = 1, 2, \ldots, n \). The following are strictly increasing functions,

\[
f(x_1, x_2, \ldots, x_n) = x_1 \lor x_2 \lor \cdots \lor x_n, \\
f(x_1, x_2, \ldots, x_n) = x_1 \land x_2 \land \cdots \land x_n, \\
f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n, \\
f(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n, \quad x_1, x_2, \ldots, x_n \geq 0.
\]

**Theorem 2.10** (Liu [95]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f \) is a continuous and strictly increasing function, then

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{2.65}
\]

has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)). \tag{2.66}
\]

**Proof:** For simplicity, we only prove the case \( n = 2 \). At first, it is clear that \( f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha)) \) is a continuous and strictly increasing function with respect to \( \alpha \). Next, we always have

\[
\{\xi \leq \Psi^{-1}(\alpha)\} = \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha))\}.
\]

On the one hand, since \( f \) is a strictly increasing function, we obtain

\[
\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.
\]

By using the independence of \( \xi_1 \) and \( \xi_2 \), we get

\[
\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \geq \mathcal{M}\{(\xi_1 \leq \Phi_1^{-1}(\alpha)) \cap (\xi_2 \leq \Phi_2^{-1}(\alpha))\}
\]
\[
= \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \land \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\}
\]
\[
= \alpha \land \alpha = \alpha.
\]

On the other hand, since \( f \) is a strictly increasing function, we obtain

\[
\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.
\]

By using the independence of \( \xi_1 \) and \( \xi_2 \), we get

\[
\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \leq \mathcal{M}\{(\xi_1 \leq \Phi_1^{-1}(\alpha)) \cup (\xi_2 \leq \Phi_2^{-1}(\alpha))\}
\]
\[
= \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \lor \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\}
\]
\[
= \alpha \lor \alpha = \alpha.
\]

It follows that \( \mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha \). That is, \( \Psi^{-1} \) is just the inverse uncertainty distribution of \( \xi \). The theorem is proved.
Exercise 2.30: Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Show that the sum
\[
\xi = \xi_1 + \xi_2 + \cdots + \xi_n
\]  
has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha).
\]

Exercise 2.31: Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent and positive uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Show that the multiplication
\[
\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n
\]  
has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha) \times \cdots \times \Phi_n^{-1}(\alpha).
\]

Exercise 2.32: Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Show that the minimum
\[
\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n
\]  
has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \cdots \wedge \Phi_n^{-1}(\alpha).
\]

Exercise 2.33: Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Show that the maximum
\[
\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n
\]  
has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \vee \Phi_2^{-1}(\alpha) \vee \cdots \vee \Phi_n^{-1}(\alpha).
\]

Example 2.14: The independence condition in Theorem 2.10 cannot be removed. For example, take an uncertainty space \( (\Gamma, \mathcal{L}, M) \) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then \( \xi_1(\gamma) = \gamma \) is a linear uncertain variable with inverse uncertainty distribution
\[
\Phi_1^{-1}(\alpha) = \alpha,
\]
and \( \xi_2(\gamma) = 1 - \gamma \) is also a linear uncertain variable with inverse uncertainty distribution
\[
\Phi_2^{-1}(\alpha) = \alpha.
\]
Note that \( \xi_1 \) and \( \xi_2 \) are not independent, and \( \xi_1 + \xi_2 \equiv 1 \) whose inverse uncertainty distribution is \( \Psi^{-1}(\alpha) \equiv 1 \). Thus

\[
\Psi^{-1}(\alpha) \neq \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha). \tag{2.77}
\]

Therefore, the independence condition cannot be removed.

**Theorem 2.11** Assume that \( \xi_1 \) and \( \xi_2 \) are independent linear uncertain variables \( \mathcal{L}(a_1, b_1) \) and \( \mathcal{L}(a_2, b_2) \), respectively. Then the sum \( \xi_1 + \xi_2 \) is also a linear uncertain variable \( \mathcal{L}(a_1 + a_2, b_1 + b_2) \), i.e.,

\[
\mathcal{L}(a_1, b_1) + \mathcal{L}(a_2, b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2). \tag{2.78}
\]

The multiplication of a linear uncertain variable \( \mathcal{L}(a, b) \) and a scalar number \( k > 0 \) is also a linear uncertain variable \( \mathcal{L}(ka, kb) \), i.e.,

\[
k \cdot \mathcal{L}(a, b) = \mathcal{L}(ka, kb). \tag{2.79}
\]

**Proof:** Assume that the uncertain variables \( \xi_1 \) and \( \xi_2 \) have uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Then

\[
\Phi_1^{-1}(\alpha) = (1 - \alpha)a_1 + \alpha b_1,
\]

\[
\Phi_2^{-1}(\alpha) = (1 - \alpha)a_2 + \alpha b_2.
\]

It follows from the operational law that the inverse uncertainty distribution of \( \xi_1 + \xi_2 \) is

\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (1 - \alpha)(a_1 + a_2) + \alpha(b_1 + b_2).
\]

Hence the sum is also a linear uncertain variable \( \mathcal{L}(a_1 + a_2, b_1 + b_2) \). The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable \( \xi \sim \mathcal{L}(a, b) \) is \( \Phi \). It follows from the operational law that when \( k > 0 \), the inverse uncertainty distribution of \( k\xi \) is

\[
\Psi^{-1}(\alpha) = k \Phi^{-1}(\alpha) = (1 - \alpha)(ka) + \alpha(kb).
\]

Hence \( k\xi \) is just a linear uncertain variable \( \mathcal{L}(ka, kb) \).

**Exercise 2.34:** Show that the multiplication of linear uncertain variables is no longer a linear one even they are independent and positive. That is,

\[
\mathcal{L}(a_1, b_1) \times \mathcal{L}(a_2, b_2) \neq \mathcal{L}(a_1 \times a_2, b_1 \times b_2). \tag{2.80}
\]

**Exercise 2.35:** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent linear uncertain variables on \([0, 1]\). Show that the inverse uncertainty distribution of \( \xi_1 \times \xi_2 \times \cdots \times \xi_n \) is

\[
\Psi^{-1}(\alpha) = \alpha^n. \tag{2.81}
\]

What is it if \( n \to \infty \)?
Theorem 2.12 Assume that $\xi_1$ and $\xi_2$ are independent zigzag uncertain variables $\mathcal{Z}(a_1, b_1, c_1)$ and $\mathcal{Z}(a_2, b_2, c_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$, i.e.,

$$\mathcal{Z}(a_1, b_1, c_1) + \mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2). \quad (2.82)$$

The multiplication of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ and a scalar number $k > 0$ is also a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$, i.e.,

$$k \cdot \mathcal{Z}(a, b, c) = \mathcal{Z}(ka, kb, kc). \quad (2.83)$$

Proof: Assume that the uncertain variables $\xi_1$ and $\xi_2$ have uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Then

$$\Phi_1^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a_1 + 2\alpha b_1, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b_1 + (2\alpha - 1)c_1, & \text{if } \alpha \geq 0.5, \end{cases}$$

$$\Phi_2^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a_2 + 2\alpha b_2, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b_2 + (2\alpha - 1)c_2, & \text{if } \alpha \geq 0.5. \end{cases}$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(a_1 + a_2) + 2\alpha(b_1 + b_2), & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)(b_1 + b_2) + (2\alpha - 1)(c_1 + c_2), & \text{if } \alpha \geq 0.5. \end{cases}$$

Hence the sum is also a zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ is $\Phi$. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(ka) + 2\alpha(kb), & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)(kb) + (2\alpha - 1)(kc), & \text{if } \alpha \geq 0.5. \end{cases}$$

Hence $k\xi$ is just a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$.

Theorem 2.13 Let $\xi_1$ and $\xi_2$ be independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1)$ and $\mathcal{N}(e_2, \sigma_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,

$$\mathcal{N}(e_1, \sigma_1) + \mathcal{N}(e_2, \sigma_2) = \mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2). \quad (2.84)$$

The multiplication of a normal uncertain variable $\mathcal{N}(e, \sigma)$ and a scalar number $k > 0$ is also a normal uncertain variable $\mathcal{N}(ke, k\sigma)$, i.e.,

$$k \cdot \mathcal{N}(e, \sigma) = \mathcal{N}(ke, k\sigma). \quad (2.85)$$
Proof: Assume that the uncertain variables $\xi_1$ and $\xi_2$ have uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Then

$$\Phi_1^{-1}(\alpha) = e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha},$$

$$\Phi_2^{-1}(\alpha) = e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$ 

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (e_1 + e_2) + \left(\frac{\sigma_1 + \sigma_2}{\pi}\right) \sqrt{\frac{3}{\pi}} \ln \frac{\alpha}{1 - \alpha}.$$ 

Hence the sum is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ is $\Phi$. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (ke) + \left(\frac{k\sigma}{\pi}\right) \sqrt{\frac{3}{\pi}} \ln \frac{\alpha}{1 - \alpha}.$$ 

Hence $k\xi$ is just a normal uncertain variable $\mathcal{N}(ke, k\sigma)$.

**Theorem 2.14** Assume that $\xi_1$ and $\xi_2$ are independent lognormal uncertain variables $\mathcal{LOGN}(e_1, \sigma_1)$ and $\mathcal{LOGN}(e_2, \sigma_2)$, respectively. Then the multiplication $\xi_1 \cdot \xi_2$ is also a lognormal uncertain variable $\mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,

$$\mathcal{LOGN}(e_1, \sigma_1) \cdot \mathcal{LOGN}(e_2, \sigma_2) = \mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2).$$ 

(2.86)

The multiplication of a lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ and a scalar number $k > 0$ is also a lognormal uncertain variable $\mathcal{LOGN}(e + \ln k, \sigma)$, i.e.,

$$k \cdot \mathcal{LOGN}(e, \sigma) = \mathcal{LOGN}(e + \ln k, \sigma).$$ 

(2.87)

**Proof:** Assume that the uncertain variables $\xi_1$ and $\xi_2$ have uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Then

$$\Phi_1^{-1}(\alpha) = \exp \left( e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right),$$

$$\Phi_2^{-1}(\alpha) = \exp \left( e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).$$ 

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 \cdot \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \cdot \Phi_2^{-1}(\alpha) = \exp \left( (e_1 + e_2) + \left(\frac{\sigma_1 + \sigma_2}{\pi}\right) \sqrt{\frac{3}{\pi}} \ln \frac{\alpha}{1 - \alpha} \right).$$
Hence the multiplication is a lognormal uncertain variable $\mathcal{LOGN}(e_1+e_2, \sigma_1+\sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{LOGN}(e, \sigma)$ is $\Phi$. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$
\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \exp \left( (e + \ln k) + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right).
$$

Hence $k\xi$ is just a lognormal uncertain variable $\mathcal{LOGN}(e + \ln k, \sigma)$.

**Remark 2.5:** Keep in mind that the sum of lognormal uncertain variables is no longer lognormal.

### Strictly Decreasing Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \ldots, x_n)$ is said to be strictly decreasing if

$$f(x_1, x_2, \ldots, x_n) \geq f(y_1, y_2, \ldots, y_n)$$

whenever $x_i \leq y_i$ for $i = 1, 2, \ldots, n$, and

$$f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n)$$

whenever $x_i < y_i$ for $i = 1, 2, \ldots, n$. If $f(x_1, x_2, \ldots, x_n)$ is a strictly increasing function, then

$$-f(x_1, x_2, \ldots, x_n)$$

is a strictly decreasing function, and

$$\frac{1}{f(x_1, x_2, \ldots, x_n)}$$

is also a strictly decreasing function provided that $f$ is positive.

**Theorem 2.15** (Liu [95]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is a continuous and strictly decreasing function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

**Proof:** For simplicity, we only prove the case $n = 2$. At first, it is clear that $f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha))$ is a continuous and strictly increasing function with respect to $\alpha$. Next, we always have

$$\{\xi \leq \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha))\}.$$
On the one hand, since $f$ is a strictly decreasing function, we obtain
\[ \{ \xi \leq \Psi^{-1}(\alpha) \} \supset \{ \xi_1 \geq \Phi_1^{-1}(1-\alpha) \} \cap \{ \xi_2 \geq \Phi_2^{-1}(1-\alpha) \}. \]

By using the independence of $\xi_1$ and $\xi_2$, we get
\[
M \{ \xi \leq \Psi^{-1}(\alpha) \} \geq M \{ (\xi_1 \geq \Phi_1^{-1}(1-\alpha)) \cap (\xi_2 \geq \Phi_2^{-1}(1-\alpha)) \}
\]
\[ = M \{ \xi_1 \geq \Phi_1^{-1}(1-\alpha) \} \land M \{ \xi_2 \geq \Phi_2^{-1}(1-\alpha) \}
\]
\[ = \alpha \land \alpha = \alpha. \]

On the other hand, since $f$ is a strictly decreasing function, we obtain
\[ \{ \xi \leq \Psi^{-1}(\alpha) \} \subset \{ \xi_1 \geq \Phi_1^{-1}(1-\alpha) \} \cup \{ \xi_2 \geq \Phi_2^{-1}(1-\alpha) \}. \]

By using the independence of $\xi_1$ and $\xi_2$, we get
\[
M \{ \xi \leq \Psi^{-1}(\alpha) \} \leq M \{ (\xi_1 \geq \Phi_1^{-1}(1-\alpha)) \cup (\xi_2 \geq \Phi_2^{-1}(1-\alpha)) \}
\]
\[ = M \{ \xi_1 \geq \Phi_1^{-1}(1-\alpha) \} \lor M \{ \xi_2 \geq \Phi_2^{-1}(1-\alpha) \}
\]
\[ = \alpha \lor \alpha = \alpha. \]

It follows that $M \{ \xi \leq \Psi^{-1}(\alpha) \} = \alpha$. That is, $\Psi^{-1}$ is just the inverse uncertainty distribution of $\xi$. The theorem is proved.

**Exercise 2.36:** Let $\xi_1$ and $\xi_2$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Show that
\[ \xi = \frac{1}{\xi_1 + \xi_2} \] (2.92)

has an inverse uncertainty distribution
\[ \Psi^{-1}(\alpha) = \frac{1}{\Phi_1^{-1}(1-\alpha) + \Phi_2^{-1}(1-\alpha)}. \] (2.93)

**Exercise 2.37:** Show that the independence condition in Theorem 2.15 cannot be removed.

**Strictly Monotone Function of Uncertain Variables**

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly monotone if it is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, that is,
\[ f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \leq f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \] (2.94)

whenever $x_i \leq y_i$ for $i = 1, 2, \cdots, m$ and $x_i \geq y_i$ for $i = m+1, m+2, \cdots, n$, and
\[ f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) < f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \] (2.95)
whenever \( x_i < y_i \) for \( i = 1, 2, \ldots, m \) and \( x_i > y_i \) for \( i = m + 1, m + 2, \ldots, n \). The following are strictly monotone functions,

\[
\begin{align*}
  f(x_1, x_2) &= x_1 - x_2, \\
  f(x_1, x_2) &= x_1/x_2, \quad x_1, x_2 > 0, \\
  f(x_1, x_2) &= x_1/(x_1 + x_2), \quad x_1, x_2 > 0.
\end{align*}
\]

Note that both strictly increasing function and strictly decreasing function are special cases of strictly monotone function.

**Theorem 2.16** (Liu [95]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f(\xi_1, \xi_2, \ldots, \xi_n) \) is continuous, strictly increasing with respect to \( \xi_1, \xi_2, \ldots, \xi_m \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \ldots, \xi_n \), then

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]

has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)). \tag{2.97}
\]

**Proof:** We only prove the case of \( m = 1 \) and \( n = 2 \). At first, it is clear that \( f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha)) \) is a continuous and strictly increasing function with respect to \( \alpha \). Next, we always have

\[
\{\xi \leq \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\}.
\]

On the one hand, since the function \( f(x_1, x_2) \) is strictly increasing with respect to \( x_1 \) and strictly decreasing with \( x_2 \), we obtain

\[
\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}.
\]

By using the independence of \( \xi_1 \) and \( \xi_2 \), we get

\[
M\{\xi \leq \Psi^{-1}(\alpha)\} \geq M\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}
= M\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \land M\{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}
= \alpha \land \alpha = \alpha.
\]

On the other hand, since the function \( f(x_1, x_2) \) is strictly increasing with respect to \( x_1 \) and strictly decreasing with \( x_2 \), we obtain

\[
\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}.
\]

By using the independence of \( \xi_1 \) and \( \xi_2 \), we get

\[
M\{\xi \leq \Psi^{-1}(\alpha)\} \leq M\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}
= M\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \lor M\{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}
= \alpha \lor \alpha = \alpha.
\]
It follows that \( M\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha \). That is, \( \Psi^{-1} \) is just the inverse uncertainty distribution of \( \xi \). The theorem is proved.

**Exercise 2.38:** Let \( \xi_1 \) and \( \xi_2 \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Show that the inverse uncertainty distribution of the difference \( \xi_1 - \xi_2 \) is
\[
\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1 - \alpha).
\] (2.98)

**Exercise 2.39:** Let \( \xi_1 \) and \( \xi_2 \) be independent linear uncertain variables \( L(a_1, b_1) \) and \( L(a_2, b_2) \), respectively. Show that the difference \( \xi_1 - \xi_2 \) is also a linear uncertain variable \( L(a_1 - b_2, b_1 - a_2) \), i.e.,
\[
L(a_1, b_1) - L(a_2, b_2) = L(a_1 - b_2, b_1 - a_2).
\] (2.99)

**Exercise 2.40:** Let \( \xi_1 \) and \( \xi_2 \) be independent zigzag uncertain variables \( Z(a_1, b_1, c_1) \) and \( Z(a_2, b_2, c_2) \), respectively. Show that the difference \( \xi_1 - \xi_2 \) is also a zigzag uncertain variable \( Z(a_1 - c_2, b_1 - b_2, c_1 - a_2) \), i.e.,
\[
Z(a_1, b_1, c_1) - Z(a_2, b_2, c_2) = Z(a_1 - c_2, b_1 - b_2, c_1 - a_2).
\] (2.100)

**Exercise 2.41:** Let \( \xi_1 \) and \( \xi_2 \) be independent normal uncertain variables \( N(e_1, \sigma_1) \) and \( N(e_2, \sigma_2) \), respectively. Show that the difference \( \xi_1 - \xi_2 \) is also a normal uncertain variable \( N(e_1 - e_2, \sigma_1 + \sigma_2) \), i.e.,
\[
N(e_1, \sigma_1) - N(e_2, \sigma_2) = N(e_1 - e_2, \sigma_1 + \sigma_2).
\] (2.101)

**Exercise 2.42:** Let \( \xi_1 \) and \( \xi_2 \) be independent and positive uncertain variables with regular uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Show that the inverse uncertainty distribution of the quotient \( \xi_1/\xi_2 \) is
\[
\Psi^{-1}(\alpha) = \frac{\Phi_1^{-1}(\alpha)}{\Phi_2^{-1}(1 - \alpha)}.
\] (2.102)

**Exercise 2.43:** Assume \( \xi_1 \) and \( \xi_2 \) are independent and positive uncertain variables with regular uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Show that the inverse uncertainty distribution of \( \xi_1/(\xi_1 + \xi_2) \) is
\[
\Psi^{-1}(\alpha) = \frac{\Phi_1^{-1}(\alpha)}{\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(1 - \alpha)}.
\] (2.103)

**Exercise 2.44:** Show that the independence condition in Theorem 2.16 cannot be removed.
2.6 Operational Law: Distribution

This section will give some operational laws for calculating the uncertainty distributions of strictly increasing function, strictly decreasing function, and strictly monotone function of uncertain variables.

Strictly Increasing Function of Uncertain Variables

**Theorem 2.17** (Liu [95]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f \) is a continuous and strictly increasing function, then

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]  

(2.104)

has an uncertainty distribution

\[
\Psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1 \leq i \leq n} \Phi_i(x_i).
\]  

(2.105)

**Proof:** For simplicity, we only prove the case \( n = 2 \). Since \( f \) is a continuous and strictly increasing function, it holds that

\[
\{f(\xi_1,\xi_2) \leq x\} = \bigcup_{f(x_1,x_2) = x} (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2).
\]

Thus the uncertainty distribution is

\[
\Psi(x) = \mathcal{M}\{f(\xi_1,\xi_2) \leq x\} = \mathcal{M}\left\{ \bigcup_{f(x_1,x_2) = x} (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2) \right\}.
\]

Note that for each given number \( x \), the event

\[
\bigcup_{f(x_1,x_2) = x} (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2)
\]

is just a polyrectangle. It follows from the polyrectangular theorem that

\[
\Psi(x) = \sup_{f(x_1,x_2) = x} \mathcal{M}\{ (\xi_1 \leq x_1) \cap (\xi_2 \leq x_2) \}
\]

\[
= \sup_{f(x_1,x_2) = x} \mathcal{M}\{\xi_1 \leq x_1\} \land \mathcal{M}\{\xi_2 \leq x_2\}
\]

\[
= \sup_{f(x_1,x_2) = x} \Phi_1(x_1) \land \Phi_2(x_2).
\]

The theorem is proved.

**Remark 2.6:** It is possible that the equation \( f(x_1, x_2, \ldots, x_n) = x \) does not have a root for some values of \( x \). In this case, if

\[
f(x_1, x_2, \ldots, x_n) < x
\]  

(2.106)
for any vector \((x_1, x_2, \cdots, x_n)\), then we set \(\Psi(x) = 1\); and if
\[
f(x_1, x_2, \cdots, x_n) > x
\]
for any vector \((x_1, x_2, \cdots, x_n)\), then we set \(\Psi(x) = 0\).

**Exercise 2.45:** Let \(\xi_1, \xi_2, \cdots, \xi_n\) be iid uncertain variables with a common uncertainty distribution \(\Phi\). Show that the sum
\[
\xi = \xi_1 + \xi_2 + \cdots + \xi_n
\]
has an uncertainty distribution
\[
\Psi(x) = \Phi \left( \frac{x}{n} \right).
\]

**Exercise 2.46:** Let \(\xi_1, \xi_2, \cdots, \xi_n\) be iid and positive uncertain variables with a common uncertainty distribution \(\Phi\). Show that the multiplication
\[
\xi = \xi_1 \xi_2 \cdots \xi_n
\]
has an uncertainty distribution
\[
\Psi(x) = \Phi \left( \sqrt[n]{x} \right).
\]

**Exercise 2.47:** Let \(\xi_1, \xi_2, \cdots, \xi_n\) be independent uncertain variables with uncertainty distributions \(\Phi_1, \Phi_2, \cdots, \Phi_n\), respectively. Show that the minimum
\[
\xi = \xi_1 \land \xi_2 \land \cdots \land \xi_n
\]
has an uncertainty distribution
\[
\Psi(x) = \Phi_1(x) \lor \Phi_2(x) \lor \cdots \lor \Phi_n(x).
\]

**Exercise 2.48:** Let \(\xi_1, \xi_2, \cdots, \xi_n\) be independent uncertain variables with uncertainty distributions \(\Phi_1, \Phi_2, \cdots, \Phi_n\), respectively. Show that the maximum
\[
\xi = \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n
\]
has an uncertainty distribution
\[
\Psi(x) = \Phi_1(x) \land \Phi_2(x) \land \cdots \land \Phi_n(x).
\]

**Example 2.15:** The independence condition in Theorem 2.17 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with
Borel algebra and Lebesgue measure. Then \( \xi_1(\gamma) = \gamma \) is a linear uncertain variable with uncertainty distribution

\[
\Phi_1(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
x, & \text{if } 0 < x \leq 1 \\
1, & \text{if } x > 1,
\end{cases}
\]  

(2.116)

and \( \xi_2(\gamma) = 1 - \gamma \) is also a linear uncertain variable with uncertainty distribution

\[
\Phi_2(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
x, & \text{if } 0 < x \leq 1 \\
1, & \text{if } x > 1.
\end{cases}
\]  

(2.117)

Note that \( \xi_1 \) and \( \xi_2 \) are not independent, and \( \xi_1 + \xi_2 \equiv 1 \) whose uncertainty distribution is

\[
\Psi(x) = \begin{cases} 
0, & \text{if } x < 1 \\
1, & \text{if } x \geq 1.
\end{cases}
\]  

(2.118)

Thus

\[
\Psi(x) \neq \sup_{x_1 + x_2 = x} \Phi_1(x_1) \wedge \Phi_2(x_2).
\]  

(2.119)

Therefore, the independence condition cannot be removed.

**Definition 2.15** (Gao-Gao-Yang [54], Order Statistic) Let \( \xi_1, \xi_2, \cdots, \xi_n \) be uncertain variables, and let \( k \) be an index with \( 1 \leq k \leq n \). Then

\[
\xi = k\text{-min}[\xi_1, \xi_2, \cdots, \xi_n]
\]  

(2.120)

is called the \( k \)th order statistic of \( \xi_1, \xi_2, \cdots, \xi_n \), where \( k\text{-min} \) represents the \( k \)th smallest value.

**Theorem 2.18** (Gao-Gao-Yang [54]) Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Then the \( k \)th order statistic of \( \xi_1, \xi_2, \cdots, \xi_n \) has an uncertainty distribution

\[
\Psi(x) = k\text{-max}[\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)]
\]  

(2.121)

where \( k\text{-max} \) represents the \( k \)th largest value.

**Proof:** Since \( f(x_1, x_2, \cdots, x_n) = k\text{-min}[x_1, x_2, \cdots, x_n] \) is a strictly increasing function, it follows from Theorem 2.17 that the \( k \)th order statistic has an uncertainty distribution

\[
\Psi(x) = \sup_{k\text{-min}[x_1, x_2, \cdots, x_n] = x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_n(x_n)
\]

\[= k\text{-max}[\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)]. \]

The theorem is proved.
Exercise 2.49: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Then

$$\xi = k \text{-max}[\xi_1, \xi_2, \ldots, \xi_n]$$

is just the $(n - k + 1)$th order statistic. Show that $\xi$ has an uncertainty distribution

$$\Psi(x) = k \text{-min}[\Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x)].$$

\[2.123\]

Theorem 2.19 (Liu [101], Extreme Value Theorem) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables. Assume that

$$S_i = \xi_1 + \xi_2 + \cdots + \xi_i$$

have uncertainty distributions $\Psi_i$ for $i = 1, 2, \ldots, n$, respectively. Then the maximum

$$S = S_1 \lor S_2 \lor \cdots \lor S_n$$

has an uncertainty distribution

$$\Upsilon(x) = \Psi_1(x) \land \Psi_2(x) \land \cdots \land \Psi_n(x);$$

\[2.126\]

and the minimum

$$S = S_1 \land S_2 \land \cdots \land S_n$$

has an uncertainty distribution

$$\Upsilon(x) = \Psi_1(x) \lor \Psi_2(x) \lor \cdots \lor \Psi_n(x).$$

\[2.128\]

Proof: Assume that the uncertainty distributions of the uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. It follows from Theorem 2.17 that

$$\Psi_i(x) = \sup_{x_1 + x_2 + \cdots + x_i = x} \Phi_1(x_1) \land \Phi_2(x_2) \land \cdots \land \Phi_i(x_i)$$

for $i = 1, 2, \ldots, n$. Define

$$f(x_1, x_2, \cdots, x_n) = x_1 \lor (x_1 + x_2) \lor \cdots \lor (x_1 + x_2 + \cdots + x_n).$$

Then $f$ is a strictly increasing function and

$$S = f(\xi_1, \xi_2, \ldots, \xi_n).$$

It follows from Theorem 2.17 that $S$ has an uncertainty distribution

$$\Upsilon(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \Phi_1(x_1) \land \Phi_2(x_2) \land \cdots \land \Phi_n(x_n)$$

$$= \min_{1 \leq i \leq n} \sup_{x_1 + x_2 + \cdots + x_i = x} \Phi_1(x_1) \land \Phi_2(x_2) \land \cdots \land \Phi_i(x_i)$$

$$= \min_{1 \leq i \leq n} \Psi_i(x).$$
Thus (2.126) is verified. Similarly, define
\[ f(x_1, x_2, \ldots, x_n) = x_1 \land (x_1 + x_2) \land \cdots \land (x_1 + x_2 + \cdots + x_n). \]
Then \( f \) is a strictly increasing function and
\[ S = f(\xi_1, \xi_2, \cdots, \xi_n). \]
It follows from Theorem 2.17 that \( S \) has an uncertainty distribution
\[ \Upsilon(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \Phi_1(x_1) \land \Phi_2(x_2) \land \cdots \land \Phi_n(x_n) \]
\[ = \max_{1 \leq i \leq n} \sup_{x_1 + x_2 + \cdots + x_i = x} \Phi_1(x_1) \land \Phi_2(x_2) \land \cdots \land \Phi_i(x_i) \]
\[ = \max_{1 \leq i \leq n} \Psi_i(x). \]
Thus (2.128) is verified.

**Strictly Decreasing Function of Uncertain Variables**

**Theorem 2.20** (Liu [95]) Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with continuous uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If \( f \) is a continuous and strictly decreasing function, then
\[ \xi = f(\xi_1, \xi_2, \cdots, \xi_n) \] (2.129)
has an uncertainty distribution
\[ \Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \min_{1 \leq i \leq n} (1 - \Phi_i(x_i)). \] (2.130)

**Proof:** For simplicity, we only prove the case \( n = 2 \). Since \( f \) is a continuous and strictly decreasing function, it holds that
\[ \{ f(\xi_1, \xi_2) \leq x \} = \bigcup_{f(x_1, x_2) = x} (\xi_1 \geq x_1) \cap (\xi_2 \geq x_2). \]
Thus the uncertainty distribution is
\[ \Psi(x) = \mathcal{M}\{ f(\xi_1, \xi_2) \leq x \} = \mathcal{M} \left\{ \bigcup_{f(x_1, x_2) = x} (\xi_1 \geq x_1) \cap (\xi_2 \geq x_2) \right\}. \]
Note that for each given number \( x \), the event
\[ \bigcup_{f(x_1, x_2) = x} (\xi_1 \geq x_1) \cap (\xi_2 \geq x_2) \]
is just a polyrectangle. It follows from the polyrectangular theorem that
\[\Psi(x) = \sup_{f(x_1,x_2)=x} M \{ (\xi_1 \geq x_1) \cap (\xi_2 \geq x_2) \} \]
\[= \sup_{f(x_1,x_2)=x} M \{ \xi_1 \geq x_1 \} \wedge M \{ \xi_2 \geq x_2 \} \]
\[= \sup_{f(x_1,x_2)=x} (1 - \Phi_1(x_1)) \wedge (1 - \Phi_2(x_2)). \]
The theorem is proved.

**Exercise 2.50:** Let \( \xi_1 \) and \( \xi_2 \) be independent and positive uncertain variables with continuous uncertainty distributions \( \Phi_1 \) and \( \Phi_2 \), respectively. Show that
\[\xi = \frac{1}{\xi_1 + \xi_2} \] has an uncertainty distribution
\[\Psi(x) = \sup_{y > 0} (1 - \Phi_1(y)) \wedge \left(1 - \Phi_2 \left(\frac{1}{x} - y\right)\right). \] (2.132)

**Exercise 2.51:** Show that the independence condition in Theorem 2.20 cannot be removed.

### Strictly Monotone Function of Uncertain Variables

**Theorem 2.21** (Liu [95]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with continuous uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f(\xi_1, \xi_2, \ldots, \xi_n) \) is continuous, strictly increasing with respect to \( \xi_1, \xi_2, \ldots, \xi_m \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \ldots, \xi_n \), then
\[\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \] has an uncertainty distribution
\[\Psi(x) = \sup_{f(x_1,x_2,\ldots,x_n)=x} \left( \min_{1 \leq i \leq m} \Phi_i(x_i) \wedge \min_{m+1 \leq i \leq n} (1 - \Phi_i(x_i)) \right). \] (2.134)

**Proof:** For simplicity, we only prove the case of \( m = 1 \) and \( n = 2 \). Since \( f(x_1,x_2) \) is continuous, strictly increasing with respect to \( x_1 \) and strictly decreasing with respect to \( x_2 \), it holds that
\[\{ f(\xi_1, \xi_2) \leq x \} = \bigcup_{f(x_1,x_2)=x} (\xi_1 \leq x_1) \cap (\xi_2 \geq x_2). \]
Thus the uncertainty distribution is
\[\Psi(x) = M \{ f(\xi_1, \xi_2) \leq x \} = M \left\{ \bigcup_{f(x_1,x_2)=x} (\xi_1 \leq x_1) \cap (\xi_2 \geq x_2) \right\}. \]
Note that for each given number $x$, the event

$$\bigcup_{f(x_1,x_2)=x} (\xi_1 \leq x_1) \cap (\xi_2 \geq x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\Psi(x) = \sup_{f(x_1,x_2)=x} M\{(\xi_1 \leq x_1) \cap (\xi_2 \geq x_2)\}$$

$$= \sup_{f(x_1,x_2)=x} M\{\xi_1 \leq x_1\} \land M\{\xi_2 \geq x_2\}$$

$$= \sup_{f(x_1,x_2)=x} \Phi_1(x_1) \land (1 - \Phi_2(x_2)) \cdot$$

The theorem is proved.

**Exercise 2.52:** Let $\xi_1$ and $\xi_2$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Show that $\xi_1 - \xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{y \in \mathbb{R}} \Phi_1(x + y) \land (1 - \Phi_2(y)). \quad (2.135)$$

**Exercise 2.53:** Let $\xi_1$ and $\xi_2$ be independent and positive uncertain variables with continuous uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Show that $\xi_1/\xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{y > 0} \Phi_1(xy) \land (1 - \Phi_2(y)). \quad (2.136)$$

**Exercise 2.54:** Let $\xi_1$ and $\xi_2$ be independent and positive uncertain variables with continuous uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Show that $\xi_1/(\xi_1 + \xi_2)$ has an uncertainty distribution

$$\Psi(x) = \sup_{y > 0} \Phi_1(xy) \land (1 - \Phi_2(y - xy)). \quad (2.137)$$

**Exercise 2.55:** Show that the independence condition in Theorem 2.21 cannot be removed.

### 2.7 Operational Law: Boolean System

A function is said to be Boolean if it is a mapping from $\{0,1\}^n$ to $\{0,1\}$. For example,

$$f(x_1, x_2, x_3) = x_1 \lor x_2 \land x_3 \quad (2.138)$$
is a Boolean function. An uncertain variable is said to be Boolean if it takes values either 0 or 1. For example, the following is a Boolean uncertain variable,

$$\xi = \begin{cases} 1 \text{ with uncertain measure } a \\ 0 \text{ with uncertain measure } 1 - a \end{cases} \quad (2.139)$$

where $a$ is a number between 0 and 1. This section introduces an operational law for Boolean system.

**Theorem 2.22** (Liu [95]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,

$$\xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases} \quad (2.140)$$

for $i = 1, 2, \cdots, n$. If $f$ is a Boolean function (not necessarily monotone), then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$M\{\xi = 1\} = \begin{cases} \sup \min \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\ 1 - \sup \min \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5 \end{cases} \quad (2.141)$$

where $x_i$ take values either 0 or 1, and $\nu_i$ are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (2.142)$$

for $i = 1, 2, \cdots, n$, respectively.

**Proof:** Let $B_1, B_2, \cdots, B_n$ be nonempty subsets of $\{0, 1\}$. In other words, they take values of $\{0\}$, $\{1\}$ or $\{0, 1\}$. Write

$$\Lambda = \{\xi = 1\}, \quad \Lambda^c = \{\xi = 0\}, \quad \Lambda_i = \{\xi_i \in B_i\}$$

for $i = 1, 2, \cdots, n$. It is easy to verify that

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{1\},$$

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda^c \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{0\}.$$
It follows from the product axiom that

\[
M\{\xi = 1\} = \begin{cases} 
\sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\}, & \text{if } \sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} > 0.5 \\
1 - \sup_{f(B_1, B_2, \ldots, B_n) = \{0\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\}, & \text{if } \sup_{f(B_1, B_2, \ldots, B_n) = \{0\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} > 0.5 \\
0.5, & \text{otherwise}
\end{cases}
\] (2.143)

Please note that

\[\nu_i(1) = M\{\xi_i = 1\}, \quad \nu_i(0) = M\{\xi_i = 0\}\]

for \(i = 1, 2, \ldots, n\). The argument breaks down into four cases. Case 1: Assume

\[\sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5.\]

Then we have

\[\sup_{f(B_1, B_2, \ldots, B_n) = \{0\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5.\]

It follows from (2.143) that

\[M\{\xi = 1\} = \sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i).\]

Case 2: Assume

\[\sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5.\]

Then we have

\[\sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5.\]

It follows from (2.143) that

\[M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i).\]

Case 3: Assume

\[\sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = 0.5, \quad \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = 0.5.\]
Then we have
\[
\sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 0.5,
\]
\[
\sup_{f(B_1, B_2, \ldots, B_n) = \{0\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 0.5.
\]
It follows from (2.143) that
\[
M\{\xi = 1\} = 0.5 = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i).
\]

Case 4: Assume
\[
\sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = 0.5,
\]
\[
\sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5.
\]
Then we have
\[
\sup_{f(B_1, B_2, \ldots, B_n) = \{1\}} \min_{1 \leq i \leq n} M\{\xi_i \in B_i\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) > 0.5.
\]
It follows from (2.143) that
\[
M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i).
\]

Hence the equation (2.141) is proved for the four cases.

**Example 2.16:** The independence condition in Theorem 2.22 cannot be removed. For example, take an uncertainty space \((\Gamma, L, M)\) to be \(\{\gamma_1, \gamma_2\}\) with power set and \(M\{\gamma_1\} = M\{\gamma_2\} = 0.5\). Then
\[
\xi_1(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2
\end{cases} \quad (2.144)
\]
is a Boolean uncertain variable with
\[
M\{\xi_1 = 1\} = 0.5, \quad (2.145)
\]
and
\[
\xi_2(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
0, & \text{if } \gamma = \gamma_2
\end{cases} \quad (2.146)
\]
is also a Boolean uncertain variable with
\[
M\{\xi_2 = 1\} = 0.5. \quad (2.147)
\]
Note that $\xi_1$ and $\xi_2$ are not independent, and $\xi_1 \land \xi_2 \equiv 0$ from which we obtain
\[ M\{\xi_1 \land \xi_2 = 1\} = 0. \tag{2.148} \]
However, by using (2.141), we get
\[ M\{\xi_1 \land \xi_2 = 1\} = 0.5. \tag{2.149} \]
Thus the independence condition cannot be removed.

**(Theorem 2.23)** *(Liu [95]), Order Statistic)* Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,
\[ \xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases} \tag{2.150} \]
for $i = 1, 2, \cdots, n$. Then the $k$th order statistic
\[ \xi = k\text{-min} [\xi_1, \xi_2, \cdots, \xi_n] \tag{2.151} \]
is a Boolean uncertain variable such that
\[ M\{\xi = 1\} = k\text{-min} [a_1, a_2, \cdots, a_n]. \tag{2.152} \]

**Proof:** The corresponding Boolean function for the $k$th order statistic is
\[ f(x_1, x_2, \cdots, x_n) = k\text{-min} [x_1, x_2, \cdots, x_n]. \tag{2.153} \]
Without loss of generality, we assume $a_1 \leq a_2 \leq \cdots \leq a_n$. Then we have
\[ \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_k \land \min_{1 \leq i < k} (a_i \lor (1 - a_i)), \]
\[ \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = (1 - a_k) \land \min_{k < i \leq n} (a_i \lor (1 - a_i)) \]
where $\nu_i(x_i)$ are defined by (2.142) for $i = 1, 2, \cdots, n$. When $a_k \geq 0.5$, we have
\[ \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5, \]
\[ \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = 1 - a_k. \]
It follows from Theorem 2.22 that
\[ M\{\xi = 1\} = 1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i) = 1 - (1 - a_k) = a_k. \]
When $a_k < 0.5$, we have
\[ \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_k < 0.5. \]
It follows from Theorem 2.22 that
\[
M\{\xi = 1\} = \sup_{f(x_1, x_2, \ldots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) = a_k.
\]
Thus \(M\{\xi = 1\}\) is always \(a_k\), i.e., the \(k\)th smallest value of \(a_1, a_2, \ldots, a_n\). The theorem is proved.

**Exercise 2.56:** Let \(\xi_1, \xi_2, \ldots, \xi_n\) be independent Boolean uncertain variables defined by (2.150). Then the minimum
\[
\xi = \xi_1 \land \xi_2 \land \cdots \land \xi_n
\]
is the 1st order statistic. Show that
\[
M\{\xi = 1\} = a_1 \land a_2 \land \cdots \land a_n.
\]

**Exercise 2.57:** Let \(\xi_1, \xi_2, \ldots, \xi_n\) be independent Boolean uncertain variables defined by (2.150). Then the maximum
\[
\xi = \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n
\]
is the \(n\)th order statistic. Show that
\[
M\{\xi = 1\} = a_1 \lor a_2 \lor \cdots \lor a_n.
\]

**Exercise 2.58:** Let \(\xi_1, \xi_2, \ldots, \xi_n\) be independent Boolean uncertain variables defined by (2.150). Then
\[
\xi = k\text{-max} [\xi_1, \xi_2, \ldots, \xi_n]
\]
is the \((n - k + 1)\)th order statistic. Show that
\[
M\{\xi = 1\} = k\text{-max} [a_1, a_2, \ldots, a_n].
\]

### 2.8 Expected Value

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

**Definition 2.16** (Liu [88]) Let \(\xi\) be an uncertain variable. Then the expected value of \(\xi\) is defined by
\[
E[\xi] = \int_{0}^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx
\]
provided that at least one of the two integrals is finite.
Theorem 2.24 (Liu [88]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx. \quad (2.161)$$

**Proof:** It follows from the measure inversion theorem that for almost all numbers $x$, we have $M\{\xi \geq x\} = 1 - \Phi(x)$ and $M\{\xi \leq x\} = \Phi(x)$. By using the definition of expected value operator, we obtain

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^0 M\{\xi \leq x\}dx$$

$$= \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx.$$

See Figure 2.12. The theorem is proved.

![Figure 2.12](image)

Figure 2.12: $E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx$

Theorem 2.25 (Liu [95]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then

$$E[\xi] = \int_{-\infty}^{+\infty} x\Phi(x). \quad (2.162)$$

**Proof:** It follows from the integration by parts and Theorem 2.24 that the expected value is

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx$$

$$= \int_0^{+\infty} x\Phi(x) + \int_{-\infty}^0 x\Phi(x) = \int_{-\infty}^{+\infty} x\Phi(x).$$

See Figure 2.13. The theorem is proved.
Theorem 2.26 (Liu [95]) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then
\[ E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \] (2.163)

**Proof:** Substituting $\Phi(x)$ with $\alpha$ and $x$ with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.25 that the expected value is
\[ E[\xi] = \int_{-\infty}^{+\infty} x d\Phi(x) = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \]

See Figure 2.13. The theorem is proved.

**Exercise 2.59:** Show that the linear uncertain variable $\xi \sim \mathcal{L}(a,b)$ has an expected value
\[ E[\xi] = \frac{a + b}{2}. \] (2.164)

**Exercise 2.60:** Show that the zigzag uncertain variable $\xi \sim \mathcal{Z}(a,b,c)$ has an expected value
\[ E[\xi] = \frac{a + 2b + c}{4}. \] (2.165)

**Exercise 2.61:** Show that the normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$ has an expected value $e$, i.e.,
\[ E[\xi] = e. \] (2.166)

**Exercise 2.62:** Show that the lognormal uncertain variable $\xi \sim \log\mathcal{N}(e,\sigma)$ has an expected value
\[ E[\xi] = \begin{cases} \sigma \sqrt{3} \exp(e) \csc(\sigma \sqrt{3}), & \text{if } \sigma < \pi/\sqrt{3} \\ +\infty, & \text{if } \sigma \geq \pi/\sqrt{3}. \end{cases} \] (2.167)
This formula was first discovered by Dr. Zhongfeng Qin with the help of Maple software, and was verified again by Dr. Kai Yao through a rigorous mathematical derivation.

**Exercise 2.63:** Let $\xi$ be an uncertain variable with empirical uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < x_1 \\
\alpha_i + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1}, \ 1 \leq i < n \\
1, & \text{if } x > x_n 
\end{cases}
$$

where $x_1 < x_2 < \cdots < x_n$ and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1$. Show that

$$
E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right) x_n. \quad (2.168)
$$

**Expected Value of Monotone Function of Uncertain Variables**

**Theorem 2.27** *(Liu-Ha [115])* Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then

$$
\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \quad (2.169)
$$

has an expected value

$$
E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha)) d\alpha. \quad (2.170)
$$

**Proof:** Since $f(x_1, x_2, \cdots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, it follows from Theorem 2.16 that the inverse uncertainty distribution of $\xi$ is

$$
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha)).
$$

By using Theorem 2.26, we obtain (2.170). The theorem is proved.

**Exercise 2.64:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $f(x)$ be a continuous and strictly monotone (increasing or decreasing) function. Show that

$$
E[f(\xi)] = \int_0^1 f(\Phi^{-1}(\alpha))d\alpha. \quad (2.171)
$$
Exercise 2.65: Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $f(x)$ be a continuous and strictly monotone (increasing or decreasing) function. Show that

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)d\Phi(x). \quad (2.172)$$

Exercise 2.66: Let $\xi$ and $\eta$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Show that

$$E[\xi \eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)d\alpha. \quad (2.173)$$

Exercise 2.67: Let $\xi$ and $\eta$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Show that

$$E \left[ \frac{\xi}{\xi + \eta} \right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1 - \alpha)}d\alpha. \quad (2.174)$$

Exercise 2.68: Assume $\xi$ and $\eta$ are independent and positive uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Show that

$$E \left[ \frac{\xi}{\xi + \eta} \right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1 - \alpha)}d\alpha. \quad (2.175)$$

Linearity of Expected Value Operator

Theorem 2.28 (Liu [95]): Let $\xi$ and $\eta$ be independent uncertain variables with finite expected values. Then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \quad (2.176)$$

Proof: Without loss of generality, suppose $\xi$ and $\eta$ have regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular.

Step 1: We first prove $E[a\xi] = aE[\xi]$. If $a = 0$, then the equation holds trivially. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

It follows from Theorem 2.26 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(\alpha)d\alpha = a\int_0^1 \Phi^{-1}(\alpha)d\alpha = aE[\xi].$$

If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1 - \alpha).$$
It follows from Theorem 2.26 that
\[ E[a\xi] = \int_0^1 a\Phi^{-1}(1 - \alpha)d\alpha = a \int_0^1 \Phi^{-1}(\alpha)d\alpha = aE[\xi]. \]
Thus we always have \( E[a\xi] = aE[\xi] \).

**STEP 2:** We prove \( E[\xi + \eta] = E[\xi] + E[\eta] \). The inverse uncertainty distribution of the sum \( \xi + \eta \) is
\[ \Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha). \]
It follows from Theorem 2.26 that
\[ E[\xi + \eta] = \int_0^1 \Upsilon^{-1}(\alpha)d\alpha = \int_0^1 \Phi^{-1}(\alpha)d\alpha + \int_0^1 \Psi^{-1}(\alpha)d\alpha = E[\xi] + E[\eta]. \]

**STEP 3:** Finally, for any real numbers \( a \) and \( b \), it follows from Steps 1 and 2 that
\[ E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta]. \]
The theorem is proved.

**Example 2.17:** Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed. For example, take an uncertainty space \( (\Gamma, \mathcal{L}, M) \) to be \( \{\gamma_1, \gamma_2, \gamma_3\} \) with power set and \( M\{\gamma_1\} = 0.6, M\{\gamma_2\} = 0.3 \) and \( M\{\gamma_3\} = 0.2 \). Define two uncertain variables as follows,
\[ \xi(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
0, & \text{if } \gamma = \gamma_2 \\
2, & \text{if } \gamma = \gamma_3,
\end{cases} \quad \eta(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
2, & \text{if } \gamma = \gamma_2 \\
3, & \text{if } \gamma = \gamma_3.
\end{cases} \]
Note that \( \xi \) and \( \eta \) are not independent, and their sum is
\[ (\xi + \eta)(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
2, & \text{if } \gamma = \gamma_2 \\
5, & \text{if } \gamma = \gamma_3.
\end{cases} \]
It is easy to verify that \( E[\xi] = 0.9, E[\eta] = 1 \) and \( E[\xi + \eta] = 2 \). Thus we have
\[ E[\xi + \eta] > E[\xi] + E[\eta]. \]
If the uncertain variables are defined by
\[ \xi(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2 \\
2, & \text{if } \gamma = \gamma_3,
\end{cases} \quad \eta(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
3, & \text{if } \gamma = \gamma_2 \\
1, & \text{if } \gamma = \gamma_3.
\end{cases} \]
Then

$$(\xi + \eta)(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
4, & \text{if } \gamma = \gamma_2 \\
3, & \text{if } \gamma = \gamma_3. 
\end{cases}$$

It is easy to verify that $E[\xi] = 0.6$, $E[\eta] = 1$ and $E[\xi + \eta] = 1.5$. Thus we have

$$E[\xi + \eta] < E[\xi] + E[\eta].$$

Therefore, the independence condition cannot be removed.

**Comonotonic Functions of Uncertain Variable**

Two real-valued functions $f$ and $g$ are said to be **comonotonic** if for any numbers $x$ and $y$, we always have

$$(f(x) - f(y))(g(x) - g(y)) \geq 0. \quad (2.177)$$

It is easy to verify that (i) any monotone increasing functions are comonotonic; and (ii) any monotone decreasing functions are comonotonic.

**Theorem 2.29** (Yang [182]) Let $f$ and $g$ be comonotonic functions. Then for any uncertain variable $\xi$, we have

$$E[f(\xi) + g(\xi)] = E[f(\xi)] + E[g(\xi)]. \quad (2.178)$$

**Proof:** Without loss of generality, suppose $f(\xi)$ and $g(\xi)$ have regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular. Since $f$ and $g$ are comonotonic functions, at least one of the following relations is true,

$$\{f(\xi) \leq \Phi^{-1}(\alpha)\} \subset \{g(\xi) \leq \Psi^{-1}(\alpha)\},$$

$$\{f(\xi) \leq \Phi^{-1}(\alpha)\} \supset \{g(\xi) \leq \Psi^{-1}(\alpha)\}.$$

On the one hand, we have

$$\mathcal{M}\{f(\xi) + g(\xi) \leq \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)\}$$

$$\geq \mathcal{M}\{(f(\xi) \leq \Phi^{-1}(\alpha)) \cap (g(\xi) \leq \Psi^{-1}(\alpha))\}$$

$$= \mathcal{M}\{f(\xi) \leq \Phi^{-1}(\alpha)\} \land \mathcal{M}\{g(\xi) \leq \Psi^{-1}(\alpha)\}$$

$$= \alpha \land \alpha = \alpha.$$

On the other hand, we have

$$\mathcal{M}\{f(\xi) + g(\xi) \leq \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)\}$$

$$\leq \mathcal{M}\{(f(\xi) \leq \Phi^{-1}(\alpha)) \cup (g(\xi) \leq \Psi^{-1}(\alpha))\}$$

$$= \mathcal{M}\{f(\xi) \leq \Phi^{-1}(\alpha)\} \lor \mathcal{M}\{g(\xi) \leq \Psi^{-1}(\alpha)\}$$

$$= \alpha \lor \alpha = \alpha.$$
It follows that

\[ M\{f(\xi) + g(\xi) \leq \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)\} = \alpha \]

holds for each \( \alpha \). That is, \( \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha) \) is the inverse uncertainty distribution of \( f(\xi) + g(\xi) \). By using Theorem 2.26, we obtain

\[
E[f(\xi) + g(\xi)] = \int_0^1 (\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha))d\alpha \\
= \int_0^1 \Phi^{-1}(\alpha)d\alpha + \int_0^1 \Psi^{-1}(\alpha)d\alpha \\
= E[f(\xi)] + E[g(\xi)].
\]

The theorem is verified.

**Exercise 2.69:** Let \( \xi \) be a positive uncertain variable. Show that \( \ln x \) and \( \exp(x) \) are comonotonic functions on \((0, +\infty)\), and

\[
E[\ln \xi + \exp(\xi)] = E[\ln \xi] + E[\exp(\xi)].
\]

**Exercise 2.70:** Let \( \xi \) be a positive uncertain variable. Show that \( x, x^2, \cdots, x^n \) are comonotonic functions on \([0, +\infty)\), and

\[
E[\xi + \xi^2 + \cdots + \xi^n] = E[\xi] + E[\xi^2] + \cdots + E[\xi^n].
\]

**Absolute Value of Uncertain Variable**

Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). Then the expected value of \(|\xi|\) is

\[
E[|\xi|] = \int_0^{+\infty} M\{|\xi| \geq x\}dx \\
= \int_0^{+\infty} M\{(\xi \geq x) \cup (\xi \leq -x)\}dx \\
\leq \int_0^{+\infty} (M\{\xi \geq x\} + M\{\xi \leq -x\})dx \\
= \int_0^{+\infty} (1 - \Phi(x) + \Phi(-x))dx.
\]

Thus we have the following stipulation.

**Stipulation 2.1** (Liu [106]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). Then the expected value of \(|\xi|\) is

\[
E[|\xi|] = \int_0^{+\infty} (1 - \Phi(x) + \Phi(-x))dx.
\]
Theorem 2.30 (Liu [106]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. Then the expected value of $|\xi|$ is

$$E[|\xi|] = \int_{-\infty}^{+\infty} |x|d\Phi(x). \quad (2.182)$$

Proof: This theorem is based on Stipulation 2.1. The change of variables and integration by parts produce

$$E[|\xi|] = \int_{0}^{+\infty} (1 - \Phi(x) + \Phi(-x))dx$$

$$= \int_{0}^{+\infty} xd\Phi(x) - \int_{0}^{+\infty} xd\Phi(-x)$$

$$= \int_{0}^{+\infty} |x|d\Phi(x) + \int_{-\infty}^{0} |x|d\Phi(x)$$

$$= \int_{-\infty}^{+\infty} |x|d\Phi(x).$$

The theorem is proved.

Theorem 2.31 (Liu [106]) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then the expected value of $|\xi|$ is

$$E[|\xi|] = \int_{0}^{1} |\Phi^{-1}(\alpha)|d\alpha. \quad (2.183)$$

Proof: Substituting $\Phi(x)$ with $\alpha$ and $x$ with $\Phi^{-1}(\alpha)$, it follows from the change of variables and Theorem 2.30 that the expected value is

$$E[|\xi|] = \int_{-\infty}^{+\infty} |x|d\Phi(x) = \int_{0}^{1} |\Phi^{-1}(\alpha)|d\alpha.$$

The theorem is verified.

Some Inequalities

Theorem 2.32 (Liu [88]) Let $\xi$ be an uncertain variable, and let $f$ be a nonnegative even function. If $f$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$, then for any given number $t > 0$, we have

$$\mathcal{M}\{|\xi| \geq t\} \leq \frac{E[f(\xi)]}{f(t)}. \quad (2.184)$$
Proof: It is clear that \(\mathcal{M}\{\vert \xi \vert \geq f^{-1}(r)\}\) is a monotone decreasing function of \(r\) on \([0, \infty)\). It follows from the nonnegativity of \(f(\xi)\) that

\[
E[f(\xi)] = \int_0^{+\infty} \mathcal{M}\{f(\xi) \geq x\}dx = \int_0^{+\infty} \mathcal{M}\{\vert \xi \vert \geq f^{-1}(x)\}dx = \int_0^{f(t)} \mathcal{M}\{\vert \xi \vert \geq f^{-1}(x)\}dx \geq \int_0^{f(t)} \mathcal{M}\{\vert \xi \vert \geq f^{-1}(f(t))\}dx = \int_0^{f(t)} \mathcal{M}\{\vert \xi \vert \geq t\}dx = f(t) \cdot \mathcal{M}\{\vert \xi \vert \geq t\}
\]

which proves the inequality.

**Theorem 2.33** (Liu [88], Markov Inequality) Let \(\xi\) be an uncertain variable. Then for any given numbers \(t > 0\) and \(p > 0\), we have

\[
\mathcal{M}\{\vert \xi \vert \geq t\} \leq \frac{E[\vert \xi \vert^p]}{t^p}. \tag{2.185}
\]

Proof: It is a special case of Theorem 2.32 when \(f(x) = \vert x \vert^p\).

**Example 2.18:** For any given positive number \(t\), we define an uncertain variable as follows,

\[
\xi = \begin{cases} 
0 & \text{with uncertain measure 1/2} \\
t & \text{with uncertain measure 1/2.}
\end{cases}
\]

Then \(E[\xi^p] = t^p/2\) and \(\mathcal{M}\{\xi \geq t\} = 1/2 = E[\xi^p]/t^p\).

**Theorem 2.34** (Liu [88], Hölder’s Inequality) Let \(p\) and \(q\) be positive numbers with \(1/p + 1/q = 1\), and let \(\xi\) and \(\eta\) be independent uncertain variables. Then

\[
E[\vert \xi \eta \vert] \leq \sqrt[p]{E[\vert \xi \vert^p]} \sqrt[q]{E[\vert \eta \vert^q]}. \tag{2.186}
\]

**Proof:** The inequality holds trivially if at least one of \(\xi\) and \(\eta\) is zero a.s. Now we assume \(E[\vert \xi \vert^p] > 0\) and \(E[\vert \eta \vert^q] > 0\). It is easy to prove that the function \(f(x, y) = \sqrt{x} \sqrt{y}\) is a concave function on \(\{(x, y) : x \geq 0, y \geq 0\}\). Thus for any point \((x_0, y_0)\) with \(x_0 > 0\) and \(y_0 > 0\), there exist two real numbers \(a\) and \(b\) such that

\[
f(x, y) - f(x_0, y_0) \leq a(x - x_0) + b(y - y_0), \quad \forall x \geq 0, y \geq 0.
\]

Letting \(x_0 = E[\vert \xi \vert^p], y_0 = E[\vert \eta \vert^q], x = \vert \xi \vert^p\) and \(y = \vert \eta \vert^q\), we have

\[
f(\vert \xi \vert^p, \vert \eta \vert^q) - f(E[\vert \xi \vert^p], E[\vert \eta \vert^q]) \leq a(\vert \xi \vert^p - E[\vert \xi \vert^p]) + b(\vert \eta \vert^q - E[\vert \eta \vert^q])
\]

Taking the expected values on both sides, we obtain

\[
E[f(\vert \xi \vert^p, \vert \eta \vert^q)] \leq f(E[\vert \xi \vert^p], E[\vert \eta \vert^q]).
\]

Hence the inequality (2.186) holds.
Theorem 2.35 (Liu [88], Minkowski Inequality) Let $p$ be a real number with $p \geq 1$, and let $\xi$ and $\eta$ be independent uncertain variables. Then

$$\sqrt[\nu]{E[|\xi + \eta|^p]} \leq \sqrt[\nu]{E[|\xi|^p]} + \sqrt[\nu]{E[|\eta|^p]}.$$  (2.187)

Proof: The inequality holds trivially if at least one of $\xi$ and $\eta$ is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^p] > 0$. It is easy to prove that the function $f(x, y) = (\sqrt[\nu]{x} + \sqrt[\nu]{y})^p$ is a concave function on \((x, y) : x \geq 0, y \geq 0\). Thus for any point \((x_0, y_0)\) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers $a$ and $b$ such that

$$f(x, y) - f(x_0, y_0) \leq a(x - x_0) + b(y - y_0), \quad \forall x \geq 0, y \geq 0.$$  

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^p]$, $x = |\xi|^p$ and $y = |\eta|^p$, we have

$$f(|\xi|^p, |\eta|^p) - f(E[|\xi|^p], E[|\eta|^p]) \leq a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^p - E[|\eta|^p]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^p)] \leq f(E[|\xi|^p], E[|\eta|^p]).$$

Hence the inequality (2.187) holds.

Theorem 2.36 (Liu [88], Jensen’s Inequality) Let $\xi$ be an uncertain variable, and let $f$ be a convex function. Then

$$E[f(\xi)] \leq f(E[\xi]).$$  (2.188)

Especially, when $f(x) = |x|^p$ and $p \geq 1$, we have $|E[\xi]|^p \leq E[|\xi|^p]$.

Proof: Since $f$ is a convex function, for each $y$, there exists a number $k$ such that $f(x) - f(y) \geq k \cdot (x - y)$. Replacing $x$ with $\xi$ and $y$ with $E[\xi]$, we obtain

$$f(\xi) - f(E[\xi]) \geq k \cdot (\xi - E[\xi]).$$

Taking the expected values on both sides, we have

$$E[f(\xi)] - f(E[\xi]) \geq k \cdot (E[\xi] - E[\xi]) = 0$$

which proves the inequality.

Exercise 2.71: (Zhang [222]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with finite expected values, and let $f$ be a convex function. Show that

$$f(E[\xi_1], E[\xi_2], \cdots, E[\xi_n]) \leq E[f(\xi_1, \xi_2, \cdots, \xi_n)].$$  (2.189)
2.9 Variance

The variance of uncertain variable provides a degree of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value.

Definition 2.17 (Liu [88]) Let $\xi$ be an uncertain variable with finite expected value $e$. Then the variance of $\xi$ is

$$V[\xi] = E[(\xi - e)^2].$$

(2.190)

This definition tells us that the variance is just the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is a nonnegative uncertain variable, we also have

$$V[\xi] = \int_{0}^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}dx.$$  

(2.191)

Theorem 2.37 (Liu [88]) If $\xi$ is an uncertain variable with finite expected value, $a$ and $b$ are real numbers, then

$$V[a \xi + b] = a^2 V[\xi].$$

(2.192)

Proof: Let $e$ be the expected value of $\xi$. Then $a \xi + b$ has an expected value $ae + b$. It follows from the definition of variance that

$$V[a \xi + b] = E [(a \xi + b - (ae + b))^2] = a^2 E[(\xi - e)^2] = a^2 V[\xi].$$

The theorem is thus verified.

Theorem 2.38 (Liu [88]) Let $\xi$ be an uncertain variable with expected value $e$. Then $V[\xi] = 0$ if and only if $\mathcal{M}\{\xi = e\} = 1$. That is, the uncertain variable $\xi$ is essentially the constant $e$.

Proof: We first assume $V[\xi] = 0$. It follows from the equation (2.191) that

$$\int_{0}^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}dx = 0$$

which implies $\mathcal{M}\{(\xi - e)^2 \geq x\} = 0$ for any $x > 0$. Hence we have

$$\mathcal{M}\{(\xi - e)^2 = 0\} = 1.$$

That is, $\mathcal{M}\{\xi = e\} = 1$. Conversely, assume $\mathcal{M}\{\xi = e\} = 1$. Then we immediately have $\mathcal{M}\{(\xi - e)^2 = 0\} = 1$ and $\mathcal{M}\{(\xi - e)^2 \geq x\} = 0$ for any $x > 0$. Thus

$$V[\xi] = \int_{0}^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}dx = 0.$$  

The theorem is proved.
Theorem 2.39 (Yao [193]) Let $\xi$ and $\eta$ be independent uncertain variables whose variances exist. Then
\begin{equation}
\sqrt{V[\xi + \eta]} \leq \sqrt{V[\xi]} + \sqrt{V[\eta]}.
\end{equation}

**Proof:** It is a special case of Theorem 2.35 when $p = 2$ and the uncertain variables $\xi$ and $\eta$ are replaced with $\xi - E[\xi]$ and $\eta - E[\eta]$, respectively.

Theorem 2.40 (Liu [88], Chebyshev Inequality) Let $\xi$ be an uncertain variable whose variance exists. Then for any given number $t > 0$, we have
\begin{equation}
M\{|\xi - E[\xi]| \geq t\} \leq \frac{V[\xi]}{t^2}.
\end{equation}

**Proof:** It is a special case of Theorem 2.32 when the uncertain variable $\xi$ is replaced with $\xi - E[\xi]$, and $f(x) = x^2$.

Example 2.19: For any given positive number $t$, we define an uncertain variable as follows,
\begin{equation}
\xi = \begin{cases} 
- t & \text{with uncertain measure } 1/2 \\
t & \text{with uncertain measure } 1/2.
\end{cases}
\end{equation}

Then $V[\xi] = t^2$ and $M\{|\xi - E[\xi]| \geq t\} = 1 = V[\xi]/t^2$.

**How to Obtain Variance from Uncertainty Distribution?**

Let $\xi$ be an uncertain variable with expected value $e$. If we only know its uncertainty distribution $\Phi$, then the variance
\begin{equation}
V[\xi] = \int_0^{+\infty} M\{(\xi - e)^2 \geq x\}dx
\end{equation}
\begin{equation}
= \int_0^{+\infty} M\{(\xi \geq e + \sqrt{x}) \cup (\xi \leq e - \sqrt{x})\}dx
\end{equation}
\begin{equation}
\leq \int_0^{+\infty} (M\{\xi \geq e + \sqrt{x}\} + M\{\xi \leq e - \sqrt{x}\})dx
\end{equation}
\begin{equation}
= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))dx.
\end{equation}

Thus we have the following stipulation.

Stipulation 2.2 (Liu [95]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$ and finite expected value $e$. Then
\begin{equation}
V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))dx.
\end{equation}
Theorem 2.41 (Liu [106]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$ and finite expected value $e$. Then

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 d\Phi(x). \quad (2.196)$$

**Proof:** This theorem is based on Stipulation 2.2 that says the variance of $\xi$ is

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))dy + \int_0^{+\infty} \Phi(e - \sqrt{y})dy.$$

Substituting $e + \sqrt{y}$ with $x$ and $y$ with $(x - e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))dy = \int_e^{+\infty} (1 - \Phi(x))d(x - e)^2 = \int_e^{+\infty} (x - e)^2 d\Phi(x).$$

Similarly, substituting $e - \sqrt{y}$ with $x$ and $y$ with $(x - e)^2$, we obtain

$$\int_0^{+\infty} \Phi(e - \sqrt{y})dy = \int_e^{-\infty} \Phi(x)d(x - e)^2 = \int_e^{-\infty} (x - e)^2 d\Phi(x).$$

It follows that the variance is

$$V[\xi] = \int_e^{+\infty} (x - e)^2 d\Phi(x) + \int_e^{-\infty} (x - e)^2 d\Phi(x) = \int_{-\infty}^{+\infty} (x - e)^2 d\Phi(x).$$

The theorem is verified.

Theorem 2.42 (Yao [193]) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$ and finite expected value $e$. Then

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha. \quad (2.197)$$

**Proof:** Substituting $\Phi(x)$ with $\alpha$ and $x$ with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.41 that the variance is

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 d\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha.$$

The theorem is verified.

**Exercise 2.72:** Show that the linear uncertain variable $\xi \sim L(a, b)$ has a variance

$$V[\xi] = \frac{(b - a)^2}{12}. \quad (2.198)$$
Exercise 2.73: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has a variance

$$V[\xi] = \sigma^2.$$  \hfill (2.199)

Exercise 2.74: Let $\xi$ and $\eta$ be independent uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Assume there exist two real numbers $a$ and $b$ such that

$$\Phi^{-1}(\alpha) = a\Psi^{-1}(\alpha) + b$$  \hfill (2.200)

for all $\alpha \in (0, 1)$. Show that

$$\sqrt{V[\xi + \eta]} = \sqrt{V[\xi]} + \sqrt{V[\eta]}$$  \hfill (2.201)

in the sense of Stipulation 2.2.

Remark 2.7: If $\xi$ and $\eta$ are independent linear uncertain variables, then the condition (2.200) is met. If they are independent normal uncertain variables, then the condition (2.200) is also met.

2.10 Moment

Definition 2.18 (Liu [88]) Let $\xi$ be an uncertain variable and let $k$ be a positive integer. Then $E[\xi^k]$ is called the $k$-th moment of $\xi$.

Theorem 2.43 (Liu [106]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $k$ be an odd number. Then the $k$-th moment of $\xi$ is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))dx - \int_{-\infty}^{0} \Phi(\sqrt[k]{x})dx.$$  \hfill (2.202)

Proof: Since $k$ is an odd number, it follows from the definition of expected value operator that

$$E[\xi^k] = \int_{-\infty}^{+\infty} \mathcal{M}\{\xi^k \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi^k \leq x\}dx$$

$$= \int_{0}^{+\infty} \mathcal{M}\{\xi \geq \sqrt[k]{x}\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq \sqrt[k]{x}\}dx$$

$$= \int_{0}^{+\infty} (1 - \Phi(\sqrt[k]{x}))dx - \int_{-\infty}^{0} \Phi(\sqrt[k]{x})dx.$$  

The theorem is proved.
However, when \( k \) is an even number, the \( k \)-th moment of \( \xi \) cannot be uniquely determined by the uncertainty distribution \( \Phi \). In this case, we have

\[
E[\xi^k] = \int_0^{+\infty} M\{\xi^k \geq x\}dx \\
= \int_0^{+\infty} M\{\xi \geq \sqrt[k]{x}\} \cup \{\xi \leq -\sqrt[k]{x}\}dx \\
\leq \int_0^{+\infty} (M\{\xi \geq \sqrt[k]{x}\} + M\{\xi \leq -\sqrt[k]{x}\})dx \\
= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))dx.
\]

Thus for the even number \( k \), we have the following stipulation.

**Stipulation 2.3** (Liu [106]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \), and let \( k \) be an even number. Then the \( k \)-th moment of \( \xi \) is

\[
E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))dx. \quad (2.203)
\]

**Theorem 2.44** (Liu [106]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \), and let \( k \) be a positive integer. Then the \( k \)-th moment of \( \xi \) is

\[
E[\xi^k] = \int_{-\infty}^{+\infty} x^k d\Phi(x). \quad (2.204)
\]

**Proof:** When \( k \) is an odd number, Theorem 2.43 says that the \( k \)-th moment is

\[
E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))dy - \int_{-\infty}^{0} \Phi(\sqrt[k]{y})dy.
\]

Substituting \( \sqrt[k]{y} \) with \( x \) and \( y \) with \( x^k \), the change of variables and integration by parts produce

\[
\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))dy = \int_0^{+\infty} (1 - \Phi(x))dx^k = \int_0^{+\infty} x^k d\Phi(x)
\]

and

\[
\int_{-\infty}^{0} \Phi(\sqrt[k]{y})dy = \int_{-\infty}^{0} \Phi(x)dx^k = -\int_{-\infty}^{0} x^k d\Phi(x).
\]

Thus we have

\[
E[\xi^k] = \int_0^{+\infty} x^k d\Phi(x) + \int_{-\infty}^{0} x^k d\Phi(x) = \int_{-\infty}^{+\infty} x^k d\Phi(x).
\]
When $k$ is an even number, the theorem is based on Stipulation 2.3 that says the $k$-th moment is
\[
E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt{y}) + \Phi(-\sqrt{y}))dy.
\]
Substituting $\sqrt{y}$ with $x$ and $y$ with $x^k$, the change of variables and integration by parts produce
\[
\int_0^{+\infty} (1 - \Phi(\sqrt{y}))dy = \int_0^{0^+} (1 - \Phi(x))dx = \int_0^{+\infty} x^k d\Phi(x).
\]
Similarly, substituting $-\sqrt{y}$ with $x$ and $y$ with $x^k$, we obtain
\[
\int_0^{+\infty} \Phi(-\sqrt{y})dy = \int_{-\infty}^0 \Phi(x)dx = \int_{-\infty}^0 x^k d\Phi(x).
\]
It follows that the $k$-th moment is
\[
E[\xi^k] = \int_0^{+\infty} x^k d\Phi(x) + \int_{-\infty}^0 x^k d\Phi(x) = \int_{-\infty}^{+\infty} x^k d\Phi(x).
\]
The theorem is thus verified for any positive integer $k$.

**Theorem 2.45** *(Sheng-Kar [155])* Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $k$ be a positive integer. Then the $k$-th moment of $\xi$ is
\[
E[\xi^k] = \int_0^1 (\Phi^{-1}(\alpha))^k d\alpha.
\] (2.205)

**Proof:** Substituting $\Phi(x)$ with $\alpha$ and $x$ with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.44 that the $k$-th moment is
\[
E[\xi^k] = \int_{-\infty}^{+\infty} x^k d\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha))^k d\alpha.
\]
The theorem is verified.

**Exercise 2.75:** Show that the second moment of linear uncertain variable $\xi \sim L(a, b)$ is
\[
E[\xi^2] = \frac{a^2 + ab + b^2}{3}.
\] (2.206)

**Exercise 2.76:** Show that the second moment of normal uncertain variable $\xi \sim N(e, \sigma)$ is
\[
E[\xi^2] = e^2 + \sigma^2.
\] (2.207)
2.11 Distance

Definition 2.19 (Liu [88]) The distance between uncertain variables $\xi$ and $\eta$ is defined as
\[
d(\xi, \eta) = E[|\xi - \eta|].
\] (2.208)

That is, the distance between $\xi$ and $\eta$ is just the expected value of $|\xi - \eta|$. Since $|\xi - \eta|$ is a nonnegative uncertain variable, we always have
\[
d(\xi, \eta) = \int_0^{+\infty} M\{|\xi - \eta| \geq x\} \, dx.
\] (2.209)

Theorem 2.46 (Liu [88]) Let $\xi, \eta, \tau$ be uncertain variables, and let $d(\cdot, \cdot)$ be the distance. Then we have
(a) (Nonnegativity) $d(\xi, \eta) \geq 0$;
(b) (Identification) $d(\xi, \eta) = 0$ if and only if $\xi = \eta$;
(c) (Symmetry) $d(\xi, \eta) = d(\eta, \xi)$;
(d) (Triangle Inequality) $d(\xi, \eta) \leq 2d(\xi, \tau) + 2d(\tau, \eta)$.

Proof: The parts (a), (b) and (c) follow immediately from the definition. Now we prove the part (d). It follows from the subadditivity axiom that
\[
d(\xi, \eta) = \int_0^{+\infty} M\{|\xi - \tau| + |\tau - \eta| \geq x\} \, dx
\leq \int_0^{+\infty} M\{|\xi - \tau| \geq x/2\} \cup \{|\tau - \eta| \geq x/2\} \, dx
\leq \int_0^{+\infty} (M\{|\xi - \tau| \geq x/2\} + M\{|\tau - \eta| \geq x/2\}) \, dx
\leq 2E[|\xi - \tau|] + 2E[|\tau - \eta|] = 2d(\xi, \tau) + 2d(\tau, \eta).
\]

Example 2.20: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Define $M\{\emptyset\} = 0$, $M\{\Gamma\} = 1$ and $M\{\Lambda\} = 1/2$ for any subset $\Lambda$ (excluding $\emptyset$ and $\Gamma$). We set uncertain variables $\xi, \eta$ and $\tau$ as follows,
\[
\xi(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2 \\
0, & \text{if } \gamma = \gamma_3,
\end{cases} 
\eta(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
-1, & \text{if } \gamma = \gamma_2 \\
-1, & \text{if } \gamma = \gamma_3,
\end{cases}
\tau(\gamma) \equiv 0.
\]

It is easy to verify that $d(\xi, \tau) = d(\tau, \eta) = 0.5$ and $d(\xi, \eta) = 1.5$. Thus
\[
d(\xi, \eta) = 1.5(d(\xi, \tau) + d(\tau, \eta)).
\]
A conjecture is $d(\xi, \eta) \leq 1.5(d(\xi, \tau) + d(\tau, \eta))$ for arbitrary uncertain variables $\xi, \eta$ and $\tau$. This is an open problem.
Chapter 2 - Uncertain Variable

**Theorem 2.47** (Liu [106]) Let $\xi$ and $\eta$ be independent uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Then the distance between $\xi$ and $\eta$ is

$$d(\xi, \eta) = \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x))dx$$  \hspace{1cm} (2.210)

where $\Upsilon(x)$ is the uncertainty distribution of $\xi - \eta$, and

$$\Upsilon(x) = \sup_{y \in \mathbb{R}} \Phi(x + y) \land (1 - \Psi(y)).$$  \hspace{1cm} (2.211)

**Proof:** The equation (2.210) follows from $d(\xi, \eta) = E[|\xi - \eta|]$ and Stipulation 2.1 immediately.

**Theorem 2.48** (Liu [106]) Let $\xi$ and $\eta$ be independent uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Then the distance between $\xi$ and $\eta$ is

$$d(\xi, \eta) = \int_{-\infty}^{+\infty} |x|d\Upsilon(x)$$  \hspace{1cm} (2.212)

where $\Upsilon(x)$ is the uncertainty distribution of $\xi - \eta$, determined by (2.211).

**Proof:** The equation (2.212) follows from $d(\xi, \eta) = E[|\xi - \eta|]$ and Theorem 2.30 immediately.

**Exercise 2.77:** Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$, and let $c$ be a constant. Show that the distance between $\xi$ and $c$ is

$$d(\xi, c) = \int_{-\infty}^{+\infty} |x - c|d\Phi(x).$$  \hspace{1cm} (2.213)

**Theorem 2.49** (Liu [106]) Let $\xi$ and $\eta$ be independent uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Then the distance between $\xi$ and $\eta$ is

$$d(\xi, \eta) = \int_0^1 |\Upsilon^{-1}(\alpha)|d\alpha$$  \hspace{1cm} (2.214)

where $\Upsilon^{-1}(\alpha)$ is the inverse uncertainty distribution of $\xi - \eta$, and

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) - \Psi^{-1}(1 - \alpha).$$  \hspace{1cm} (2.215)

**Proof:** The equation (2.214) follows from $d(\xi, \eta) = E[|\xi - \eta|]$ and Theorem 2.31 immediately.

**Exercise 2.78:** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$, and let $c$ be a constant. Show that the distance between $\xi$ and $c$ is

$$d(\xi, c) = \int_0^1 |\Phi^{-1}(\alpha) - c|d\alpha.$$  \hspace{1cm} (2.216)
2.12 Entropy

This section defines an entropy as the degree of difficulty of predicting the realization of an uncertain variable.

**Definition 2.20** *(Liu [91])* Suppose that \( \xi \) is an uncertain variable with uncertainty distribution \( \Phi \). Then its entropy is defined by

\[
H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x)) \, dx
\]  

(2.217)

where \( S(t) = -t \ln t - (1 - t) \ln(1 - t) \).

---

Figure 2.14: Function \( S(t) = -t \ln t - (1 - t) \ln(1 - t) \). It is easy to verify that \( S(t) \) is a symmetric function about \( t = 0.5 \), strictly increasing on the interval \( [0, 0.5] \), strictly decreasing on the interval \( [0.5, 1] \), and reaches its unique maximum \( \ln 2 \) at \( t = 0.5 \).

**Example 2.21**: Let \( \xi \) be an uncertain variable with uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < a \\
1, & \text{if } x \geq a.
\end{cases}
\]  

(2.218)

Essentially, \( \xi \) is a constant \( a \). It follows from the definition of entropy that

\[
H[\xi] = -\int_{-\infty}^{a} (0 \ln 0 + 1 \ln 1) \, dx - \int_{a}^{+\infty} (1 \ln 1 + 0 \ln 0) \, dx = 0.
\]

This means a constant has entropy 0.

**Example 2.22**: Let \( \xi \) be a linear uncertain variable \( \mathcal{L}(a, b) \). Then its entropy is

\[
H[\xi] = -\int_{a}^{b} \left( \frac{x - a}{b - a} \ln \frac{x - a}{b - a} + \frac{b - x}{b - a} \ln \frac{b - x}{b - a} \right) \, dx = \frac{b - a}{2}.
\]  

(2.219)
Exercise 2.79: Show that the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an entropy
\[ H[\xi] = \frac{c - a}{2}. \] (2.220)

Exercise 2.80: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an entropy
\[ H[\xi] = \frac{\pi \sigma}{\sqrt{3}}. \] (2.221)

Theorem 2.50 Let $\xi$ be an uncertain variable. Then $H[\xi] \geq 0$ and equality holds if $\xi$ is essentially a constant.

Proof: The nonnegativity is clear. In addition, when an uncertain variable tends to a constant, its entropy tends to the minimum 0.

Theorem 2.51 Let $\xi$ be an uncertain variable taking values on the interval $[a, b]$. Then
\[ H[\xi] \leq (b - a) \ln 2 \] (2.222)
and equality holds if $\xi$ has an uncertainty distribution $\Phi(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum $\ln 2$ at $t = 0.5$.

Theorem 2.52 Let $\xi$ be an uncertain variable, and let $c$ be a real number. Then
\[ H[\xi + c] = H[\xi]. \] (2.223)
That is, the entropy is invariant under arbitrary translations.

Proof: Write the uncertainty distribution of $\xi$ by $\Phi$. Then the uncertain variable $\xi + c$ has an uncertainty distribution $\Phi(x - c)$. It follows from the definition of entropy that
\[ H[\xi + c] = \int_{-\infty}^{+\infty} S(\Phi(x - c)) \, dx = \int_{-\infty}^{+\infty} S(\Phi(x)) \, dx = H[\xi]. \]
The theorem is proved.

Theorem 2.53 (Dai-Chen [21]) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then
\[ H[\xi] = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} \, d\alpha. \] (2.224)
Proof: It is clear that $S(\alpha)$ is a derivable function whose derivative has the form

$$S'(\alpha) = -\ln \frac{\alpha}{1 - \alpha}.$$ 

Since

$$S(\Phi(x)) = \int_0^{\Phi(x)} S'(\alpha)d\alpha = -\int_{\Phi(x)}^1 S'(\alpha)d\alpha,$$

we have

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx = \int_{-\infty}^{0} \int_0^{\Phi(x)} S'(\alpha)d\alpha dx - \int_0^{+\infty} \int_{\Phi(x)}^1 S'(\alpha)d\alpha dx.$$ 

It follows from Fubini theorem that

$$H[\xi] = \int_0^{\Phi(0)} \int_{\Phi^{-1}(\alpha)}^{0} S'(\alpha)d\alpha dx - \int_{\Phi(0)}^{1} \int_{\Phi^{-1}(\alpha)}^{0} S'(\alpha)d\alpha dx$$

$$= -\int_0^{\Phi(0)} \Phi^{-1}(\alpha)S'(\alpha)d\alpha - \int_{\Phi(0)}^{1} \Phi^{-1}(\alpha)S'(\alpha)d\alpha$$

$$= -\int_0^{1} \Phi^{-1}(\alpha)S'(\alpha)d\alpha = \int_0^{1} \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} d\alpha.$$ 

The theorem is verified.

Theorem 2.54 (Dai-Chen [21]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \ldots, \xi_m)$ is continuous, strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \quad (2.225)$$

has an entropy

$$H[\xi] = \int_0^{1} f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) \ln \frac{\alpha}{1 - \alpha} d\alpha.$$ 

Proof: Since $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, it follows from Theorem 2.16 that the inverse uncertainty distribution of $\xi$ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

By using Theorem 2.53, we get the entropy formula.

Exercise 2.81: Let $\xi$ and $\eta$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Show that

$$H[\xi|\eta] = \int_0^{1} \Phi^{-1}(\alpha)\Psi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} d\alpha.$$
Chapter 2 - Uncertain Variable

Exercise 2.82: Let $\xi$ and $\eta$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Show that

$$H\left[ \frac{\xi}{\eta} \right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \, d\alpha.$$ 

Exercise 2.83: Let $\xi$ and $\eta$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Show that

$$H\left[ \frac{\xi}{\xi+\eta} \right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \, d\alpha.$$ 

Theorem 2.55 (Dai-Chen [21]) Let $\xi$ and $\eta$ be independent uncertain variables. Then for any real numbers $a$ and $b$, we have

$$H[a\xi + b\eta] = |a| H[\xi] + |b| H[\eta]. \tag{2.226}$$

Proof: Without loss of generality, suppose $\xi$ and $\eta$ have regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular.

Step 1: We prove $H[a\xi] = |a| H[\xi]$. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a \Phi^{-1}(\alpha).$$

It follows from Theorem 2.53 that

$$H[a\xi] = \int_0^1 a \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \, d\alpha = a \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \, d\alpha = |a| H[\xi].$$

If $a = 0$, then we immediately have $H[a\xi] = 0 = |a| H[\xi]$. If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a \Phi^{-1}(1-\alpha).$$

It follows from Theorem 2.53 that

$$H[a\xi] = \int_0^1 a \Phi^{-1}(1-\alpha) \ln \frac{\alpha}{1-\alpha} \, d\alpha = (-a) \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \, d\alpha = |a| H[\xi].$$

Thus we always have $H[a\xi] = |a| H[\xi]$.

Step 2: We prove $H[\xi + \eta] = H[\xi] + H[\eta]$. Note that the inverse uncertainty distribution of $\xi + \eta$ is

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

It follows from Theorem 2.53 that

$$H[\xi + \eta] = \int_0^1 (\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \, d\alpha = H[\xi] + H[\eta].$$
Step 3: Finally, for any real numbers \(a\) and \(b\), it follows from Steps 1 and 2 that
\[
H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta].
\]
The theorem is proved.

**Example 2.23:** The independence condition in Theorem 2.55 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then \(\xi(\gamma) = \gamma\) is a linear uncertain variable \(\mathcal{L}(0, 1)\) with entropy
\[
H[\xi] = 0.5,
\]
and \(\eta(\gamma) = 1 - \gamma\) is also a linear uncertain variable \(\mathcal{L}(0, 1)\) with entropy
\[
H[\eta] = 0.5.
\]
Note that \(\xi\) and \(\eta\) are not independent, and \(\xi + \eta = 1\) whose entropy is
\[
H[\xi + \eta] = 0.
\]
Thus
\[
H[\xi + \eta] \neq H[\xi] + H[\eta].
\]
Therefore, the independence condition cannot be removed.

**Maximum Entropy Principle**

Given some constraints, for example, expected value and variance, there are usually multiple compatible uncertainty distributions. Which uncertainty distribution shall we take? The *maximum entropy principle* attempts to select the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints.

**Theorem 2.56** (Chen-Dai [9]) Let \(\xi\) be an uncertain variable whose uncertainty distribution is arbitrary but the expected value \(e\) and variance \(\sigma^2\). Then
\[
H[\xi] \leq \frac{\pi\sigma}{\sqrt{3}}
\]
and the equality holds if \(\xi\) is a normal uncertain variable \(\mathcal{N}(e, \sigma)\).

**Proof:** Let \(\Phi(x)\) be the uncertainty distribution of \(\xi\) and write \(\Psi(x) = \Phi(2e - x)\) for \(x \geq e\). It follows from the stipulation (2.2) and the change of variable of integral that the variance is
\[
V[\xi] = 2\int_{e}^{+\infty} (x - e)(1 - \Phi(x))dx + 2\int_{e}^{+\infty} (x - e)\Psi(x)dx = \sigma^2.
\]
Thus there exists a real number $\kappa$ such that
\[
2 \int_{e}^{+\infty} (x-e)(1-\Phi(x))dx = \kappa \sigma^2,
\]
\[
2 \int_{e}^{+\infty} (x-e)\Psi(x)dx = (1-\kappa)\sigma^2.
\]
The maximum entropy distribution $\Phi$ should maximize the entropy
\[
H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx = \int_{e}^{+\infty} S(\Phi(x))dx + \int_{e}^{+\infty} S(\Psi(x))dx
\]
subject to the above two constraints. The Lagrangian is
\[
L = \int_{e}^{+\infty} S(\Phi(x))dx + \int_{e}^{+\infty} S(\Psi(x))dx
\]
\[
-\alpha \left( 2 \int_{e}^{+\infty} (x-e)(1-\Phi(x))dx - \kappa \sigma^2 \right)
\]
\[
-\beta \left( 2 \int_{e}^{+\infty} (x-e)\Psi(x)dx - (1-\kappa)\sigma^2 \right).
\]
The maximum entropy distribution meets Euler-Lagrange equations
\[
\ln \Phi(x) - \ln(1-\Phi(x)) = 2\alpha(x-e),
\]
\[
\ln \Psi(x) - \ln(1-\Psi(x)) = 2\beta(e-x).
\]
Thus $\Phi$ and $\Psi$ have the forms
\[
\Phi(x) = \left( 1 + \exp(2\alpha(e-x)) \right)^{-1},
\]
\[
\Psi(x) = \left( 1 + \exp(2\beta(x-e)) \right)^{-1}.
\]
Substituting them into the variance constraints, we get
\[
\Phi(x) = \left( 1 + \exp \left( \frac{\pi(e-x)}{\sqrt{6\kappa\sigma}} \right) \right)^{-1},
\]
\[
\Psi(x) = \left( 1 + \exp \left( \frac{\pi(x-e)}{\sqrt{6(1-\kappa)\sigma}} \right) \right)^{-1}.
\]
Then the entropy is
\[
H[\xi] = \frac{\pi\sigma\sqrt{\kappa}}{\sqrt{6}} + \frac{\pi\sigma\sqrt{1-\kappa}}{\sqrt{6}}
\]
which achieves the maximum when $\kappa = 1/2$. Thus the maximum entropy distribution is just the normal uncertainty distribution $\mathcal{N}(e, \sigma)$. 
2.13 Conditional Uncertainty Distribution

Definition 2.21 (Liu [88]) The conditional uncertainty distribution $\Phi$ of an uncertain variable $\xi$ given $A$ is defined by

$$\Phi(x|A) = M\{\xi \leq x|A\}$$

(2.232)

provided that $M\{A\} > 0$.

Theorem 2.57 (Liu [95]) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi(x)$, and let $t$ be a real number with $\Phi(t) < 1$. Then the conditional uncertainty distribution of $\xi$ given $\xi > t$ is

$$\Phi(x|(t, +\infty)) = \begin{cases} 
0, & \text{if } \Phi(x) \leq \Phi(t) \\
\frac{\Phi(x)}{1 - \Phi(t)} \land 0.5, & \text{if } \Phi(t) < \Phi(x) \leq (1 + \Phi(t))/2 \\
\frac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}, & \text{if } (1 + \Phi(t))/2 < \Phi(x) \end{cases}$$

Proof: It follows from $\Phi(x|(t, +\infty)) = M\{\xi \leq x|\xi > t\}$ and the definition of conditional uncertain measure that

$$\Phi(x|(t, +\infty)) = \begin{cases} 
\frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}}, & \text{if } \frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} < 0.5 \\
1 - \frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}}, & \text{if } \frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}} < 0.5 \\
0.5, & \text{otherwise}.
\end{cases}$$

When $\Phi(x) \leq \Phi(t)$, we have $x \leq t$, and

$$\frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{M\{\emptyset\}}{1 - \Phi(t)} = 0 < 0.5.$$ 

Thus

$$\Phi(x|(t, +\infty)) = \frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} = 0.$$

When $\Phi(t) < \Phi(x) \leq (1 + \Phi(t))/2$, we have $x > t$, and

$$\frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{M\{t < \xi \leq x\}}{M\{\xi > t\}} \leq \frac{\Phi(x)}{1 - \Phi(t)},$$

$$\frac{M\{(\xi > x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{1 - \Phi(x)}{1 - \Phi(t)} \geq \frac{1 - (1 + \Phi(t))/2}{1 - \Phi(t)} = 0.5.$$ 

If $\Phi(x)/(1 - \Phi(t)) < 0.5$, then the maximum uncertainty principle implies

$$\Phi(x|(t, +\infty)) = \frac{M\{(\xi \leq x) \cap (\xi > t)\}}{M\{\xi > t\}} = \frac{\Phi(x)}{1 - \Phi(t)}.$$
If $\Phi(x)/(1 - \Phi(t)) \geq 0.5$, then we have to assign
\[
\Phi(x|(t, +\infty)) = 0.5.
\]
The above two results can be combined into
\[
\Phi(x|(t, +\infty)) = \frac{\Phi(x)}{1 - \Phi(t)} \wedge 0.5.
\]
When $(1 + \Phi(t))/2 < \Phi(x)$, we have $x > t$, and
\[
\frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1 - \Phi(x)}{1 - \Phi(t)} < \frac{1 - (1 + \Phi(t))/2}{1 - \Phi(t)} = 0.5.
\]
Thus
\[
\Phi(x|(t, +\infty)) = 1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 1 - \frac{1 - \Phi(x)}{1 - \Phi(t)} = \frac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}.
\]
The theorem is proved.

**Exercise 2.84:** Let $\xi$ be a linear uncertain variable $\mathcal{L}(a, b)$, and let $t$ be a real number with $a < t < b$. Show that the conditional uncertainty distribution of $\xi$ given $\xi > t$ is
\[
\Phi(x|(t, +\infty)) = \begin{cases} 
0, & \text{if } x \leq t \\
\frac{x - a}{b - t} \wedge 0.5, & \text{if } t < x \leq (b + t)/2 \\
\frac{x - t}{b - t} \wedge 1, & \text{if } (b + t)/2 \leq x.
\end{cases}
\]

![Figure 2.15: Conditional Uncertainty Distribution $\Phi(x|(t, +\infty))$](image)
**Theorem 2.58** (Liu [95]) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi(x)$, and let $t$ be a real number with $\Phi(t) > 0$. Then the conditional uncertainty distribution of $\xi$ given $\xi \leq t$ is

$$
\Phi(x|(-\infty, t]) = \begin{cases} 
\frac{\Phi(x)}{\Phi(t)}, & \text{if } \Phi(x) < \Phi(t)/2 \\
\frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)} \vee 0.5, & \text{if } \Phi(t)/2 \leq \Phi(x) < \Phi(t) \\
1, & \text{if } \Phi(t) \leq \Phi(x).
\end{cases}
$$

**Proof:** It follows from $\Phi(x|(-\infty, t]) = M\{\xi \leq x|\xi \leq t\}$ and the definition of conditional uncertain measure that

$$
\Phi(x|(-\infty, t]) = \begin{cases} 
\frac{M\{(\xi \leq x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}}, & \text{if } \frac{M\{(\xi \leq x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} < 0.5 \\
1 - \frac{M\{(\xi > x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}}, & \text{if } \frac{M\{(\xi > x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
$$

When $\Phi(x) < \Phi(t)/2$, we have $x < t$, and

$$
\frac{M\{(\xi \leq x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} = \frac{\Phi(x)}{\Phi(t)} < \frac{\Phi(t)/2}{\Phi(t)} = 0.5.
$$

Thus

$$
\Phi(x|(-\infty, t]) = \frac{M\{(\xi \leq x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} = \frac{\Phi(x)}{\Phi(t)}.
$$

When $\Phi(t)/2 \leq \Phi(x) < \Phi(t)$, we have $x < t$, and

$$
\frac{M\{(\xi \leq x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} = \frac{\Phi(x)}{\Phi(t)} > \frac{\Phi(t)/2}{\Phi(t)} = 0.5,
$$

$$
\frac{M\{(\xi > x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} = \frac{M\{x < \xi \leq t\}}{M\{\xi \leq t\}} \leq \frac{1 - \Phi(x)}{\Phi(t)}.
$$

i.e.,

$$
1 - \frac{M\{(\xi > x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} \geq \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)}.
$$

If $(\Phi(x) + \Phi(t) - 1)/\Phi(t) > 0.5$, then the maximum uncertainty principle implies

$$
\Phi(x|(-\infty, t]) = 1 - \frac{M\{(\xi > x) \cap (\xi \leq t)\}}{M\{\xi \leq t\}} = \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)}.
$$

If $(\Phi(x) + \Phi(t) - 1)/\Phi(t) \leq 0.5$, then we have to assign

$$
\Phi(x|(-\infty, t)) = 0.5.
$$
The above two results can be combined into
\[ \Phi(x|(-\infty, t]) = \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)} \vee 0.5. \]

When \( \Phi(t) \leq \Phi(x) \), we have \( x \geq t \), and
\[ \frac{\mathcal{M}\{ (\xi > x) \cap (\xi \leq t) \}}{\mathcal{M}\{ \xi \leq t \}} = \frac{\mathcal{M}\{ \emptyset \}}{\Phi(t)} = 0 < 0.5. \]
Thus
\[ \Phi(x|(-\infty, t]) = 1 - \frac{\mathcal{M}\{ (\xi > x) \cap (\xi \leq t) \}}{\mathcal{M}\{ \xi \leq t \}} = 1 - 0 = 1. \]
The theorem is proved.

**Exercise 2.85:** Let \( \xi \) be a linear uncertain variable \( \mathcal{L}(a, b) \), and let \( t \) be a real number with \( a < t < b \). Show that the conditional uncertainty distribution of \( \xi \) given \( \xi \leq t \) is
\[ \Phi(x|(-\infty, t]) = \begin{cases} 
\frac{x - a}{t - a} \vee 0, & \text{if } x \leq (a + t)/2 \\
\left(1 - \frac{b - x}{t - a}\right) \vee 0.5, & \text{if } (a + t)/2 \leq x < t \\
1, & \text{if } x \geq t.
\end{cases} \]

\[ \Phi(x|(-\infty, t]) \]

Figure 2.16: Conditional Uncertainty Distribution \( \Phi(x|(-\infty, t]) \)

### 2.14 Uncertain Sequence

Uncertain sequence is a sequence of uncertain variables indexed by integers. This section introduces four convergence concepts of uncertain sequence: convergence almost surely (a.s.), convergence in measure, convergence in mean, and convergence in distribution.
Table 2.1: Relationship among Convergence Concepts

<table>
<thead>
<tr>
<th>Convergence in Mean</th>
<th>Convergence in Measure</th>
<th>Convergence in Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
<td></td>
</tr>
<tr>
<td>Convergence Almost Surely</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Definition 2.22** (Liu [88]) The uncertain sequence $\{\xi_i\}$ is said to be convergent a.s. to $\xi$ if there exists an event $\Lambda$ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$
\lim_{i \to \infty} |\xi_i(\gamma) - \xi(\gamma)| = 0
$$

(2.233)

for every $\gamma \in \Lambda$. In that case we write $\xi_i \to \xi$, a.s.

**Definition 2.23** (Liu [88]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in measure to $\xi$ if

$$
\lim_{i \to \infty} \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = 0
$$

(2.234)

for every $\varepsilon > 0$.

**Definition 2.24** (Liu [88]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in mean to $\xi$ if

$$
\lim_{i \to \infty} \mathbb{E}[|\xi_i - \xi|] = 0.
$$

(2.235)

**Definition 2.25** (Liu [88]) Let $\Phi, \Phi_1, \Phi_2, \cdots$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. We say the uncertain sequence $\{\xi_i\}$ converges in distribution to $\xi$ if

$$
\lim_{i \to \infty} \Phi_i(x) = \Phi(x)
$$

(2.236)

for all $x$ at which $\Phi(x)$ is continuous.

**Convergence in Mean vs. Convergence in Measure**

**Theorem 2.59** (Liu [88]) If the uncertain sequence $\{\xi_i\}$ converges in mean to $\xi$, then $\{\xi_i\}$ converges in measure to $\xi$.

**Proof:** Since $\{\xi_i\}$ converges in mean to $\xi$, we have $\mathbb{E}[|\xi_i - \xi|] \to 0$ as $i \to \infty$. For any given number $\varepsilon > 0$, it follows from Markov inequality that

$$
\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} \leq \frac{\mathbb{E}[|\xi_i - \xi|]}{\varepsilon} \to 0
$$

as $i \to \infty$. Thus $\{\xi_i\}$ converges in measure to $\xi$. The theorem is proved.
Example 2.24: Convergence in measure does not imply convergence in mean. Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2, \cdots\}\) with power set and
\[
\mathcal{M}\{A\} = \sum_{\gamma_j \in A} \frac{1}{2^j}.
\]
Define uncertain variables as
\[
\xi_i(\gamma_j) = \begin{cases} 
2^i, & \text{if } j = i \\
0, & \text{otherwise}
\end{cases}
\]
for \(i = 1, 2, \cdots\) and \(\xi \equiv 0\). For any small number \(\varepsilon > 0\), we have
\[
\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{2^i} \rightarrow 0
\]
as \(i \rightarrow \infty\). That is, the sequence \(\{\xi_i\}\) converges in measure to \(\xi\). However, for each \(i\), we have
\[
E[|\xi_i - \xi|] = 1.
\]
That is, the sequence \(\{\xi_i\}\) does not converge in mean to \(\xi\).

Convergence in Measure vs. Convergence in Distribution

Theorem 2.60 (Liu [88]) If the uncertain sequence \(\{\xi_i\}\) converges in measure to \(\xi\), then \(\{\xi_i\}\) converges in distribution to \(\xi\).

Proof: Let \(x\) be a continuity point of the uncertainty distribution \(\Phi\). On the one hand, for any \(y > x\), we have
\[
\{\xi_i \leq x\} = \{\xi_i \leq x, \xi \leq y\} \cup \{\xi_i \leq x, \xi > y\} \subseteq \{\xi \leq y\} \cup \{|\xi_i - \xi| \geq y - x\}.
\]
It follows from the subadditivity axiom that
\[
\Phi_i(x) \leq \Phi(y) + \mathcal{M}\{|\xi_i - \xi| \geq y - x\}.
\]
Since \(\{\xi_i\}\) converges in measure to \(\xi\), we have \(\mathcal{M}\{|\xi_i - \xi| \geq y - x\} \rightarrow 0\) as \(i \rightarrow \infty\). Thus we obtain \(\limsup_{i \rightarrow \infty} \Phi_i(x) \leq \Phi(y)\) for any \(y > x\). Letting \(y \rightarrow x\), we get
\[
\limsup_{i \rightarrow \infty} \Phi_i(x) \leq \Phi(x).
\]
On the other hand, for any \(z < x\), we have
\[
\{\xi \leq z\} = \{\xi_i \leq x, \xi \leq z\} \cup \{\xi_i > x, \xi \leq z\} \subseteq \{\xi_i \leq x\} \cup \{|\xi_i - \xi| \geq x - z\}
\]
which implies that
\[
\Phi(z) \leq \Phi_i(x) + \mathcal{M}\{|\xi_i - \xi| \geq x - z\}.
\]
Since $M\{|\xi_i - \xi| \geq x - z\} \to 0$, we obtain $\Phi(z) \leq \lim\inf_{i \to \infty} \Phi_i(x)$ for any $z < x$. Letting $z \to x$, we get

$$\Phi(x) \leq \lim\inf_{i \to \infty} \Phi_i(x). \tag{2.238}$$

It follows from (2.237) and (2.238) that $\Phi_i(x) \to \Phi(x)$ as $i \to \infty$. The theorem is proved.

**Example 2.25:** Convergence in distribution does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2\}$ with power set and $M\{\gamma_1\} = M\{\gamma_2\} = 1/2$. Define uncertain variables as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2, \end{cases}$$

and $\xi_i = -\xi$ for $i = 1, 2, \cdots$. Then $\xi_i$ and $\xi$ have the same uncertainty distribution. Thus $\{\xi_i\}$ converges in distribution to $\xi$. However, for some small number $\varepsilon > 0$, we have

$$M\{|\xi_i - \xi| \geq \varepsilon\} = 1.$$ 

That is, the sequence $\{\xi_i\}$ does not converge in measure to $\xi$.

**Convergence Almost Surely vs. Convergence in Measure**

**Example 2.26:** Convergence a.s. does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and

$$M\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise}. \end{cases}$$

Define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then the sequence $\{\xi_i\}$ converges a.s. to $\xi$. However, for some small number $\varepsilon > 0$, we have

$$M\{|\xi_i - \xi| \geq \varepsilon\} = 0.5$$

for each $i$. That is, the sequence $\{\xi_i\}$ does not converge in measure to $\xi$.

**Example 2.27:** Convergence in measure does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $[0, 1]$ with Borel algebra and
Lebesgue measure. For any positive integer $i$, there is an integer $j$ such that $i = 2^j + k$, where $k$ is an integer between 0 and $2^j - 1$. Define uncertain variables as

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \leq \gamma \leq (k + 1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then for any small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in measure to $\xi$. However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[k/2^j, (k + 1)/2^j]$ containing $\gamma$. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to $\xi$.

**Convergence Almost Surely vs. Convergence in Mean**

**Example 2.28:** Convergence a.s. does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_j \in \Lambda} \frac{1}{2^j}.$$  

Define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} 2^i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then $\xi_i$ converges a.s. to $\xi$. However, the sequence $\{\xi_i\}$ does not converge in mean to $\xi$ because $E[|\xi_i - \xi|] = 1$ for each $i$.

**Example 2.29:** Convergence in mean does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer $i$, there is an integer $j$ such that $i = 2^j + k$, where $k$ is an integer between 0 and $2^j - 1$. Define uncertain variables as

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \leq \gamma \leq (k + 1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then

$$E[|\xi_i - \xi|] = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in mean to $\xi$. However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[k/2^j, (k + 1)/2^j]$ containing $\gamma$. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to $\xi$. 
Convergence Almost Surely vs. Convergence in Distribution

**Example 2.30:** Convergence in distribution does not imply convergence a.s. Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2\}\) with power set and \(\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2\). Define uncertain variables as
\[
\xi(\gamma) = \begin{cases} 
-1, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2
\end{cases}
\]
and \(\xi_i = -\xi\) for \(i = 1, 2, \cdots\). Then \(\xi_i\) and \(\xi\) have the same uncertainty distribution. Thus \(\{\xi_i\}\) converges in distribution to \(\xi\). However, the sequence \(\{\xi_i\}\) does not converge a.s. to \(\xi\).

**Example 2.31:** Convergence a.s. does not imply convergence in distribution. Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2, \cdots\}\) with power set and
\[
\mathcal{M}\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
0.5, & \text{otherwise}
\end{cases}
\]
Define uncertain variables as
\[
\xi_i(\gamma_j) = \begin{cases} 
i, & \text{if } j = i \\
0, & \text{otherwise}
\end{cases}
\]
for \(i = 1, 2, \cdots\) and \(\xi \equiv 0\). Then the sequence \(\{\xi_i\}\) converges a.s. to \(\xi\). However, the uncertainty distributions of \(\xi_i\) are
\[
\Phi_i(x) = \begin{cases} 
0, & \text{if } x < 0 \\
0.5, & \text{if } 0 \leq x < i \\
1, & \text{if } x \geq i
\end{cases}
\]
for \(i = 1, 2, \cdots\), respectively, and the uncertainty distribution of \(\xi\) is
\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1, & \text{if } x \geq 0.
\end{cases}
\]
It is clear that \(\Phi_i(x)\) does not converge to \(\Phi(x)\) at \(x > 0\). That is, the sequence \(\{\xi_i\}\) does not converge in distribution to \(\xi\).

### 2.15 Uncertain Vector

As an extension of uncertain variable, this section introduces a concept of uncertain vector whose components are uncertain variables.
Definition 2.26 (Liu [88]) A k-dimensional uncertain vector is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k-dimensional real vectors such that $\{\xi \in B\}$ is an event for any Borel set $B$ of k-dimensional real vectors.

Theorem 2.61 (Liu [88]) The vector $(\xi_1, \xi_2, \cdots, \xi_k)$ is an uncertain vector if and only if $\xi_1, \xi_2, \cdots, \xi_k$ are uncertain variables.

Proof: Write $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$. Suppose that $\xi$ is an uncertain vector on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set $B$ of real numbers, the set $B \times \mathbb{R}^{k-1}$ is a Borel set of k-dimensional real vectors. Thus the set

$$\{\xi_1 \in B\} = \{\xi_1 \in B, \xi_2 \in \mathbb{R}, \cdots, \xi_k \in \mathbb{R}\} = \{\xi \in B \times \mathbb{R}^{k-1}\}$$

is an event. Hence $\xi_1$ is an uncertain variable. A similar process may prove that $\xi_2, \xi_3, \cdots, \xi_k$ are uncertain variables.

Conversely, suppose that all $\xi_1, \xi_2, \cdots, \xi_k$ are uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We define

$$B = \{B \subset \mathbb{R}^k \mid \{\xi \in B\} \text{ is an event}\}.$$

The vector $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$ is proved to be an uncertain vector if we can prove that $B$ contains all Borel sets of k-dimensional real vectors. First, the class $B$ contains all open intervals of $\mathbb{R}^k$ because

$$\left\{\xi \in \prod_{i=1}^{k}(a_i, b_i)\right\} = \bigcap_{i=1}^{k}\{\xi_i \in (a_i, b_i)\}$$

is an event. Next, the class $B$ is a $\sigma$-algebra over $\mathbb{R}^k$ because (i) we have $\mathbb{R}^k \in B$ since $\{\xi \in \mathbb{R}^k\} = \Gamma$; (ii) if $B \in B$, then $\{\xi \in B\}$ is an event, and

$$\{\xi \in B^c\} = \{\xi \in B\}^c$$

is an event. This means that $B^c \in B$; (iii) if $B_i \in B$ for $i = 1, 2, \cdots$, then $\{\xi \in B_i\}$ are events and

$$\left\{\xi \in \bigcup_{i=1}^{\infty} B_i\right\} = \bigcup_{i=1}^{\infty}\{\xi \in B_i\}$$

is an event. This means that $\bigcup_i B_i \in B$. Since the smallest $\sigma$-algebra containing all open intervals of $\mathbb{R}^k$ is just the Borel algebra over $\mathbb{R}^k$, the class $B$ contains all Borel sets of k-dimensional real vectors. The theorem is proved.

Definition 2.27 (Liu [103]) The k-dimensional uncertain vectors $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if for any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, we have

$$\mathcal{M}\left(\bigcap_{i=1}^{n}(\xi_i \in B_i)\right) = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}.$$ (2.239)
Exercise 2.86: Let \((\xi_1, \xi_2, \xi_3)\) and \((\eta_1, \eta_2, \eta_3)\) be independent uncertain vectors. Show that \(\xi_1\) and \(\eta_2\) are independent uncertain variables.

Exercise 2.87: Let \((\xi_1, \xi_2, \xi_3)\) and \((\eta_1, \eta_2, \eta_3)\) be independent uncertain vectors. Show that \((\xi_1; \xi_2)\) and \((\eta_2; \eta_3)\) are independent uncertain vectors.

Theorem 2.62 (Liu [103]) The \(k\)-dimensional uncertain vectors \(\xi_1, \xi_2, \ldots, \xi_n\) are independent if and only if

\[
\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \quad (2.240)
\]

for any Borel sets \(B_1, B_2, \ldots, B_n\) of \(k\)-dimensional real vectors.

Proof: It follows from the duality of uncertain measure that \(\xi_1, \xi_2, \ldots, \xi_n\) are independent if and only if

\[
\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i^c)\right\} = 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}.
\]

The theorem is thus proved.

Theorem 2.63 Let \(\xi_1, \xi_2, \ldots, \xi_n\) be independent uncertain vectors, and let \(f_1, f_2, \ldots, f_n\) be vector-valued measurable functions. Then \(f_1(\xi_1), f_2(\xi_2), \ldots, f_n(\xi_n)\) are also independent uncertain vectors.

Proof: For any Borel sets \(B_1, B_2, \ldots, B_n\) of \(k\)-dimensional real vectors, it follows from the definition of independence that

\[
\mathcal{M}\left\{\bigcap_{i=1}^{n}(f_i(\xi_i) \in B_i)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in f_i^{-1}(B_i))\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{f_i(\xi_i) \in B_i\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in f_i^{-1}(B_i)\}.
\]

Thus \(f_1(\xi_1), f_2(\xi_2), \ldots, f_n(\xi_n)\) are independent uncertain variables.

2.16 Bibliographic Notes

As a fundamental concept in uncertainty theory, the uncertain variable was presented by Liu [88] in 2007. In order to describe uncertain variable, Liu [88] also introduced the uncertainty distribution. Later, Peng-Iwamura [137] proved a sufficient and necessary condition for uncertainty distribution. In
addition, Liu [95] proposed the inverse uncertainty distribution, and Liu [100] verified a sufficient and necessary condition for it. Furthermore, Liu [88] proposed the conditional uncertainty distribution, and derived some formulas for calculating it.

Following the independence concept of uncertain variables proposed by Liu [91], the operational law was given by Liu [95] for calculating the uncertainty distribution and inverse uncertainty distribution of strictly monotone function of independent uncertain variables.

In order to rank uncertain variables, Liu [88] proposed the expected value operator. In addition, the linearity of expected value operator was verified by Liu [95]. As an important contribution, Liu-Ha [115] derived a useful formula for calculating the expected values of strictly monotone functions of independent uncertain variables. Based on the expected value operator, Liu [88] presented the variance, moments and distance between uncertain variables.

The entropy was proposed by Liu [91] as the degree of difficulty of predicting the realization of an uncertain variable. Chen-Dai [9] discussed the maximum entropy principle in order to select the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints. Especially, normal uncertainty distribution is proved to have maximum entropy when the expected value and variance are fixed in advance.

Uncertain sequence was presented by Liu [88] with convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution. Furthermore, Gao [46], You [207], Zhang [222], and Chen-Li-Ralescu [16] developed some other concepts of convergence and investigated their mathematical properties.

Uncertain vector was defined by Liu [88] as a measurable function from an uncertainty space to the set of real vectors. In addition, Liu [103] discussed the independence of uncertain vectors.
Chapter 3

Uncertain Programming

Uncertain programming was founded by Liu [90] in 2009. This chapter will provide the theory of uncertain programming, and present some uncertain programming models for machine scheduling problem, vehicle routing problem, and project scheduling problem.

3.1 Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Assume that $x$ is a decision vector, and $\xi$ is an uncertain vector. Since an uncertain objective function $f(x, \xi)$ cannot be directly minimized, we may minimize its expected value, i.e.,

$$\min_{x} E[f(x, \xi)].$$

(3.1)

In addition, since the uncertain constraints $g_j(x, \xi) \leq 0, j = 1, 2, \cdots, p$ do not define a crisp feasible set, it is naturally desired that the uncertain constraints hold with confidence levels $\alpha_1, \alpha_2, \cdots, \alpha_p$. Then we have a set of chance constraints,

$$M\{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p.$$  

(3.2)

In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu [90] proposed the following uncertain programming model,

$$\begin{cases}
\min_{x} E[f(x, \xi)] \\
\text{subject to:} \\
M\{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p.
\end{cases}$$

(3.3)
Definition 3.1 (Liu [90]) A vector \( \mathbf{x} \) is called a feasible solution to the uncertain programming model (3.3) if
\[
\mathcal{M}\{g_j(\mathbf{x}, \mathbf{\xi}) \leq 0\} \geq \alpha_j
\]
for \( j = 1, 2, \cdots, p \).

Definition 3.2 (Liu [90]) A feasible solution \( \mathbf{x}^* \) is called an optimal solution to the uncertain programming model (3.3) if
\[
E[f(\mathbf{x}^*, \mathbf{\xi})] \leq E[f(\mathbf{x}, \mathbf{\xi})]
\]
for any feasible solution \( \mathbf{x} \).

Theorem 3.1 Assume the objective function \( f(\mathbf{x}, \xi_1, \xi_2, \cdots, \xi_n) \) is continuous, strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_m \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \cdots, \xi_n \). If \( \xi_1, \xi_2, \cdots, \xi_n \) are independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively, then the expected objective function \( E[f(\mathbf{x}, \xi_1, \xi_2, \cdots, \xi_n)] \) is equal to
\[
\int_0^1 f(\mathbf{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))d\alpha.
\]

Proof: It follows from Theorem 2.27 immediately.

Exercise 3.1: Assume \( f(\mathbf{x}, \mathbf{\xi}) = h_1(\mathbf{x})\xi_1 + h_2(\mathbf{x})\xi_2 + \cdots + h_n(\mathbf{x})\xi_n + h_0(\mathbf{x}) \) where \( h_1(\mathbf{x}), h_2(\mathbf{x}), \cdots, h_n(\mathbf{x}), h_0(\mathbf{x}) \) are real-valued functions and \( \xi_1, \xi_2, \cdots, \xi_n \) are independent uncertain variables. Show that
\[
E[f(\mathbf{x}, \mathbf{\xi})] = h_1(\mathbf{x})E[\xi_1] + h_2(\mathbf{x})E[\xi_2] + \cdots + h_n(\mathbf{x})E[\xi_n] + h_0(\mathbf{x}).
\]

Theorem 3.2 Assume the constraint function \( g(\mathbf{x}, \xi_1, \xi_2, \cdots, \xi_n) \) is continuous, strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_k \) and strictly decreasing with respect to \( \xi_{k+1}, \xi_{k+2}, \cdots, \xi_n \). If \( \xi_1, \xi_2, \cdots, \xi_n \) are independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively, then the chance constraint
\[
\mathcal{M}\{g(\mathbf{x}, \xi_1, \xi_2, \cdots, \xi_n) \leq 0\} \geq \alpha
\]
holds if and only if
\[
g(\mathbf{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \leq 0.
\]

Proof: It follows from the operational law of uncertain variables that the inverse uncertainty distribution of \( g(\mathbf{x}, \xi_1, \xi_2, \cdots, \xi_n) \) is
\[
\Psi^{-1}(\alpha) = g(\mathbf{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).
\]
Thus (3.8) holds if and only if \( \Psi^{-1}(\alpha) \leq 0 \). The theorem is thus verified.
Exercise 3.2: Assume $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n, \xi$ are independent linear uncertain variables $\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2), \cdots, \mathcal{L}(a_n, b_n), \mathcal{L}(a, b)$, respectively. Show that for any confidence level $\alpha \in (0, 1)$, the chance constraint
\[
\mathcal{M}\left\{ \sum_{i=1}^{n} \xi_i x_i \leq \xi \right\} \geq \alpha
\]  
holds if and only if
\[
\sum_{i=1}^{n} ((1 - \alpha)a_i + \alpha b_i)x_i \leq \alpha a + (1 - \alpha)b.
\]  

Exercise 3.3: Assume $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n, \xi$ are independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1), \mathcal{N}(e_2, \sigma_2), \cdots, \mathcal{N}(e_n, \sigma_n), \mathcal{N}(e, \sigma)$, respectively. Show that for any confidence level $\alpha \in (0, 1)$, the chance constraint
\[
\mathcal{M}\left\{ \sum_{i=1}^{n} \xi_i x_i \leq \xi \right\} \geq \alpha
\]  
holds if and only if
\[
\sum_{i=1}^{n} \left( e_i + \frac{\sigma_i \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) x_i \leq e - \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]  

Exercise 3.4: Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $h_1(x), h_2(x), \cdots, h_n(x), h_0(x)$ are real-valued functions. Show that
\[
\mathcal{M}\left\{ \sum_{i=1}^{n} h_i(x) \xi_i \leq h_0(x) \right\} \geq \alpha
\]  
holds if and only if
\[
\sum_{i=1}^{n} h_i^+(x)\Phi_i^{-1}(\alpha) - \sum_{i=1}^{n} h_i^-(x)\Phi_i^{-1}(1 - \alpha) \leq h_0(x)
\]  
where
\[
h_i^+(x) = \begin{cases} h_i(x), & \text{if } h_i(x) > 0 \\ 0, & \text{if } h_i(x) \leq 0, \end{cases}
\]
\[
h_i^-(x) = \begin{cases} -h_i(x), & \text{if } h_i(x) < 0 \\ 0, & \text{if } h_i(x) \geq 0 \end{cases}
\]  
for $i = 1, 2, \cdots, n.$
Theorem 3.3 Assume \( f(x, \xi_1, \xi_2, \cdots, \xi_n) \) is continuous, strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_n \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \cdots, \xi_n \), and \( g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \) are continuous, strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_k \) and strictly decreasing with respect to \( \xi_{k+1}, \xi_{k+2}, \cdots, \xi_n \) for \( j = 1, 2, \cdots, p \). If \( \xi_1, \xi_2, \cdots, \xi_n \) are independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively, then the uncertain programming

\[
\begin{align*}
\min_x & \ E[f(x, \xi_1, \xi_2, \cdots, \xi_n)] \\
\text{subject to:} & \quad M\{g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p
\end{align*}
\]

is equivalent to the crisp mathematical programming

\[
\begin{align*}
\min_x & \ \int_0^1 f(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))d\alpha \\
\text{subject to:} & \quad g_j(x, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \Phi_{k+1}^{-1}(1-\alpha_j), \cdots, \Phi_n^{-1}(1-\alpha_j)) \leq 0, \quad j = 1, 2, \cdots, p.
\end{align*}
\]

Proof: It follows from Theorems 3.1 and 3.2 immediately.

3.2 Numerical Method

When the objective functions and constraint functions are monotone with respect to the uncertain parameters, the uncertain programming model may be converted to a crisp mathematical programming.

It is fortunate for us that almost all objective and constraint functions in practical problems are indeed monotone with respect to the uncertain parameters (not decision variables).

From the mathematical viewpoint, there is no difference between crisp mathematical programming and classical mathematical programming except for an integral. Thus we may solve it by simplex method, branch-and-bound method, cutting plane method, implicit enumeration method, interior point method, gradient method, genetic algorithm, particle swarm optimization, neural networks, tabu search, and so on.

Example 3.1: Assume that \( x_1, x_2, x_3 \) are nonnegative decision variables, \( \xi_1, \xi_2, \xi_3 \) are independent linear uncertain variables \( \mathcal{L}(1, 2), \mathcal{L}(2, 3), \mathcal{L}(3, 4) \), and \( \eta_1, \eta_2, \eta_3 \) are independent zigzag uncertain variables \( \mathcal{Z}(1, 2, 3), \mathcal{Z}(2, 3, 4) \),
Consider the uncertain programming,

$$\max_{x_1, x_2, x_3} \mathbb{E} \left[ \sqrt{x_1 + \xi_1} + \sqrt{x_2 + \xi_2} + \sqrt{x_3 + \xi_3} \right]$$

subject to:

$$\mathcal{M}\{(x_1 + \eta_1)^2 + (x_2 + \eta_2)^2 + (x_3 + \eta_3)^2 \leq 100\} \geq 0.9$$

$$x_1, x_2, x_3 \geq 0.$$ 

Note that $\sqrt{x_1 + \xi_1} + \sqrt{x_2 + \xi_2} + \sqrt{x_3 + \xi_3}$ is a strictly increasing function with respect to $\xi_1, \xi_2, \xi_3$, and $(x_1 + \eta_1)^2 + (x_2 + \eta_2)^2 + (x_3 + \eta_3)^2$ is a strictly increasing function with respect to $\eta_1, \eta_2, \eta_3$. It is easy to verify that the uncertain programming model can be converted to the crisp model,

$$\max_{x_1, x_2, x_3} \int_0^1 \left( \sqrt{x_1 + \Phi_1^{-1}(\alpha)} + \sqrt{x_2 + \Phi_2^{-1}(\alpha)} + \sqrt{x_3 + \Phi_3^{-1}(\alpha)} \right) d\alpha$$

subject to:

$$(x_1 + \Psi_1^{-1}(0.9))^2 + (x_2 + \Psi_2^{-1}(0.9))^2 + (x_3 + \Psi_3^{-1}(0.9))^2 \leq 100$$

$$x_1, x_2, x_3 \geq 0$$

where $\Phi_1^{-1}, \Phi_2^{-1}, \Phi_3^{-1}, \Psi_1^{-1}, \Psi_2^{-1}, \Psi_3^{-1}$ are inverse uncertainty distributions of uncertain variables $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$, respectively. The optimal solution is

$$(x_1^*, x_2^*, x_3^*) = (2.9735, 1.9735, 0.9735)$$

whose objective value is 6.3419.

**Example 3.2:** Assume that $x_1$ and $x_2$ are decision variables, $\xi_1$ and $\xi_2$ are iid linear uncertain variables $\mathcal{L}(0, \pi/2)$. Consider the uncertain programming,

$$\min_{x_1, x_2} \mathbb{E} \left[ x_1 \sin(x_1 - \xi_1) - x_2 \cos(x_2 + \xi_2) \right]$$

subject to:

$$0 \leq x_1 \leq \frac{\pi}{2}, \quad 0 \leq x_2 \leq \frac{\pi}{2}.$$ 

It is clear that $x_1 \sin(x_1 - \xi_1) - x_2 \cos(x_2 + \xi_2)$ is strictly decreasing with respect to $\xi_1$ and strictly increasing with respect to $\xi_2$. Thus the uncertain programming is equivalent to the crisp model,

$$\min_{x_1, x_2} \int_0^1 \left( x_1 \sin(x_1 - \Phi_1^{-1}(1 - \alpha)) - x_2 \cos(x_2 + \Phi_2^{-1}(\alpha)) \right) d\alpha$$

subject to:

$$0 \leq x_1 \leq \frac{\pi}{2}, \quad 0 \leq x_2 \leq \frac{\pi}{2}$$

where $\Phi_1^{-1}, \Phi_2^{-1}$ are inverse uncertainty distributions of $\xi_1, \xi_2$, respectively. The optimal solution is

$$(x_1^*, x_2^*) = (0.4026, 0.4026)$$

whose objective value is $-0.2708.$


Chapter 3 - Uncertain Programming

3.3 Machine Scheduling Problem

Machine scheduling problem is concerned with finding an efficient schedule during an uninterrupted period of time for a set of machines to process a set of jobs. A lot of research work has been done on this type of problem. The study of machine scheduling problem with uncertain processing times was started by Liu [95] in 2010.

Machine

\[ \begin{array}{ccc} 
M_3 & J_6 & J_7 \\
M_2 & J_4 & J_5 \\
M_1 & J_1 & J_2 & J_3 
\end{array} \]

Time

\[ \text{Makespan} \]

Figure 3.1: A Machine Schedule with 3 Machines and 7 Jobs

In a machine scheduling problem, we assume that (a) each job can be processed on any machine without interruption; (b) each machine can process only one job at a time; and (c) the processing times are uncertain variables with known uncertainty distributions. We also use the following indices and parameters:

- \( i = 1, 2, \ldots, n \): jobs;
- \( k = 1, 2, \ldots, m \): machines;
- \( \xi_{ik} \): uncertain processing time of job \( i \) on machine \( k \);
- \( \Phi_{ik} \): uncertainty distribution of \( \xi_{ik} \).

How to Represent a Schedule?

Liu [86] suggested that a schedule should be represented by two decision vectors \( \mathbf{x} \) and \( \mathbf{y} \), where

- \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \): integer decision vector representing \( n \) jobs with \( 1 \leq x_i \leq n \) and \( x_i \neq x_j \) for all \( i \neq j, i, j = 1, 2, \ldots, n \). That is, the sequence \( \{x_1, x_2, \ldots, x_n\} \) is a rearrangement of \( \{1, 2, \ldots, n\} \);
- \( \mathbf{y} = (y_1, y_2, \ldots, y_{m-1}) \): integer decision vector with \( y_0 \equiv 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \equiv y_m \).

We note that the schedule is fully determined by the decision vectors \( \mathbf{x} \) and \( \mathbf{y} \) in the following way. For each \( k \) (\( 1 \leq k \leq m \)), if \( y_k = y_{k-1} \), then the machine \( k \) is not used; if \( y_k > y_{k-1} \), then the machine \( k \) is used and processes jobs \( x_{y_{k-1}+1}, x_{y_{k-1}+2}, \ldots, x_{y_k} \) in turn. Thus the schedule of all machines is
as follows,

Machine 1: \( x_{y_0+1} \rightarrow x_{y_0+2} \rightarrow \cdots \rightarrow x_{y_1} \);

Machine 2: \( x_{y_1+1} \rightarrow x_{y_1+2} \rightarrow \cdots \rightarrow x_{y_2} \);

\[ \vdots \]

Machine \( m \): \( x_{y_{m-1}+1} \rightarrow x_{y_{m-1}+2} \rightarrow \cdots \rightarrow x_{y_m} \).

(3.19)

\[ \begin{align*}
& y_0 \quad y_1 \quad y_2 \quad y_3 \\
& \begin{array}{cccc}
& x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
& \text{M-1} & \text{M-2} & \text{M-3} \\
\end{array}
\end{align*} \]

Figure 3.2: Formulation of Schedule in which Machine 1 processes Jobs \( x_1, x_2 \), Machine 2 processes Jobs \( x_3, x_4 \) and Machine 3 processes Jobs \( x_5, x_6, x_7 \).

Completion Times

Let \( C_i(x, y, \xi) \) be the completion times of jobs \( i, i = 1, 2, \ldots, n \), respectively. For each \( k \) with \( 1 \leq k \leq m \), if the machine \( k \) is used (i.e., \( y_k > y_{k-1} \)), then we have

\[ C_{x_{y_{k-1}+1}}(x, y, \xi) = \xi_{x_{y_{k-1}+1}k} \]  

(3.20)

and

\[ C_{x_{y_{k-1}+j}}(x, y, \xi) = C_{x_{y_{k-1}+j-1}}(x, y, \xi) + \xi_{x_{y_{k-1}+j}k} \]  

(3.21)

for \( 2 \leq j \leq y_k - y_{k-1} \).

If the machine \( k \) is used, then the completion time \( C_{x_{y_{k-1}+1}}(x, y, \xi) \) of job \( x_{y_{k-1}+1} \) is an uncertain variable whose inverse uncertainty distribution is

\[ \Psi^{-1}_{x_{y_{k-1}+1}}(x, y, \alpha) = \Phi^{-1}_{x_{y_{k-1}+1}k}(\alpha). \]  

(3.22)

Generally, suppose the completion time \( C_{x_{y_{k-1}+j-1}}(x, y, \xi) \) has an inverse uncertainty distribution \( \Psi^{-1}_{x_{y_{k-1}+j-1}}(x, y, \alpha) \). Then the completion time \( C_{x_{y_{k-1}+j}}(x, y, \xi) \) has an inverse uncertainty distribution

\[ \Psi^{-1}_{x_{y_{k-1}+j}}(x, y, \alpha) = \Psi^{-1}_{x_{y_{k-1}+j-1}}(x, y, \alpha) + \Phi^{-1}_{x_{y_{k-1}+j}k}(\alpha). \]  

(3.23)

This recursive process may produce all inverse uncertainty distributions of completion times of jobs.
Makespan

Note that, for each \( k \) \( (1 \leq k \leq m) \), the value \( C_{x_ky_k}(x, y, \xi) \) is just the time that the machine \( k \) finishes all jobs assigned to it. Thus the makespan of the schedule \((x, y)\) is determined by

\[
f(x, y, \xi) = \max_{1 \leq k \leq m} C_{x_ky_k}(x, y, \xi)
\]
whose inverse uncertainty distribution is

\[
\Upsilon^{-1}(x, y, \alpha) = \max_{1 \leq k \leq m} \Psi_{x_ky_k}^{-1}(x, y, \alpha).
\]

Machine Scheduling Model

In order to minimize the expected makespan \( E[f(x, y, \xi)] \), we have the following machine scheduling model,

\[
\begin{align*}
\min_{x, y} & \quad E[f(x, y, \xi)] \\
\text{subject to:} & \quad 1 \leq x_i \leq n, \quad i = 1, 2, \ldots, n \\
& \quad x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\
& \quad 0 \leq y_1 \leq y_2 \cdots \leq y_{m-1} \leq n \\
& \quad x_i, y_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m-1, \quad \text{integers.}
\end{align*}
\]

Since \( \Upsilon^{-1}(x, y, \alpha) \) is the inverse uncertainty distribution of \( f(x, y, \xi) \), the machine scheduling model is simplified as follows,

\[
\begin{align*}
\min_{x, y} & \quad \int_{0}^{1} \Upsilon^{-1}(x, y, \alpha)d\alpha \\
\text{subject to:} & \quad 1 \leq x_i \leq n, \quad i = 1, 2, \ldots, n \\
& \quad x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\
& \quad 0 \leq y_1 \leq y_2 \cdots \leq y_{m-1} \leq n \\
& \quad x_i, y_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m-1, \quad \text{integers.}
\end{align*}
\]

Numerical Experiment

Assume that there are 3 machines and 7 jobs with the following linear uncertain processing times

\[
\xi_{ik} \sim \mathcal{L}(i, i+k), \quad i = 1, 2, \ldots, 7, \quad k = 1, 2, 3
\]

where \( i \) is the index of jobs and \( k \) is the index of machines. The optimal solution is

\[
x^* = (1, 4, 5, 3, 7, 2, 6), \quad y^* = (3, 5).
\]
In other words, the optimal machine schedule is

- Machine 1: 1 → 4 → 5
- Machine 2: 3 → 7
- Machine 3: 2 → 6

whose expected makespan is 12.

### 3.4 Vehicle Routing Problem

Vehicle routing problem (VRP) is concerned with finding efficient routes, beginning and ending at a central depot, for a fleet of vehicles to serve a number of customers.

![Figure 3.3: A Vehicle Routing Plan with Single Depot and 7 Customers](image)

Due to its wide applicability and economic importance, vehicle routing problem has been extensively studied. Liu [95] first introduced uncertainty theory into the research area of vehicle routing problem in 2010. In this section, vehicle routing problem will be modelled by uncertain programming in which the travel times are assumed to be uncertain variables with known uncertainty distributions.

We assume that (a) a vehicle will be assigned for only one route on which there may be more than one customer; (b) a customer will be visited by one and only one vehicle; (c) each route begins and ends at the depot; and (d) each customer specifies its time window within which the delivery is permitted or preferred to start.

Let us first introduce the following indices and model parameters:

- $i = 0$: depot;
- $i = 1, 2, \cdots, n$: customers;
- $k = 1, 2, \cdots, m$: vehicles;
- $D_{ij}$: travel distance from customers $i$ to $j$, $i, j = 0, 1, 2, \cdots, n$;
- $T_{ij}$: uncertain travel time from customers $i$ to $j$, $i, j = 0, 1, 2, \cdots, n$;
- $\Phi_{ij}$: uncertainty distribution of $T_{ij}$, $i, j = 0, 1, 2, \cdots, n$;
[a_i, b_i]: time window of customer i, i = 1, 2, \ldots, n.

Operational Plan

Liu [86] suggested that an operational plan should be represented by three decision vectors x, y and t, where

x = (x_1, x_2, \ldots, x_n): integer decision vector representing n customers with 1 \leq x_i \leq n and x_i \neq x_j for all i \neq j, i, j = 1, 2, \ldots, n. That is, the sequence \{x_1, x_2, \ldots, x_n\} is a rearrangement of \{1, 2, \ldots, n\};

y = (y_1, y_2, \ldots, y_m-1): integer decision vector with y_0 \equiv 0 \leq y_1 \leq y_2 \leq \ldots \leq y_{m-1} \leq n \equiv y_m;

t = (t_1, t_2, \ldots, t_m): each t_k represents the starting time of vehicle k at the depot, k = 1, 2, \ldots, m.

We note that the operational plan is fully determined by the decision vectors x, y and t in the following way. For each k (1 \leq k \leq m), if y_k = y_{k-1}, then vehicle k is not used; if y_k > y_{k-1}, then vehicle k is used and starts from the depot at time t_k, and the tour of vehicle k is 0 \rightarrow x_{y_{k-1}+1} \rightarrow x_{y_{k-1}+2} \rightarrow \ldots \rightarrow x_{y_k} \rightarrow 0. Thus the tours of all vehicles are as follows:

Vehicle 1: 0 \rightarrow x_{y_0+1} \rightarrow x_{y_0+2} \rightarrow \ldots \rightarrow x_{y_1} \rightarrow 0;

Vehicle 2: 0 \rightarrow x_{y_1+1} \rightarrow x_{y_1+2} \rightarrow \ldots \rightarrow x_{y_2} \rightarrow 0;

\ldots

Vehicle m: 0 \rightarrow x_{y_{m-1}+1} \rightarrow x_{y_{m-1}+2} \rightarrow \ldots \rightarrow x_{y_m} \rightarrow 0.

Figure 3.4: Formulation of Operational Plan in which Vehicle 1 visits Customers x_1, x_2, Vehicle 2 visits Customers x_3, x_4 and Vehicle 3 visits Customers x_5, x_6, x_7.

It is clear that this type of representation is intuitive, and the total number of decision variables is n + 2m - 1. We also note that the above decision variables x, y and t ensure that: (a) each vehicle will be used at most one time; (b) all tours begin and end at the depot; (c) each customer will be visited by one and only one vehicle; and (d) there is no subtour.

Arrival Times

Let f_i(x, y, t) be the arrival time function of some vehicles at customers i for i = 1, 2, \ldots, n. We remind readers that f_i(x, y, t) are determined by the decision variables x, y and t, i = 1, 2, \ldots, n. Since unloading can start either
immediately, or later, when a vehicle arrives at a customer, the calculation of \( f_i(x, y, t) \) is heavily dependent on the operational strategy. Here we assume that the customer does not permit a delivery earlier than the time window. That is, the vehicle will wait to unload until the beginning of the time window if it arrives before the time window. If a vehicle arrives at a customer after the beginning of the time window, unloading will start immediately. For each \( k \) with \( 1 \leq k \leq m \), if vehicle \( k \) is used (i.e., \( y_k > y_{k-1} \)), then we have
\[
f_{x_{y_k-1}+1}(x, y, t) = t_k + T_{0_{x_{y_k-1}+1}}
\]
and
\[
f_{x_{y_k-1}+j}(x, y, t) = f_{x_{y_k-1}+j-1}(x, y, t) \vee a_{x_{y_k-1}+j-1} + T_{x_{y_k-1}+j-1x_{y_k-1}+j}
\]
for \( 2 \leq j \leq y_k - y_{k-1} \). If the vehicle \( k \) is used, i.e., \( y_k > y_{k-1} \), then the arrival time \( f_{x_{y_k-1}+1}(x, y, t) \) at the customer \( x_{y_k-1}+1 \) is an uncertain variable whose inverse uncertainty distribution is
\[
\Psi^{-1}_{x_{y_{k-1}+1}}(x, y, t, \alpha) = t_k + \Phi^{-1}_{0_{x_{y_{k-1}+1}}}(\alpha).
\]
Generally, suppose the arrival time \( f_{x_{y_k-1}+j-1}(x, y, t) \) has an inverse uncertainty distribution \( \Psi^{-1}_{x_{y_k-1}+j-1}(x, y, t, \alpha) \). Then \( f_{x_{y_k-1}+j}(x, y, t) \) has an inverse uncertainty distribution
\[
\Psi^{-1}_{x_{y_k-1}+j}(x, y, t, \alpha) = \Psi^{-1}_{x_{y_k-1}+j-1}(x, y, t, \alpha) \vee a_{x_{y_k-1}+j-1} + \Phi^{-1}_{x_{y_k-1}+j-1x_{y_k-1}+j}(\alpha)
\]
for \( 2 \leq j \leq y_k - y_{k-1} \). This recursive process may produce all inverse uncertainty distributions of arrival times at customers.

**Travel Distance**

Let \( g(x, y) \) be the total travel distance of all vehicles. Then we have
\[
g(x, y) = \sum_{k=1}^{m} g_k(x, y) \tag{3.29}
\]
where
\[
g_k(x, y) = \begin{cases} 
D_{0_{x_{y_k-1}+1}} + \sum_{j=y_{k-1}+1}^{y_k-1} D_{x_jx_{j+1}} + D_{x_{y_k}0}, & \text{if } y_k > y_{k-1} \\
0, & \text{if } y_k = y_{k-1}
\end{cases}
\]
for \( k = 1, 2, \cdots, m \).
Vehicle Routing Model

If we hope that each customer \( i (1 \leq i \leq n) \) is visited within its time window \([a_i, b_i]\) with confidence level \( \alpha_i \) (i.e., the vehicle arrives at customer \( i \) before time \( b_i \)), then we have the following chance constraint,

\[
M\{f_i(x, y, t) \leq b_i\} \geq \alpha_i.
\] (3.30)

If we want to minimize the total travel distance of all vehicles subject to the time window constraint, then we have the following vehicle routing model,

\[
\begin{aligned}
\min_{x, y, t} & \quad g(x, y) \\
\text{subject to:} & \\
& M\{f_i(x, y, t) \leq b_i\} \geq \alpha_i, \quad i = 1, 2, \cdots, n \\
& 1 \leq x_i \leq n, \quad i = 1, 2, \cdots, n \\
& x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \cdots, n \\
& 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \\
& x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers}
\end{aligned}
\] (3.31)

which is equivalent to

\[
\begin{aligned}
\min_{x, y, t} & \quad g(x, y) \\
\text{subject to:} & \\
& \Psi_i^{-1}(x, y, t, \alpha_i) \leq b_i, \quad i = 1, 2, \cdots, n \\
& 1 \leq x_i \leq n, \quad i = 1, 2, \cdots, n \\
& x_i \neq x_j, \quad i \neq j, \quad i, j = 1, 2, \cdots, n \\
& 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \\
& x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers}
\end{aligned}
\] (3.32)

where \( \Psi_i^{-1}(x, y, t, \alpha) \) are the inverse uncertainty distributions of \( f_i(x, y, t) \) for \( i = 1, 2, \cdots, n \), respectively.

Numerical Experiment

Assume that there are 3 vehicles and 7 customers with time windows shown in Table 3.1, and each customer is visited within time windows with confidence level 0.90.

We also assume that the distances are \( D_{ij} = |i - j| \) for \( i, j = 0, 1, 2, \cdots, 7 \), and the travel times are normal uncertain variables

\[
T_{ij} \sim \mathcal{N}(2|i - j|, 1), \quad i, j = 0, 1, 2, \cdots, 7.
\]

The optimal solution is

\[
\begin{aligned}
x^* & = (1, 3, 2, 5, 7, 4, 6), \\
y^* & = (2, 5), \\
t^* & = (6 : 18, 4 : 18, 8 : 18).
\end{aligned}
\] (3.33)
Table 3.1: Time Windows of Customers

<table>
<thead>
<tr>
<th>Node</th>
<th>Window</th>
<th>Node</th>
<th>Window</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[7:00, 9:00]</td>
<td>5</td>
<td>[15:00, 17:00]</td>
</tr>
<tr>
<td>2</td>
<td>[7:00, 9:00]</td>
<td>6</td>
<td>[19:00, 21:00]</td>
</tr>
<tr>
<td>3</td>
<td>[15:00, 17:00]</td>
<td>7</td>
<td>[19:00, 21:00]</td>
</tr>
<tr>
<td>4</td>
<td>[15:00, 17:00]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In other words, the optimal operational plan is

Vehicle 1: depot → 1 → 3 → depot (the latest starting time is 6:18)
Vehicle 2: depot → 2 → 5 → 7 → depot (the latest starting time is 4:18)
Vehicle 3: depot → 4 → 6 → depot (the latest starting time is 8:18)

whose total travel distance is 32.

3.5 Project Scheduling Problem

Project scheduling problem is to determine the schedule of allocating resources so as to balance the total cost and the completion time. The study of project scheduling problem with uncertain factors was started by Liu [95] in 2010. This section presents an uncertain programming model for project scheduling problem in which the duration times are assumed to be uncertain variables with known uncertainty distributions.

Project scheduling is usually represented by a directed acyclic network where nodes correspond to milestones, and arcs to activities which are basically characterized by the times and costs consumed.

Let \((V, A)\) be a directed acyclic graph, where \(V = \{1, 2, \ldots, n, n + 1\}\) is the set of nodes, \(A\) is the set of arcs, \((i, j) \in A\) is the arc of the graph \((V, A)\) from nodes \(i\) to \(j\). It is well-known that we can rearrange the indexes of the nodes in \(V\) such that \(i < j\) for all \((i, j) \in A\).
Before we begin to study project scheduling problem with uncertain activity duration times, we first make some assumptions: (a) all of the costs needed are obtained via loans with some given interest rate; and (b) each activity can be processed only if the loan needed is allocated and all the foregoing activities are finished.

In order to model the project scheduling problem, we introduce the following indices and parameters:

- \( \xi_{ij} \): uncertain duration time of activity \((i,j)\) in \(A\);
- \( \Phi_{ij} \): uncertainty distribution of \( \xi_{ij} \);
- \( c_{ij} \): cost of activity \((i,j)\) in \(A\);
- \( r \): interest rate;
- \( x_i \): integer decision variable representing the allocating time of all loans needed for all activities \((i,j)\) in \(A\).

**Starting Times**

For simplicity, we write \( \xi = \{\xi_{ij} : (i,j) \in A\} \) and \( x = (x_1, x_2, \ldots, x_n) \). Let \( T_i(x, \xi) \) denote the starting time of all activities \((i,j)\) in \(A\). According to the assumptions, the starting time of the total project (i.e., the starting time of all activities \((1,j)\) in \(A\)) should be

\[
T_1(x, \xi) = x_1
\]

whose inverse uncertainty distribution may be written as

\[
\Psi^{-1}_1(x, \alpha) = x_1.
\]

From the starting time \( T_1(x, \xi) \), we deduce that the starting time of activity \((2,5)\) is

\[
T_2(x, \xi) = x_2 \lor (x_1 + \xi_{12})
\]

whose inverse uncertainty distribution may be written as

\[
\Psi^{-1}_2(x, \alpha) = x_2 \lor (x_1 + \Phi^{-1}_{12}(\alpha)).
\]

Generally, suppose that the starting time \( T_k(x, \xi) \) of all activities \((k,i)\) in \(A\) has an inverse uncertainty distribution \( \Psi^{-1}_k(x, \alpha) \). Then the starting time \( T_i(x, \xi) \) of all activities \((i,j)\) in \(A\) should be

\[
T_i(x, \xi) = x_i \lor \max_{(k,i) \in A} (T_k(x, \xi) + \xi_{ki})
\]

whose inverse uncertainty distribution is

\[
\Psi^{-1}_i(x, \alpha) = x_i \lor \max_{(k,i) \in A} \left( \Psi^{-1}_k(x, \alpha) + \Phi^{-1}_{ki}(\alpha) \right).
\]

This recursive process may produce all inverse uncertainty distributions of starting times of activities.
Completion Time

The completion time $T(x, \xi)$ of the total project (i.e., the finish time of all activities $(k, n + 1)$ in $A$) is

$$T(x, \xi) = \max_{(k, n + 1) \in A} (T_k(x, \xi) + \xi_{k, n+1})$$  \hspace{1cm} (3.40)

whose inverse uncertainty distribution is

$$\Psi^{-1}(x, \alpha) = \max_{(k, n + 1) \in A} \left( \Psi_k^{-1}(x, \alpha) + \Phi_{k, n+1}(\alpha) \right).$$  \hspace{1cm} (3.41)

Total Cost

Based on the completion time $T(x, \xi)$, the total cost of the project can be written as

$$C(x, \xi) = \sum_{(i, j) \in A} c_{ij} (1 + r)^{\left\lceil T(x, \xi) - x_i \right\rceil}$$  \hspace{1cm} (3.42)

where $\lceil a \rceil$ represents the minimal integer greater than or equal to $a$. Note that $C(x, \xi)$ is a discrete uncertain variable whose inverse uncertainty distribution is

$$\Upsilon^{-1}(x, \alpha) = \sum_{(i, j) \in A} c_{ij} (1 + r)^{\left\lceil \Psi^{-1}(x, \alpha) - x_i \right\rceil}$$  \hspace{1cm} (3.43)

for $0 < \alpha < 1$.

Project Scheduling Model

In order to minimize the expected cost of the project under the completion time constraint, we may construct the following project scheduling model,

$$\begin{cases} \min E[C(x, \xi)] \\ \text{subject to:} \\ M\{T(x, \xi) \leq T_0\} \geq \alpha_0 \\ x \geq 0, \text{ integer vector} \end{cases}$$  \hspace{1cm} (3.44)

where $T_0$ is a due date of the project, $\alpha_0$ is a predetermined confidence level, $T(x, \xi)$ is the completion time defined by (3.40), and $C(x, \xi)$ is the total cost defined by (3.42). This model is equivalent to

$$\begin{cases} \min \int_0^1 \Upsilon^{-1}(x, \alpha) d\alpha \\ \text{subject to:} \\ \Psi^{-1}(x, \alpha_0) \leq T_0 \\ x \geq 0, \text{ integer vector} \end{cases}$$  \hspace{1cm} (3.45)
where $\Psi^{-1}(x, \alpha)$ is the inverse uncertainty distribution of $T(x, \xi)$ determined by (3.41) and $\Upsilon^{-1}(x, \alpha)$ is the inverse uncertainty distribution of $C(x, \xi)$ determined by (3.43).

**Numerical Experiment**

Consider a project scheduling problem shown by Figure 3.5 in which there are 8 milestones and 11 activities. Assume that all duration times of activities are linear uncertain variables,

$$\xi_{ij} \sim L(3i, 3j), \quad \forall (i, j) \in \mathcal{A}$$

and the costs of activities are

$$c_{ij} = i + j, \quad \forall (i, j) \in \mathcal{A}.$$  

In addition, we also suppose that the interest rate is $r = 0.02$, the due date is $T_0 = 60$, and the confidence level is $\alpha_0 = 0.85$. The optimal solution is

$$x^* = (7, 24, 17, 16, 35, 33, 30).$$ (3.46)

In other words, the optimal allocating times of all loans needed for all activities are shown in Table 3.2 whose expected total cost is 190.6, and

$$M\{T(x^*, \xi) \leq 60\} = 0.88.$$  

<table>
<thead>
<tr>
<th>Date</th>
<th>7</th>
<th>16</th>
<th>17</th>
<th>24</th>
<th>30</th>
<th>33</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Loan</td>
<td>12</td>
<td>11</td>
<td>27</td>
<td>7</td>
<td>15</td>
<td>14</td>
<td>13</td>
</tr>
</tbody>
</table>

3.6 Uncertain Multiobjective Programming

It has been increasingly recognized that many real decision-making problems involve multiple, noncommensurable, and conflicting objectives which should be considered simultaneously. In order to optimize multiple objectives, multiobjective programming has been well developed and applied widely. For modelling multiobjective decision-making problems with uncertain parameters, Liu-Chen [107] presented the following uncertain multiobjective programming,

$$\begin{align*}
\min_{x} & \quad (E[f_1(x, \xi)], E[f_2(x, \xi)], \ldots, E[f_m(x, \xi)]) \\
\text{subject to:} & \quad M\{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \ldots, p
\end{align*}$$  

(3.47)
where \( f_i(x, \xi) \) are objective functions for \( i = 1, 2, \ldots, m \), \( g_j(x, \xi) \) are constraint functions, and \( \alpha_j \) are confidence levels for \( j = 1, 2, \ldots, p \).

Since the objectives are usually in conflict, there is no optimal solution that simultaneously minimizes all the objective functions. In this case, we have to introduce the concept of Pareto solution, which means that it is impossible to improve any one objective without sacrificing on one or more of the other objectives.

**Definition 3.3** A feasible solution \( x^* \) is said to be Pareto to the uncertain multiobjective programming (3.47) if there is no feasible solution \( x \) such that

\[
E[f_i(x, \xi)] \leq E[f_i(x^*, \xi)], \quad i = 1, 2, \ldots, m
\]

and

\[
E[f_j(x, \xi)] < E[f_j(x^*, \xi)] \quad \text{for at least one index } j.
\]

If the decision maker has a real-valued preference function aggregating the \( m \) objective functions, then we may minimize the aggregating preference function subject to the same set of chance constraints. This model is referred to as a compromise model whose solution is called a compromise solution. It has been proved that the compromise solution is Pareto to the original multiobjective model.

The first well-known compromise model is set up by weighting the objective functions, i.e.,

\[
\begin{aligned}
\min_{x} & \quad \sum_{i=1}^{m} \lambda_i E[f_i(x, \xi)] \\
\text{subject to:} & \quad M\{g_j(x, \xi) \leq 0 \} \geq \alpha_j, \quad j = 1, 2, \ldots, p
\end{aligned}
\]  

(3.49)

where the weights \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are nonnegative numbers with \( \lambda_1 + \lambda_2 + \cdots + \lambda_m = 1 \), for example, \( \lambda_i \equiv \frac{1}{m} \) for \( i = 1, 2, \ldots, m \).

The second way is related to minimizing the distance function from a solution

\[
(E[f_1(x, \xi)], E[f_2(x, \xi)], \ldots, E[f_m(x, \xi)]
\]

(3.50)

to an ideal vector \((f_1^*, f_2^*, \ldots, f_m^*)\), where \( f_i^* \) are the optimal values of the \( i \)th objective functions without considering other objectives, \( i = 1, 2, \ldots, m \), respectively. That is,

\[
\begin{aligned}
\min_{x} & \quad \sum_{i=1}^{m} \lambda_i (E[f_i(x, \xi)] - f_i^*)^2 \\
\text{subject to:} & \quad M\{g_j(x, \xi) \leq 0 \} \geq \alpha_j, \quad j = 1, 2, \ldots, p
\end{aligned}
\]  

(3.51)

where the weights \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are nonnegative numbers with \( \lambda_1 + \lambda_2 + \cdots + \lambda_m = 1 \), for example, \( \lambda_i \equiv \frac{1}{m} \) for \( i = 1, 2, \ldots, m \).

By the third way a compromise solution can be found via an interactive approach consisting of a sequence of decision phases and computation phases. Various interactive approaches have been developed.
3.7 Uncertain Goal Programming

The concept of goal programming was presented by Charnes-Cooper [4] in 1961 and subsequently studied by many researchers. Goal programming can be regarded as a special compromise model for multiobjective optimization and has been applied in a wide variety of real-world problems. In multiobjective decision-making problems, we assume that the decision-maker is able to assign a target level for each goal and the key idea is to minimize the deviations (positive, negative, or both) from the target levels. In the real-world situation, the goals are achievable only at the expense of other goals and these goals are usually incompatible. In order to balance multiple conflicting objectives, a decision-maker may establish a hierarchy of importance among these incompatible goals so as to satisfy as many goals as possible in the order specified. For multiobjective decision-making problems with uncertain parameters, Liu-Chen [107] proposed an uncertain goal programming,

\[
\begin{align*}
\min_{\mathbf{x}} & \sum_{j=1}^{l} P_j \sum_{i=1}^{m} (u_{ij} d_{ij}^+ + v_{ij} d_{ij}^-) \\
\text{subject to:} & \quad E[f_i(\mathbf{x}, \xi)] + d_i^- - d_i^+ = b_i, \quad i = 1, 2, \ldots, m \\
& \quad \mathcal{M}\{g_j(\mathbf{x}, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \ldots, p \\
& \quad d_i^+, d_i^- \geq 0, \quad i = 1, 2, \ldots, m
\end{align*}
\]

(3.52)

where \(P_j\) are the preemptive priority factors, \(u_{ij}\) and \(v_{ij}\) are the weighting factors, \(d_i^+\) are the positive deviations, \(d_i^-\) are the negative deviations, \(f_i\) are the functions in goal constraints, \(g_j\) are the functions in real constraints, \(b_i\) are the target values, \(\alpha_j\) are the confidence levels, \(l\) is the number of priorities, \(m\) is the number of goal constraints, and \(p\) is the number of real constraints. Note that the positive and negative deviations are calculated by

\[
d_i^+ = \begin{cases} 
E[f_i(\mathbf{x}, \xi)] - b_i, & \text{if } E[f_i(\mathbf{x}, \xi)] > b_i \\
0, & \text{otherwise}
\end{cases} \quad \text{(3.53)}
\]

and

\[
d_i^- = \begin{cases} 
b_i - E[f_i(\mathbf{x}, \xi)], & \text{if } E[f_i(\mathbf{x}, \xi)] < b_i \\
0, & \text{otherwise}
\end{cases} \quad \text{(3.54)}
\]

for each \(i\). Sometimes, the objective function in the goal programming model is written as follows,

\[
\text{lexmin} \left\{ \sum_{i=1}^{m} (u_{i1} d_{i1}^+ + v_{i1} d_{i1}^-), \sum_{i=1}^{m} (u_{i2} d_{i2}^+ + v_{i2} d_{i2}^-), \ldots, \sum_{i=1}^{m} (u_{il} d_{il}^+ + v_{il} d_{il}^-) \right\}
\]

where \text{lexmin} represents lexicographically minimizing the objective vector.
3.8 Uncertain Multilevel Programming

Multilevel programming offers a means of studying decentralized decision systems in which we assume that the leader and followers may have their own decision variables and objective functions, and the leader can only influence the reactions of followers through his own decision variables, while the followers have full authority to decide how to optimize their own objective functions in view of the decisions of the leader and other followers.

Assume that in a decentralized two-level decision system there is one leader and $m$ followers. Let $x$ and $y_i$ be the control vectors of the leader and the $i$th followers, $i = 1, 2, \cdots, m$, respectively. We also assume that the objective functions of the leader and $i$th followers are $F(x, y_1, \cdots, y_m, \xi)$ and $f_i(x, y_1, \cdots, y_m, \xi)$, $i = 1, 2, \cdots, m$, respectively, where $\xi$ is an uncertain vector.

Let the feasible set of control vector $x$ of the leader be defined by the chance constraint
\[
M\{G(x, \xi) \leq 0\} \geq \alpha
\]
where $G$ is a constraint function, and $\alpha$ is a predetermined confidence level. Then for each decision $x$ chosen by the leader, the feasibility of control vectors $y_i$ of the $i$th followers should be dependent on not only $x$ but also $y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_m$, and generally represented by the chance constraints,
\[
M\{g_i(x, y_1, y_2, \cdots, y_m, \xi) \leq 0\} \geq \alpha_i
\]
where $g_i$ are constraint functions, and $\alpha_i$ are predetermined confidence levels, $i = 1, 2, \cdots, m$, respectively.

Assume that the leader first chooses his control vector $x$, and the followers determine their control array $(y_1, y_2, \cdots, y_m)$ after that. In order to minimize the expected objective of the leader, Liu-Yao [108] proposed the following uncertain multilevel programming,

\[
\begin{aligned}
& \min_x E[F(x, y_1^*, y_2^*, \cdots, y_m^*, \xi)] \\
& \text{subject to:} \\
& M\{G(x, \xi) \leq 0\} \geq \alpha \\
& \quad (y_1^*, y_2^*, \cdots, y_m^*) \text{ solves problems } (i = 1, 2, \cdots, m) \\
& \quad \min_{y_i} E[f_i(x, y_1^*, \cdots, y_i^*, y_{i+1}, \cdots, y_m^*, \xi)] \\
& \quad \text{subject to:} \\
& M\{g_i(x, y_1, y_2, \cdots, y_m, \xi) \leq 0\} \geq \alpha_i.
\end{aligned}
\]

**Definition 3.4** Let $x$ be a feasible control vector of the leader. A Nash equilibrium of followers is the feasible array $(y_1^*, y_2^*, \cdots, y_m^*)$ with respect to $x$ if
\[
E[f_i(x, y_1^*, \cdots, y_{i-1}^*, y_i^*, y_{i+1}^*, \cdots, y_m^*, \xi)] \\
\geq E[f_i(x, y_1^*, \cdots, y_{i-1}^*, y_i^*, y_{i+1}^*, \cdots, y_m^*, \xi)]
\]

(3.58)
for any feasible array \((y_1^*, \cdots, y_i^*, y_{i+1}^*, \cdots, y_m^*)\) and \(i = 1, 2, \cdots, m\).

**Definition 3.5** Suppose that \(x^*\) is a feasible control vector of the leader and \((y_1^*, y_2^*, \cdots, y_m^*)\) is a Nash equilibrium of followers with respect to \(x^*\). We call the array \((x^*, y_1^*, y_2^*, \cdots, y_m^*)\) a Stackelberg-Nash equilibrium to the uncertain multilevel programming (3.57) if

\[
E[F(x, y_1, y_2, \cdots, y_m, \xi)] \geq E[F(x^*, y_1^*, y_2^*, \cdots, y_m^*, \xi)]
\]

(3.59)

for any feasible control vector \(x\) and the Nash equilibrium \((y_1, y_2, \cdots, y_m)\) with respect to \(x\).

### 3.9 Bibliographic Notes

Uncertain programming was founded by Liu [90] in 2009 and was applied to machine scheduling problem, vehicle routing problem and project scheduling problem by Liu [95] in 2010.

As extensions of uncertain programming theory, Liu-Chen [107] developed an uncertain multiobjective programming and an uncertain goal programming. In addition, Liu-Yao [108] suggested an uncertain multilevel programming for modeling decentralized decision systems with uncertain factors. After that, the uncertain programming has obtained fruitful results in both theory and practice.
Chapter 4

Uncertain Risk Analysis

The term *risk* has been used in different ways in literature. Here the risk is defined as the “accidental loss” plus “uncertain measure of such loss”. Uncertain risk analysis is a tool to quantify risk via uncertainty theory. One main feature of this topic is to model events that almost never occur. This chapter will introduce a definition of risk index and provide some useful formulas for calculating risk index. This chapter will also discuss structural risk analysis and investment risk analysis in uncertain environments.

4.1 Loss Function

A system usually contains some factors $\xi_1, \xi_2, \cdots, \xi_n$ that may be understood as lifetime, strength, demand, production rate, cost, profit, and resource. Generally speaking, some specified loss is dependent on those factors. Although loss is a problem-dependent concept, usually such a loss may be represented by a loss function.

**Definition 4.1** Consider a system with factors $\xi_1, \xi_2, \cdots, \xi_n$. A function $f$ is called a loss function if some specified loss occurs if and only if

$$f(\xi_1, \xi_2, \cdots, \xi_n) > 0.$$

(4.1)

**Example 4.1:** Consider a series system in which there are $n$ elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. Such a system works whenever all elements work. Thus the system lifetime is

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$  

(4.2)

If the loss is understood as the case that the system fails before the time $T$, then we have a loss function

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$  

(4.3)
Hence the system fails if and only if $f(\xi_1, \xi_2, \ldots, \xi_n) > 0$.

**Example 4.2:** Consider a parallel system in which there are $n$ elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$. Such a system works whenever at least one element works. Thus the system lifetime is

$$\xi = \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n.$$  \hspace{1cm} (4.4)

If the loss is understood as the case that the system fails before the time $T$, then the loss function is

$$f(\xi_1, \xi_2, \ldots, \xi_n) = T - \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n.$$  \hspace{1cm} (4.5)

Hence the system fails if and only if $f(\xi_1, \xi_2, \ldots, \xi_n) > 0$.

**Example 4.3:** Consider a $k$-out-of-$n$ system in which there are $n$ elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$. Such a system works whenever at least $k$ of $n$ elements work. Thus the system lifetime is

$$\xi = k-\max[\xi_1, \xi_2, \ldots, \xi_n].$$  \hspace{1cm} (4.6)

If the loss is understood as the case that the system fails before the time $T$, then the loss function is

$$f(\xi_1, \xi_2, \ldots, \xi_n) = T - k-\max[\xi_1, \xi_2, \ldots, \xi_n].$$  \hspace{1cm} (4.7)

Hence the system fails if and only if $f(\xi_1, \xi_2, \ldots, \xi_n) > 0$. Note that a series system is an $n$-out-of-$n$ system, and a parallel system is a 1-out-of-$n$ system.

**Example 4.4:** Consider a standby system in which there are $n$ redundant elements whose lifetimes are $\xi_1, \xi_2, \ldots, \xi_n$. For this system, only one element is active, and one of the redundant elements begins to work only when the active element fails. Thus the system lifetime is

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n.$$  \hspace{1cm} (4.8)
If the loss is understood as the case that the system fails before the time $T$, then the loss function is

$$f(\xi_1, \xi_2, \ldots, \xi_n) = T - (\xi_1 + \xi_2 + \cdots + \xi_n). \quad (4.9)$$

Hence the system fails if and only if $f(\xi_1, \xi_2, \ldots, \xi_n) > 0$.

![Figure 4.3: A Standby System](image)

### 4.2 Risk Index

In practice, the factors $\xi_1, \xi_2, \ldots, \xi_n$ of a system are usually uncertain variables rather than known constants. Thus the risk index is defined as the uncertain measure that some specified loss occurs.

**Definition 4.2** (Liu [94]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \ldots, \xi_n$ and has a loss function $f$. Then the risk index is the uncertain measure that the system is loss-positive, i.e.,

$$\text{Risk} = \mathbb{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) > 0\}. \quad (4.10)$$

**Theorem 4.1** Assume that a system contains uncertain factors $\xi_1, \xi_2, \ldots, \xi_n$, and has a loss function $f$. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ has an uncertainty distribution $\Phi$, then the risk index is

$$\text{Risk} = 1 - \Phi(0). \quad (4.11)$$

**Proof:** It follows from the definition of risk index and the duality axiom that

$$\text{Risk} = \mathbb{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) > 0\} = 1 - \mathbb{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) \leq 0\} = 1 - \Phi(0).$$

The theorem is proved.

**Theorem 4.2** (Liu [94], Risk Index Theorem) Assume a system contains independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \ldots, \xi_n)$ is continuous, strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then the risk index is just the root $\alpha$ of the equation

$$f(\Phi_1^{-1}(1-\alpha), \ldots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) = 0. \quad (4.12)$$
Proof: It follows from Theorem 2.16 that \( f(\xi_1, \xi_2, \cdots, \xi_n) \) has an inverse uncertainty distribution

\[
\Phi^{-1}(\alpha) = f(\Phi^{-1}_1(\alpha), \cdots, \Phi^{-1}_m(\alpha), \Phi^{-1}_{m+1}(1-\alpha), \cdots, \Phi^{-1}_n(1-\alpha)).
\]

Since \( \text{Risk} = 1 - \Phi(0) \), it is the solution \( \alpha \) of the equation \( \Phi^{-1}(1-\alpha) = 0 \). The theorem is thus proved.

Remark 4.1: Since \( f(\Phi^{-1}_1(1-\alpha), \cdots, \Phi^{-1}_m(1-\alpha), \Phi^{-1}_{m+1}(\alpha), \cdots, \Phi^{-1}_n(\alpha)) \) is a continuous and strictly decreasing function with respect to \( \alpha \), its root may be estimated by the bisection method:

Step 1. Set \( a = 0 \), \( b = 1 \) and \( c = (a + b)/2 \).

Step 2. If \( f(\Phi^{-1}_1(1-c), \cdots, \Phi^{-1}_m(1-c), \Phi^{-1}_{m+1}(c), \cdots, \Phi^{-1}_n(c)) > 0 \), then set \( a = c \). Otherwise, set \( b = c \).

Step 3. If \( |b - a| > \varepsilon \) (a predetermined precision), then set \( c = (b - a)/2 \) and go to Step 2. Otherwise, output \( c \) as the root.

Remark 4.2: Keep in mind that sometimes the equation (4.12) may not have a root. In this case, if

\[
f(\Phi^{-1}_1(1-\alpha), \cdots, \Phi^{-1}_m(1-\alpha), \Phi^{-1}_{m+1}(\alpha), \cdots, \Phi^{-1}_n(\alpha)) < 0 \quad (4.13)
\]

for all \( \alpha \), then we set the root \( \alpha = 0 \); and if

\[
f(\Phi^{-1}_1(1-\alpha), \cdots, \Phi^{-1}_m(1-\alpha), \Phi^{-1}_{m+1}(\alpha), \cdots, \Phi^{-1}_n(\alpha)) > 0 \quad (4.14)
\]

for all \( \alpha \), then we set the root \( \alpha = 1 \).

4.3 Series System

Consider a series system in which there are \( n \) elements whose lifetimes are independent uncertain variables \( \xi_1, \xi_2, \cdots, \xi_n \) with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is

\[
f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \land \xi_2 \land \cdots \land \xi_n \quad (4.15)
\]

and the risk index is

\[
\text{Risk} = M\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \quad (4.16)
\]

Since \( f \) is a strictly decreasing function with respect to \( \xi_1, \xi_2, \cdots, \xi_n \), the risk index theorem says that the risk index is just the root \( \alpha \) of the equation

\[
\Phi^{-1}_1(\alpha) \land \Phi^{-1}_2(\alpha) \land \cdots \land \Phi^{-1}_n(\alpha) = T. \quad (4.17)
\]

It is easy to verify that

\[
\text{Risk} = \Phi_1(T) \lor \Phi_2(T) \lor \cdots \lor \Phi_n(T). \quad (4.18)
\]
4.4 Parallel System

Consider a parallel system in which there are \( n \) elements whose lifetimes are independent uncertain variables \( \xi_1, \xi_2, \cdots, \xi_n \) with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is

\[
f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \lor \xi_2 \lor \cdots \lor \xi_n \quad (4.19)
\]

and the risk index is

\[
Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \quad (4.20)
\]

Since \( f \) is a strictly decreasing function with respect to \( \xi_1, \xi_2, \cdots, \xi_n \), the risk index theorem says that the risk index is just the root \( \alpha \) of the equation

\[
\Phi_1^{-1}(\alpha) \lor \Phi_2^{-1}(\alpha) \lor \cdots \lor \Phi_n^{-1}(\alpha) = T. \quad (4.21)
\]

It is easy to verify that

\[
Risk = \Phi_1(T) \land \Phi_2(T) \land \cdots \land \Phi_n(T). \quad (4.22)
\]

4.5 \( k \)-out-of-\( n \) System

Consider a \( k \)-out-of-\( n \) system in which there are \( n \) elements whose lifetimes are independent uncertain variables \( \xi_1, \xi_2, \cdots, \xi_n \) with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is

\[
f(\xi_1, \xi_2, \cdots, \xi_n) = T - k\max[\xi_1, \xi_2, \cdots, \xi_n] \quad (4.23)
\]

and the risk index is

\[
Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \quad (4.24)
\]

Since \( f \) is a strictly decreasing function with respect to \( \xi_1, \xi_2, \cdots, \xi_n \), the risk index theorem says that the risk index is just the root \( \alpha \) of the equation

\[
k\max[\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)] = T. \quad (4.25)
\]

It is easy to verify that

\[
Risk = k\min[\Phi_1(T), \Phi_2(T), \cdots, \Phi_n(T)]. \quad (4.26)
\]

Note that a series system is essentially an \( n \)-out-of-\( n \) system. In this case, the risk index formula (4.26) becomes (4.18). In addition, a parallel system is essentially a 1-out-of-\( n \) system. In this case, the risk index formula (4.26) becomes (4.22).
4.6 Standby System

Consider a standby system in which there are \( n \) elements whose lifetimes are independent uncertain variables \( \xi_1, \xi_2, \ldots, \xi_n \) with regular uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is

\[
f(\xi_1, \xi_2, \ldots, \xi_n) = T - (\xi_1 + \xi_2 + \cdots + \xi_n)
\]

and the risk index is

\[
Risk = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) > 0\}.
\]

Since \( f \) is a strictly decreasing function with respect to \( \xi_1, \xi_2, \ldots, \xi_n \), the risk index theorem says that the risk index is just the root \( \alpha \) of the equation

\[
\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha) = T.
\]

4.7 Structural Risk Analysis

Uncertain structural risk analysis was first investigated by Liu [106]. Consider a structural system in which the strengths and loads are assumed to be uncertain variables. We will suppose that a structural system fails whenever for each rod, the load variable exceeds its strength variable. If the structural risk index is defined as the uncertain measure that the structural system fails, then

\[
Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta_i)\right\}
\]

where \( \xi_1, \xi_2, \ldots, \xi_n \) are strength variables, and \( \eta_1, \eta_2, \ldots, \eta_n \) are load variables of the \( n \) rods.

**Example 4.5:** (The Simplest Case) Assume there is only a single strength variable \( \xi \) and a single load variable \( \eta \) with regular uncertainty distributions \( \Phi \) and \( \Psi \), respectively. In this case, the structural risk index is

\[
Risk = \mathcal{M}\{\xi < \eta\}.
\]

It follows from the risk index theorem that the risk index is just the root \( \alpha \) of the equation

\[
\Phi^{-1}(\alpha) = \Psi^{-1}(1 - \alpha).
\]

Especially, if the strength variable \( \xi \) has a normal uncertainty distribution \( \mathcal{N}(e_s, \sigma_s) \) and the load variable \( \eta \) has a normal uncertainty distribution \( \mathcal{N}(e_l, \sigma_l) \), then the structural risk index is

\[
Risk = \left(1 + \exp\left(\frac{\pi(e_s - e_l)}{\sqrt{3}(\sigma_s + \sigma_l)}\right)\right)^{-1}.
\]
Example 4.6: (Constant Loads) Assume the uncertain strength variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent and have continuous uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. In many cases, the load variables $\eta_1, \eta_2, \ldots, \eta_n$ degenerate to crisp values $c_1, c_2, \ldots, c_n$ (for example, weight limits allowed by the legislation), respectively. In this case, it follows from (4.30) and independence that the structural risk index is

$$\text{Risk} = M \left\{ \bigcup_{i=1}^{n} (\xi_i < c_i) \right\} = \bigvee_{i=1}^{n} M\{\xi_i < c_i\}.$$ 

That is,

$$\text{Risk} = \Phi_1(c_1) \lor \Phi_2(c_2) \lor \cdots \lor \Phi_n(c_n).$$

(4.33)

Example 4.7: (Independent Load Variables) Assume the uncertain strength variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent and have regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Also assume the uncertain load variables $\eta_1, \eta_2, \ldots, \eta_n$ are independent and have regular uncertainty distributions $\Psi_1, \Psi_2, \ldots, \Psi_n$, respectively. In this case, it follows from (4.30) and independence that the structural risk index is

$$\text{Risk} = M \left\{ \bigcup_{i=1}^{n} (\xi_i < \eta_i) \right\} = \bigvee_{i=1}^{n} M\{\xi_i < \eta_i\}.$$ 

That is,

$$\text{Risk} = \alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_n$$

(4.34)

where $\alpha_i$ are the roots of the equations

$$\Phi_i^{-1}(\alpha) = \Psi_i^{-1}(1 - \alpha)$$

(4.35)

for $i = 1, 2, \ldots, n$, respectively.

However, generally speaking, the load variables $\eta_1, \eta_2, \ldots, \eta_n$ are neither constants nor independent. For examples, the load variables $\eta_1, \eta_2, \ldots, \eta_n$ may be functions of independent uncertain variables $\tau_1, \tau_2, \ldots, \tau_m$. In this case, the formula (4.34) is no longer valid. Thus we have to deal with those structural systems case by case.

Example 4.8: (Series System) Consider a structural system shown in Figure 4.4 that consists of $n$ rods in series and an object. Assume that the strength variables of the $n$ rods are uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. We also assume that the gravity of the object is an uncertain variable $\eta$ with regular uncertainty distribution $\Psi$. For each $i$ ($1 \leq i \leq n$), the load variable of the rod $i$ is just the gravity $\eta$ of the object. Thus the structural system fails
whenever the load variable \( \eta \) exceeds at least one of the strength variables \( \xi_1, \xi_2, \cdots, \xi_n \). Hence the structural risk index is

\[
Risk = M \left\{ \bigcup_{i=1}^{n} (\xi_i < \eta) \right\} = M\{\xi_1 \land \xi_2 \land \cdots \land \xi_n < \eta\}.
\]

Define the loss function as

\[
f(\xi_1, \xi_2, \cdots, \xi_n, \eta) = \eta - \xi_1 \land \xi_2 \land \cdots \land \xi_n.
\]

Then

\[
Risk = M\{f(\xi_1, \xi_2, \cdots, \xi_n, \eta) > 0\}.
\]

Since the loss function \( f \) is strictly increasing with respect to \( \eta \) and strictly decreasing with respect to \( \xi_1, \xi_2, \cdots, \xi_n \), it follows from the risk index theorem that the risk index is just the root \( \alpha \) of the equation

\[
\Psi^{-1}(1 - \alpha) - \Phi_1^{-1}(\alpha) \land \Phi_2^{-1}(\alpha) \land \cdots \land \Phi_n^{-1}(\alpha) = 0. \tag{4.36}
\]

Or equivalently, let \( \alpha_i \) be the roots of the equations

\[
\Psi^{-1}(1 - \alpha) = \Phi_i^{-1}(\alpha) \tag{4.37}
\]

for \( i = 1, 2, \cdots, n \), respectively. Then the structural risk index is

\[
Risk = \alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_n. \tag{4.38}
\]

---

**Figure 4.4:** A Structural System with \( n \) Rods and an Object

**Example 4.9:** Consider a structural system shown in Figure 4.5 that consists of 2 rods and an object. Assume that the strength variables of the left and
right rods are uncertain variables $\xi_1$ and $\xi_2$ with uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. We also assume that the gravity of the object is an uncertain variable $\eta$ with regular uncertainty distribution $\Psi$. In this case, the load variables of left and right rods are respectively equal to 
\[
\frac{\eta \sin \theta_2}{\sin(\theta_1 + \theta_2)}, \quad \frac{\eta \sin \theta_1}{\sin(\theta_1 + \theta_2)}.
\]
Thus the structural system fails whenever for any one rod, the load variable exceeds its strength variable. Hence the structural risk index is 
\[
Risk = M\left\{ \left( \xi_1 < \frac{\eta \sin \theta_2}{\sin(\theta_1 + \theta_2)} \right) \cup \left( \xi_2 < \frac{\eta \sin \theta_1}{\sin(\theta_1 + \theta_2)} \right) \right\}
\]
\[
= M\left\{ \left( \frac{\xi_1}{\sin \theta_2} < \frac{\eta}{\sin(\theta_1 + \theta_2)} \right) \cup \left( \frac{\xi_2}{\sin \theta_1} < \frac{\eta}{\sin(\theta_1 + \theta_2)} \right) \right\}
\]
\[
= M\left\{ \frac{\xi_1}{\sin \theta_2} \wedge \frac{\xi_2}{\sin \theta_1} < \frac{\eta}{\sin(\theta_1 + \theta_2)} \right\}
\]
Define the loss function as 
\[
f(\xi_1, \xi_2, \eta) = \frac{\eta}{\sin(\theta_1 + \theta_2)} - \frac{\xi_1}{\sin \theta_2} \wedge \frac{\xi_2}{\sin \theta_1}.
\]
Then 
\[
Risk = M\{f(\xi_1, \xi_2, \eta) > 0\}.
\]
Since the loss function $f$ is strictly increasing with respect to $\eta$ and strictly decreasing with respect to $\xi_1, \xi_2$, it follows from the risk index theorem that the risk index is just the root $\alpha$ of the equation 
\[
\frac{\Psi^{-1}(1 - \alpha)}{\sin(\theta_1 + \theta_2)} - \frac{\Phi_1^{-1}(\alpha)}{\sin \theta_2} \wedge \frac{\Phi_2^{-1}(\alpha)}{\sin \theta_1} = 0. \tag{4.39}
\]
Or equivalently, let $\alpha_1$ be the root of the equation 
\[
\frac{\Psi^{-1}(1 - \alpha)}{\sin(\theta_1 + \theta_2)} = \frac{\Phi_1^{-1}(\alpha)}{\sin \theta_2} \tag{4.40}
\]
and let $\alpha_2$ be the root of the equation 
\[
\frac{\Psi^{-1}(1 - \alpha)}{\sin(\theta_1 + \theta_2)} = \frac{\Phi_2^{-1}(\alpha)}{\sin \theta_1}. \tag{4.41}
\]
Then the structural risk index is 
\[
Risk = \alpha_1 \lor \alpha_2. \tag{4.42}
\]
4.8 Value-at-Risk

As a substitute of risk index (4.10), a concept of value-at-risk is given by the following definition.

**Definition 4.3** (Peng [136]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$ and has a loss function $f$. Then the value-at-risk is defined as

$$\text{VaR}(\alpha) = \sup \{ x \mid M \{ f(\xi_1, \xi_2, \cdots, \xi_n) \geq x \} \geq \alpha \}. \quad (4.43)$$

Note that $\text{VaR}(\alpha)$ represents the maximum possible loss when $\alpha$ percent of the right tail distribution is ignored. In other words, the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ will exceed $\text{VaR}(\alpha)$ with uncertain measure $\alpha$. See Figure 4.6. If the uncertainty distribution $\Phi(x)$ of $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, then

$$\text{VaR}(\alpha) = \sup \{ x \mid \Phi(x) \leq 1 - \alpha \}. \quad (4.44)$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then

$$\text{VaR}(\alpha) = \Phi^{-1}(1 - \alpha). \quad (4.45)$$

It is also easy to show that $\text{VaR}(\alpha)$ is a monotone decreasing function with respect to $\alpha$.

**Theorem 4.3** (Peng [136], Value-at-Risk Theorem) Assume a system contains independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then

$$\text{VaR}(\alpha) = f(\Phi_1^{-1}(1 - \alpha), \cdots, \Phi_m^{-1}(1 - \alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \quad (4.46)$$
**Proof:** It follows from the operational law of uncertain variables that the loss \( f(\xi_1, \xi_2, \cdots, \xi_n) \) has an inverse uncertainty distribution

\[
\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).
\]

The theorem follows from (4.45) immediately.

### 4.9 Expected Loss

Liu-Ralescu [123] proposed a concept of expected loss that is the expected value of the loss \( f(\xi_1, \xi_2, \cdots, \xi_n) \) given \( f(\xi_1, \xi_2, \cdots, \xi_n) > 0 \). A formal definition is given below.

**Definition 4.4** (Liu-Ralescu [123]) Assume that a system contains uncertain factors \( \xi_1, \xi_2, \cdots, \xi_n \) and has a loss function \( f \). Then the expected loss is defined as

\[
L = \int_{0}^{+\infty} \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\}dx. \tag{4.47}
\]

If \( \Phi(x) \) is the uncertainty distribution of the loss \( f(\xi_1, \xi_2, \cdots, \xi_n) \), then we immediately have

\[
L = \int_{0}^{+\infty} (1 - \Phi(x))dx. \tag{4.48}
\]

If its inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) exists, then the expected loss is

\[
L = \int_{0}^{1} (\Phi^{-1}(\alpha))^+ d\alpha. \tag{4.49}
\]

**Theorem 4.4** (Liu-Ralescu [123], Expected Loss Theorem) Assume that a system contains independent uncertain variables \( \xi_1, \xi_2, \cdots, \xi_n \) with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If the loss function
f(ξ₁, ξ₂, · · · , ξₙ) is continuous, strictly increasing with respect to ξ₁, ξ₂, · · · , ξₘ and strictly decreasing with respect to ξₘ₊₁, ξₘ₊₂, · · · , ξₙ, then the expected loss is

\[ L = \int_0^1 f^+(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))d\alpha. \]  (4.50)

**Proof:** It follows from the operational law of uncertain variables that the loss \( f(ξ₁, ξ₂, · · · , ξₙ) \) has an inverse uncertainty distribution

\[ \Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \]

The theorem follows from (4.49) immediately.

### 4.10 Investment Risk Analysis

Uncertain investment risk analysis was first studied by Liu [106]. Assume that an investor has \( n \) projects whose returns are uncertain variables \( ξ₁, ξ₂, · · · , ξₙ \). If the loss is understood as the case that total return \( ξ₁ + ξ₂ + · · · + ξₙ \) is below a predetermined value \( c \) (e.g., the interest rate), then the investment risk index is

\[ Risk = M\{ξ₁ + ξ₂ + · · · + ξₙ < c\}. \]  (4.51)

If \( ξ₁, ξ₂, · · · , ξₙ \) are independent uncertain variables with regular uncertainty distributions \( Φ_1, Φ_2, · · · , Φ_n \), respectively, then the investment risk index is just the root \( α \) of the equation

\[ Φ_1^{-1}(α) + Φ_2^{-1}(α) + · · · + Φ_n^{-1}(α) = c. \]  (4.52)

### 4.11 Bibliographic Notes

Uncertain risk analysis was proposed by Liu [94] in 2010 in which the risk index was defined as the uncertain measure that some specified loss occurs,
and a risk index theorem was proved. This tool was also successfully applied by Liu [106] to structural risk analysis.

As a substitute of risk index, Peng [136] suggested the concept of value-at-risk that is the maximum possible loss when the right tail distribution is ignored. In addition, Liu-Ralescu [123] investigated the concept of expected loss that takes into account not only the uncertain measure of the loss but also its severity.
Chapter 5

Uncertain Reliability Analysis

Uncertain reliability analysis is a tool to deal with system reliability via uncertainty theory. This chapter will introduce a definition of reliability index and provide some useful formulas for calculating the reliability index.

5.1 Structure Function

Many real systems may be simplified to a Boolean system in which each element (including the system itself) has two states: working and failure. We denote the states of elements $i$ by the Boolean variables

$$x_i = \begin{cases} 
1, & \text{if element } i \text{ works} \\
0, & \text{if element } i \text{ fails},
\end{cases} \quad (5.1)$$

$i = 1, 2, \cdots, n$, respectively. We also denote the state of the system by the Boolean variable

$$X = \begin{cases} 
1, & \text{if the system works} \\
0, & \text{if the system fails}.
\end{cases} \quad (5.2)$$

Usually, the state of the system is completely determined by the states of its elements via the so-called structure function.

Definition 5.1 Assume that $X$ is a Boolean system containing elements $x_1, x_2, \cdots, x_n$. A Boolean function $f$ is called a structure function of $X$ if

$$X = 1 \text{ if and only if } f(x_1, x_2, \cdots, x_n) = 1. \quad (5.3)$$

It is obvious that $X = 0$ if and only if $f(x_1, x_2, \cdots, x_n) = 0$ whenever $f$ is indeed the structure function of the system.
Example 5.1: For a series system, the structure function is a mapping from \(\{0, 1\}^n\) to \(\{0, 1\}\), i.e.,
\[
f(x_1, x_2, \cdots, x_n) = x_1 \land x_2 \land \cdots \land x_n.
\] (5.4)

Example 5.2: For a parallel system, the structure function is a mapping from \(\{0, 1\}^n\) to \(\{0, 1\}\), i.e.,
\[
f(x_1, x_2, \cdots, x_n) = x_1 \lor x_2 \lor \cdots \lor x_n.
\] (5.5)

Example 5.3: For a \(k\)-out-of-\(n\) system that works whenever at least \(k\) of the \(n\) elements work, the structure function is a mapping from \(\{0, 1\}^n\) to \(\{0, 1\}\), i.e.,
\[
f(x_1, x_2, \cdots, x_n) = k\text{-max} \{x_1, x_2, \cdots, x_n\}.
\] (5.6)

Especially, when \(k = 1\), it is a parallel system; when \(k = n\), it is a series system.

5.2 Reliability Index

The element in a Boolean system is usually represented by a Boolean uncertain variable, i.e.,
\[
\xi = \begin{cases} 
1 & \text{with uncertain measure } a \\
0 & \text{with uncertain measure } 1 - a.
\end{cases}
\] (5.7)

In this case, we will say \(\xi\) is an uncertain element with reliability \(a\). Reliability index is defined as the uncertain measure that the system is working.
Definition 5.2 (Liu [94]) Assume a Boolean system has uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ and a structure function $f$. Then the reliability index is the uncertain measure that the system is working, i.e.,

\[
\text{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}. \tag{5.8}
\]

Theorem 5.1 (Liu [94], Reliability Index Theorem) Assume that a system contains uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$, and has a structure function $f$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain elements with reliabilities $a_1, a_2, \cdots, a_n$, respectively, then the reliability index is

\[
\text{Reliability} = \begin{cases} 
\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\
1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i), & \text{if } \sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5
\end{cases} \tag{5.9}
\]

where $x_i$ take values either 0 or 1, and $\nu_i$ are defined by

\[
\nu_i(x_i) = \begin{cases} 
a_i, & \text{if } x_i = 1 \\
1 - a_i, & \text{if } x_i = 0
\end{cases} \tag{5.10}
\]

for $i = 1, 2, \cdots, n$, respectively.

\textbf{Proof:} Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables and $f$ is a Boolean function, the equation (5.9) follows from Definition 5.2 and Theorem 2.22 immediately.

5.3 Series System

Consider a series system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function is

\[
f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{5.11}
\]

It follows from the reliability index theorem that the reliability index is

\[
\text{Reliability} = \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n. \tag{5.12}
\]

5.4 Parallel System

Consider a parallel system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function is

\[
f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{5.13}
\]
It follows from the reliability index theorem that the reliability index is

\[
\text{Reliability} = M\{\xi_1 \lor \xi_2 \lor \cdots \lor \xi_n = 1\} = a_1 \lor a_2 \lor \cdots \lor a_n.
\] (5.14)

5.5 \textbf{\textit{k}}-out-of-\textit{n} System

Consider a \textit{k}-out-of-\textit{n} system having independent uncertain elements \(\xi_1, \xi_2, \cdots, \xi_n\) with reliabilities \(a_1, a_2, \cdots, a_n\), respectively. Note that the structure function has a Boolean form,

\[f(x_1, x_2, \cdots, x_n) = k\text{-max}\{x_1, x_2, \cdots, x_n\}.\] (5.15)

It follows from the reliability index theorem that the reliability index is the \textit{k}th largest value of \(a_1, a_2, \cdots, a_n\), i.e.,

\[
\text{Reliability} = k\text{-max}\{a_1, a_2, \cdots, a_n\}.
\] (5.16)

Note that a series system is essentially an \textit{n}-out-of-\textit{n} system. In this case, the reliability index formula (5.16) becomes (5.12). In addition, a parallel system is essentially a 1-out-of-\textit{n} system. In this case, the reliability index formula (5.16) becomes (5.14).

5.6 General System

It is almost impossible to find an analytic formula of reliability risk for general systems. In this case, we have to employ a numerical method.

Consider a bridge system shown in Figure 5.3 that consists of 5 independent uncertain elements whose states are denoted by \(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\). Assume each path works if and only if all elements on which are working and the system works if and only if there is a path of working elements. Then the structure function of the bridge system is

\[f(x_1, x_2, x_3, x_4, x_5) = (x_1 \land x_4) \lor (x_2 \land x_5) \lor (x_1 \land x_3 \land x_5) \lor (x_2 \land x_3 \land x_4).\]
Assume the 5 independent uncertain elements have reliabilities

\[ 0.91, 0.92, 0.93, 0.94, 0.95 \]

in uncertain measure. The reliability index is

\[ \text{Reliability} = M\{ f(\xi_1, \xi_2, \cdots, \xi_5) = 1 \} = 0.92 \]

in uncertain measure.

### 5.7 Bibliographic Notes

Uncertain reliability analysis was proposed by Liu [94] in 2010 in which the reliability index was defined as the uncertain measure that the system is working, and a reliability index theorem was proved. After that, uncertain reliability was studied by Zeng-Wen-Kang [212], Gao-Yao [39], Zeng-Kang-Wen-Zio [213][214], Zu-Kang-Wen-Zhang [230] and Gao-Yao-Zhou-Ke [36].
Propositional logic, originated from the work of Aristotle (384-322 BC), is a branch of logic that studies the properties of complex propositions composed of simpler propositions and logical connectives. Note that the propositions considered in propositional logic are not arbitrary statements but are the ones that are either true or false and not both.

Uncertain propositional logic is a generalization of propositional logic in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. This chapter will deal with uncertain propositional logic, including uncertain proposition, truth value definition, and truth value theorem. This chapter will also introduce uncertain predicate logic.

6.1 Uncertain Proposition

**Definition 6.1** (Li-Liu [82]) An uncertain proposition is a statement whose truth value is quantified by an uncertain measure.

That is, if we use $X$ to express an uncertain proposition and use $\alpha$ to express its truth value in uncertain measure, then the uncertain proposition $X$ is essentially a Boolean uncertain variable

$$X = \begin{cases} 
1 & \text{with uncertain measure } \alpha \\
0 & \text{with uncertain measure } 1 - \alpha
\end{cases} \quad (6.1)$$

where $X = 1$ means $X$ is true and $X = 0$ means $X$ is false.

**Example 6.1:** “Tom is tall with truth value 0.7” is an uncertain proposition, where “Tom is tall” is a statement, and its truth value is 0.7 in uncertain measure.
**Example 6.2:** “John is young with truth value 0.8” is an uncertain proposition, where “John is young” is a statement, and its truth value is 0.8 in uncertain measure.

**Example 6.3:** “Beijing is a big city with truth value 0.9” is an uncertain proposition, where “Beijing is a big city” is a statement, and its truth value is 0.9 in uncertain measure.

**Connective Symbols**

In addition to the proposition symbols $X$ and $Y$, we also need the negation symbol $\neg$, conjunction symbol $\land$, disjunction symbol $\lor$, conditional symbol $\rightarrow$, and biconditional symbol $\leftrightarrow$. Note that

\begin{align*}
\neg X & \text{ means “not } X\text{”;} \\
X \land Y & \text{ means “} X \text{ and } Y \text{”;} \\
X \lor Y & \text{ means “} X \text{ or } Y \text{”;} \\
X \rightarrow Y & = (\neg X) \lor Y \text{ means “if } X \text{ then } Y\text{”,} \\
X \leftrightarrow Y & = (X \rightarrow Y) \land (Y \rightarrow X) \text{ means “} X \text{ if and only if } Y \text{”.}
\end{align*}

**Boolean Function of Uncertain Propositions**

Assume $X_1, X_2, \cdots, X_n$ are uncertain propositions. Then their Boolean function

$$Z = f(X_1, X_2, \cdots, X_n)$$

(6.7)

is a Boolean uncertain variable. Thus $Z$ is also an uncertain proposition provided that it makes sense. Usually, such a Boolean function is a finite sequence of uncertain propositions and connective symbols. For example,

$$Z = \neg X_1, \quad Z = X_1 \land (\neg X_2), \quad Z = X_1 \rightarrow X_2$$

(6.8)

are all uncertain propositions.

**Independence of Uncertain Propositions**

Uncertain propositions are called *independent* if they are independent uncertain variables. Assume $X_1, X_2, \cdots, X_n$ are independent uncertain propositions. Then

$$f_1(X_1), f_2(X_2) \cdots, f_n(X_n)$$

(6.9)

are also independent uncertain propositions for any Boolean functions $f_1, f_2, \cdots, f_n$. For example, if $X_1, X_2, \cdots, X_5$ are independent uncertain propositions, then $\neg X_1, X_2 \lor X_3, X_4 \rightarrow X_5$ are also independent.


6.2 Truth Value

Truth value is a key concept in uncertain propositional logic, and is defined as the uncertain measure that the uncertain proposition is true.

**Definition 6.2** (Li-Liu [82]) Let $X$ be an uncertain proposition. Then the truth value of $X$ is defined as the uncertain measure that $X$ is true, i.e.,

$$T(X) = M\{X = 1\}. \quad (6.10)$$

**Example 6.4:** Let $X$ be an uncertain proposition with truth value $\alpha$. Then

$$T(\neg X) = M\{X = 0\} = 1 - \alpha. \quad (6.11)$$

**Example 6.5:** Let $X$ and $Y$ be two independent uncertain propositions with truth values $\alpha$ and $\beta$, respectively. Then

$$
T(X \land Y) = M\{X \land Y = 1\} = M\{(X = 1) \cap (Y = 1)\} = \alpha \land \beta, \quad (6.12)
$$

$$
T(X \lor Y) = M\{X \lor Y = 1\} = M\{(X = 1) \cup (Y = 1)\} = \alpha \lor \beta, \quad (6.13)
$$

$$
T(X \rightarrow Y) = T(\neg X \lor Y) = (1 - \alpha) \lor \beta. \quad (6.14)
$$

**Theorem 6.1** (Law of Excluded Middle) Let $X$ be an uncertain proposition. Then $X \lor \neg X$ is a tautology, i.e.,

$$T(X \lor \neg X) = 1. \quad (6.15)$$

**Proof:** It follows from the definition of truth value and the property of uncertain measure that

$$T(X \lor \neg X) = M\{X \lor \neg X = 1\} = M\{(X = 1) \cup (X = 0)\} = M\{\Gamma\} = 1.$$

The theorem is proved.

**Theorem 6.2** (Law of Contradiction) Let $X$ be an uncertain proposition. Then $X \land \neg X$ is a contradiction, i.e.,

$$T(X \land \neg X) = 0. \quad (6.16)$$

**Proof:** It follows from the definition of truth value and the property of uncertain measure that

$$T(X \land \neg X) = M\{X \land \neg X = 1\} = M\{(X = 1) \cap (X = 0)\} = M\{\emptyset\} = 0.$$

The theorem is proved.
Theorem 6.3 (Law of Truth Conservation) Let $X$ be an uncertain proposition. Then we have
\[ T(X) + T(\neg X) = 1. \] (6.17)

**Proof:** It follows from the duality axiom of uncertain measure that
\[ T(\neg X) = M\{\neg X = 1\} = M\{X = 0\} = 1 - M\{X = 1\} = 1 - T(X). \]
The theorem is proved.

Theorem 6.4 Let $X$ be an uncertain proposition. Then $X \rightarrow X$ is a tautology, i.e.,
\[ T(X \rightarrow X) = 1. \] (6.18)

**Proof:** It follows from the definition of conditional symbol and the law of excluded middle that
\[ T(X \rightarrow X) = T(\neg X \lor X) = 1. \]
The theorem is proved.

Theorem 6.5 Let $X$ be an uncertain proposition. Then we have
\[ T(X \rightarrow \neg X) = 1 - T(X). \] (6.19)

**Proof:** It follows from the definition of conditional symbol and the law of truth conservation that
\[ T(X \rightarrow \neg X) = T(\neg X \lor \neg X) = T(\neg X) = 1 - T(X). \]
The theorem is proved.

Theorem 6.6 (De Morgan’s Law) For any uncertain propositions $X$ and $Y$, we have
\[ T(\neg(X \land Y)) = T((\neg X) \lor (\neg Y)), \] (6.20)
\[ T(\neg(X \lor Y)) = T((\neg X) \land (\neg Y)). \] (6.21)

**Proof:** It follows from the basic properties of uncertain measure that
\[ T(\neg(X \land Y)) = M\{X \land Y = 0\} = M\{X = 0 \lor Y = 0\} \]
\[ = M\{\neg X \lor \neg Y = 1\} = T((\neg X) \lor (\neg Y)) \]
which proves the first equality. A similar way may verify the second equality.

Theorem 6.7 (Law of Contraposition) For any uncertain propositions $X$ and $Y$, we have
\[ T(X \rightarrow Y) = T(\neg Y \rightarrow \neg X). \] (6.22)

**Proof:** It follows from the definition of conditional symbol and basic properties of uncertain measure that
\[ T(X \rightarrow Y) = M\{(\neg X) \lor Y = 1\} = M\{(X = 0) \lor (Y = 1)\} \]
\[ = M\{Y \lor (\neg X) = 1\} = T(\neg Y \rightarrow \neg X). \]
The theorem is proved.
6.3 Chen-Ralescu Theorem

An important contribution to uncertain propositional logic is the Chen-Ralescu theorem that provides a numerical method for calculating the truth values of uncertain propositions.

**Theorem 6.8** (Chen-Ralescu Theorem [8]) Assume that $X_1, X_2, \cdots, X_n$ are independent uncertain propositions with truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Then for a Boolean function $f$, the uncertain proposition

$$Z = f(X_1, X_2, \cdots, X_n).$$

has a truth value

$$T(Z) = \begin{cases} 
\sup_{f(x_1, x_2, \cdots, x_n) = 1} \min_{1 \leq i \leq n} \nu_i(x_i), \\
1 - \sup_{f(x_1, x_2, \cdots, x_n) = 0} \min_{1 \leq i \leq n} \nu_i(x_i), \\
\end{cases}$$

(6.24)

where $x_i$ take values either 0 or 1, and $\nu_i$ are defined by

$$\nu_i(x_i) = \begin{cases} 
\alpha_i, & \text{if } x_i = 1 \\
1 - \alpha_i, & \text{if } x_i = 0 \\
\end{cases}$$

(6.25)

for $i = 1, 2, \cdots, n$, respectively.

**Proof:** Since $Z = 1$ if and only if $f(X_1, X_2, \cdots, X_n) = 1$, we immediately have

$$T(Z) = \mathcal{M}\{f(X_1, X_2, \cdots, X_n) = 1\}.$$

Thus the equation (6.24) follows from Theorem 2.22 immediately.

**Example 6.6:** Let $X_1$ and $X_2$ be independent uncertain propositions with truth values $\alpha_1$ and $\alpha_2$, respectively. Then

$$Z = X_1 \leftrightarrow X_2$$

(6.26)

is an uncertain proposition. It is clear that $Z = f(X_1, X_2)$ if we define

$$f(1, 1) = 1, \quad f(1, 0) = 0, \quad f(0, 1) = 0, \quad f(0, 0) = 1.$$

At first, we have

$$\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = \max\{\alpha_1 \land \alpha_2, (1 - \alpha_1) \land (1 - \alpha_2)\},$$
\[
\sup_{f(x_1, x_2) = 0} \min_{1 \leq i \leq 2} \nu_i(x_i) = \max\{(1 - \alpha_1) \land \alpha_2, \alpha_1 \land (1 - \alpha_2)\}.
\]

When \(\alpha_1 \geq 0.5\) and \(\alpha_2 \geq 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = \alpha_1 \land \alpha_2 \geq 0.5.
\]

It follows from Chen-Ralescu theorem that
\[
T(Z) = 1 - \sup_{f(x_1, x_2) = 0} \min_{1 \leq i \leq 2} \nu_i(x_i) = 1 - (1 - \alpha_1) \lor (1 - \alpha_2) = \alpha_1 \land \alpha_2.
\]

When \(\alpha_1 \geq 0.5\) and \(\alpha_2 < 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = (1 - \alpha_1) \lor \alpha_2 \leq 0.5.
\]

It follows from Chen-Ralescu theorem that
\[
T(Z) = \sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = (1 - \alpha_1) \lor \alpha_2.
\]

When \(\alpha_1 < 0.5\) and \(\alpha_2 \geq 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = \alpha_1 \lor (1 - \alpha_2) \leq 0.5.
\]

It follows from Chen-Ralescu theorem that
\[
T(Z) = \sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = \alpha_1 \lor (1 - \alpha_2).
\]

When \(\alpha_1 < 0.5\) and \(\alpha_2 < 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq 2} \nu_i(x_i) = (1 - \alpha_1) \land (1 - \alpha_2) > 0.5.
\]

It follows from Chen-Ralescu theorem that
\[
T(Z) = 1 - \sup_{f(x_1, x_2) = 0} \min_{1 \leq i \leq 2} \nu_i(x_i) = 1 - \alpha_1 \lor \alpha_2 = (1 - \alpha_1) \land (1 - \alpha_2).
\]

Thus we have
\[
T(Z) = \begin{cases} 
\alpha_1 \land \alpha_2, & \text{if } \alpha_1 \geq 0.5 \text{ and } \alpha_2 \geq 0.5 \\
(1 - \alpha_1) \lor \alpha_2, & \text{if } \alpha_1 \geq 0.5 \text{ and } \alpha_2 < 0.5 \\
\alpha_1 \lor (1 - \alpha_2), & \text{if } \alpha_1 < 0.5 \text{ and } \alpha_2 \geq 0.5 \\
(1 - \alpha_1) \land (1 - \alpha_2), & \text{if } \alpha_1 < 0.5 \text{ and } \alpha_2 < 0.5.
\end{cases} \tag{6.27}
\]

**Example 6.7:** The independence condition in Theorem 6.8 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2\}\) with power set and \(\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5\). Then
\[
X_1(\gamma) = \begin{cases} 
0, & \text{if } \gamma = \gamma_1 \\
1, & \text{if } \gamma = \gamma_2
\end{cases} \tag{6.28}
\]
is an uncertain proposition with truth value
\[ T(X_1) = 0.5, \] (6.29)
and
\[ X_2(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_1 \\
0, & \text{if } \gamma = \gamma_2
\end{cases} \] (6.30)
is also an uncertain proposition with truth value
\[ T(X_2) = 0.5. \] (6.31)
Note that \( X_1 \) and \( X_2 \) are not independent, and \( X_1 \lor X_2 \equiv 1 \) from which we obtain
\[ T(X_1 \lor X_2) = 1. \] (6.32)
However, by using (6.24), we get
\[ T(X_1 \lor X_2) = 0.5. \] (6.33)
Thus the independence condition cannot be removed.

**Exercise 6.1:** Let \( X_1, X_2, \cdots, X_n \) be independent uncertain propositions with truth values \( \alpha_1, \alpha_2, \cdots, \alpha_n \), respectively. Then
\[ Z = X_1 \land X_2 \land \cdots \land X_n \] (6.34)
is an uncertain proposition. Show that the truth value of \( Z \) is
\[ T(Z) = \alpha_1 \land \alpha_2 \land \cdots \land \alpha_n. \] (6.35)

**Exercise 6.2:** Let \( X_1, X_2, \cdots, X_n \) be independent uncertain propositions with truth values \( \alpha_1, \alpha_2, \cdots, \alpha_n \), respectively. Then
\[ Z = X_1 \lor X_2 \lor \cdots \lor X_n \] (6.36)
is an uncertain proposition. Show that the truth value of \( Z \) is
\[ T(Z) = \alpha_1 \lor \alpha_2 \lor \cdots \lor \alpha_n. \] (6.37)

**Exercise 6.3:** Let \( X_1 \) and \( X_2 \) be independent uncertain propositions with truth values \( \alpha_1 \) and \( \alpha_2 \), respectively. (i) What is the truth value of \( (X_1 \land X_2) \rightarrow X_2 \)? (ii) What is the truth value of \( (X_1 \lor X_2) \rightarrow X_2 \)? (iii) What is the truth value of \( X_1 \rightarrow (X_1 \land X_2) \)? (iv) What is the truth value of \( X_1 \rightarrow (X_1 \lor X_2) \)?

**Exercise 6.4:** Let \( X_1, X_2, X_3 \) be independent uncertain propositions with truth values \( \alpha_1, \alpha_2, \alpha_3 \), respectively. What is the truth value of
\[ X_1 \land (X_1 \lor X_2) \land (X_1 \lor X_2 \lor X_3)? \] (6.38)
6.4 Uncertain Predicate Logic

Consider the following propositions: “Beijing is a big city”, and “Tianjin is a big city”. Uncertain propositional logic treats them as unrelated propositions. However, uncertain predicate logic represents them by a predicate proposition $X(a)$. If $a$ represents Beijing, then

$$X(a) = \text{“Beijing is a big city”}.$$  \hspace{1cm} (6.39)

If $a$ represents Tianjin, then

$$X(a) = \text{“Tianjin is a big city”}.$$  \hspace{1cm} (6.40)

**Definition 6.3** (Zhang-Li [220]) Uncertain predicate proposition is a sequence of uncertain propositions indexed by one or more parameters.

In order to deal with uncertain predicate propositions, we need a universal quantifier $\forall$ and an existential quantifier $\exists$. If $X(a)$ is an uncertain predicate proposition defined by (6.39) and (6.40), then

$$\forall a X(a) = \text{“Both Beijing and Tianjin are big cities”},$$  \hspace{1cm} (6.41)

$$\exists a X(a) = \text{“At least one of Beijing and Tianjin is a big city”}.$$  \hspace{1cm} (6.42)

**Theorem 6.9** (Zhang-Li [220], Law of Excluded Middle) Let $X(a)$ be an uncertain predicate proposition. Then

$$T((\forall a) X(a) \lor (\exists a) \neg X(a)) = 1.$$  \hspace{1cm} (6.43)

**Proof:** Since $\neg(\forall a) X(a) = (\exists a) \neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\forall a) X(a) \lor (\exists a) \neg X(a)) = M\{(\forall a) X(a) = 1 \cup (\forall a) X(a) = 0\} = 1.$$

The theorem is proved.

**Theorem 6.10** (Zhang-Li [220], Law of Contradiction) Let $X(a)$ be an uncertain predicate proposition. Then

$$T((\forall a) X(a) \land (\exists a) \neg X(a)) = 0.$$  \hspace{1cm} (6.44)

**Proof:** Since $\neg(\forall a) X(a) = (\exists a) \neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\forall a) X(a) \land (\exists a) \neg X(a)) = M\{(\forall a) X(a) = 1 \cap (\forall a) X(a) = 0\} = 0.$$

The theorem is proved.
Theorem 6.11 (Zhang-Li [220], Law of Truth Conservation) Let $X(a)$ be an uncertain predicate proposition. Then

$$T((\forall a)X(a)) + T((\exists a)\neg X(a)) = 1. \quad (6.45)$$

**Proof:** Since $\neg (\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\exists a)\neg X(a)) = 1 - M\{(\forall a)X(a) = 1\} = 1 - T((\forall a)X(a)).$$

The theorem is proved.

Theorem 6.12 (Zhang-Li [220]) Let $X(a)$ be an uncertain predicate proposition. Then for any given $b$, we have

$$T((\forall a)X(a) \rightarrow X(b)) = 1. \quad (6.46)$$

**Proof:** The argument breaks into two cases. Case 1: If $X(b) = 0$, then $(\forall a)X(a) = 0$ and $\neg (\forall a)X(a) = 1$. Thus

$$(\forall a)X(a) \rightarrow X(b) = \neg (\forall a)X(a) \lor X(b) = 1.$$ 

Case II: If $X(b) = 1$, then we immediately have

$$(\forall a)X(a) \rightarrow X(b) = \neg (\forall a)X(a) \lor X(b) = 1.$$ 

Thus we always have (6.46). The theorem is proved.

Theorem 6.13 (Zhang-Li [220]) Let $X(a)$ be an uncertain predicate proposition. Then for any given $b$, we have

$$T(X(b) \rightarrow (\exists a)X(a)) = 1. \quad (6.47)$$

**Proof:** The argument breaks into two cases. Case 1: If $X(b) = 0$, then $\neg X(b) = 1$ and

$$X(b) \rightarrow (\forall a)X(a) = \neg X(b) \lor (\exists a)X(a) = 1.$$ 

Case II: If $X(b) = 1$, then $(\exists a)X(a) = 1$ and

$$X(b) \rightarrow (\exists a)X(a) = \neg X(b) \lor (\exists a)X(a) = 1.$$ 

Thus we always have (6.47). The theorem is proved.

Theorem 6.14 (Zhang-Li [220]) Let $X(a)$ be an uncertain predicate proposition. Then

$$T((\forall a)X(a) \rightarrow (\exists a)X(a)) = 1. \quad (6.48)$$
Theorem 6.15 (Zhang-Li [220]) Let $X(a)$ be an uncertain predicate proposition such that $\{X(a)|a \in A\}$ is a class of independent uncertain propositions. Then
\[
T((\forall a)X(a)) = \inf_{a \in A} T(X(a)), \tag{6.49}
\]
\[
T((\exists a)X(a)) = \sup_{a \in A} T(X(a)). \tag{6.50}
\]

Proof: For each uncertain predicate proposition $X(a)$, by the meaning of universal quantifier, we obtain
\[
T((\forall a)X(a)) = M\{ (\forall a)X(a) = 1 \} = M\left( \bigcap_{a \in A} (X(a) = 1) \right).
\]
Since $\{X(a)|a \in A\}$ is a class of independent uncertain propositions, we get
\[
T((\forall a)X(a)) = \inf_{a \in A} M\{X(a) = 1\} = \inf_{a \in A} T(X(a)).
\]
The first equation is verified. Similarly, by the meaning of existential quantifier, we obtain
\[
T((\exists a)X(a)) = M\{ (\exists a)X(a) = 1 \} = M\left( \bigcup_{a \in A} (X(a) = 1) \right).
\]
Since $\{X(a)|a \in A\}$ is a class of independent uncertain propositions, we get
\[
T((\exists a)X(a)) = \sup_{a \in A} M\{X(a) = 1\} = \sup_{a \in A} T(X(a)).
\]
The second equation is proved.

Theorem 6.16 (Zhang-Li [220]) Let $X(a,b)$ be an uncertain predicate proposition such that $\{X(a,b)|a \in A, b \in B\}$ is a class of independent uncertain propositions. Then
\[
T((\forall a)(\exists b)X(a,b)) = \inf_{a \in A} \sup_{b \in B} T(X(a,b)), \tag{6.51}
\]
\[
T((\exists a)(\forall b)X(a,b)) = \sup_{a \in A} \inf_{b \in B} T(X(a,b)). \tag{6.52}
\]
Proof: Since \( \{X(a, b) | a \in A, b \in B\} \) is a class of independent uncertain propositions, both \( \{ (\exists b)X(a, b) | a \in A \} \) and \( \{ (\forall b)X(a, b) | a \in A \} \) are two classes of independent uncertain propositions. It follows from Theorem 6.15 that
\[
T((\forall a)(\exists b)X(a, b)) = \inf_{a \in A} T((\exists b)X(a, b)) = \inf_{a \in A, b \in B} T(X(a, b)),
\]
\[
T((\exists a)(\forall b)X(a, b)) = \sup_{a \in A} T((\forall b)X(a, b)) = \sup_{a \in A, b \in B} \inf_{b \in B} T(X(a, b)).
\]
The theorem is proved.

6.5 Bibliographic Notes

Uncertain propositional logic was designed by Li-Liu [82] in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. An important contribution is Chen-Ralescu theorem [8] that provides a numerical method for calculating the truth value of uncertain propositions.

Another topic is the uncertain predicate logic developed by Zhang-Li [220] in which an uncertain predicate proposition is defined as a sequence of uncertain propositions indexed by one or more parameters.
Chapter 7

Uncertain Entailment

Uncertain entailment is a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. In some sense, uncertain propositional logic and uncertain entailment are mutually inverse, the former attempts to compose a complex proposition from simpler ones, while the latter attempts to decompose a complex proposition into simpler ones.

This chapter will present an uncertain entailment model. In addition, uncertain modus ponens, uncertain modus tollens and uncertain hypothetical syllogism are deduced from the uncertain entailment model.

7.1 Uncertain Entailment Model

Assume $X_1, X_2, \cdots, X_n$ are independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Also assume that

$$Y_j = f_j(X_1, X_2, \cdots, X_n)$$

are uncertain propositions with known truth values $c_j$, $j = 1, 2, \cdots, m$, respectively. Now let

$$Z = f(X_1, X_2, \cdots, X_n)$$

be an additional uncertain proposition. What is the truth value of $Z$? This is just the uncertain entailment problem. In order to solve it, let us consider what values $\alpha_1, \alpha_2, \cdots, \alpha_n$ may take. The first constraint is

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \cdots, n.$$  

The second type of constraints is represented by

$$T(Y_j) = c_j$$
where $T(Y_j)$ are determined by $\alpha_1, \alpha_2, \cdots, \alpha_n$ via

\[
T(Y_j) = \begin{cases} 
\sup \min \nu_i(x_i), & \text{if } \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\
1 - \sup \min \nu_i(x_i), & \text{if } \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5
\end{cases}
\tag{7.5}
\]

for $j = 1, 2, \cdots, m$ and

\[
\nu_i(x_i) = \begin{cases} 
\alpha_i, & \text{if } x_i = 1 \\
1 - \alpha_i, & \text{if } x_i = 0
\end{cases}
\tag{7.6}
\]

for $i = 1, 2, \cdots, n$. Please note that the additional uncertain proposition $Z = f(X_1, X_2, \cdots, X_n)$ has a truth value

\[
T(Z) = \begin{cases} 
\sup \min \nu_i(x_i), & \text{if } \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \leq i \leq n} \nu_i(x_i) < 0.5 \\
1 - \sup \min \nu_i(x_i), & \text{if } \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1 \leq i \leq n} \nu_i(x_i) \geq 0.5
\end{cases}
\tag{7.7}
\]

Since the truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not uniquely determined, the truth value $T(Z)$ is not unique too. In this case, we have to use the maximum uncertainty principle to determine the truth value $T(Z)$. That is, $T(Z)$ should be assigned the value as close to 0.5 as possible. In other words, we should minimize the value $|T(Z) - 0.5|$ via choosing appreciable values of $\alpha_1, \alpha_2, \cdots, \alpha_n$. The uncertain entailment model is thus written by Liu [92] as follows,

\[
\begin{cases} 
\min |T(Z) - 0.5| \\
\text{subject to:} \\
0 \leq \alpha_i \leq 1, \quad i = 1, 2, \cdots, n \\
T(Y_j) = c_j, \quad j = 1, 2, \cdots, m
\end{cases}
\tag{7.8}
\]

where $T(Z), T(Y_j), j = 1, 2, \cdots, m$ are functions of unknown truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$.

**Example 7.1:** Let $A$ and $B$ be independent uncertain propositions. It is known that

\[
T(A \lor B) = a, \quad T(A \land B) = b.
\tag{7.9}
\]
What is the truth value of $A \rightarrow B$? Denote the truth values of $A$ and $B$ by $\alpha_1$ and $\alpha_2$, respectively, and write

$$Y_1 = A \lor B, \quad Y_2 = A \land B, \quad Z = A \rightarrow B.$$ 

It is clear that

$$T(Y_1) = \alpha_1 \lor \alpha_2 = a,$$

$$T(Y_2) = \alpha_1 \land \alpha_2 = b,$$

$$T(Z) = (1 - \alpha_1) \lor \alpha_2.$$ 

In this case, the uncertain entailment model (7.8) becomes

$$\begin{align*}
\min & \quad |(1 - \alpha_1) \lor \alpha_2 - 0.5| \\
\text{subject to:} & \\
0 & \leq \alpha_1 \leq 1 \\
0 & \leq \alpha_2 \leq 1 \\
\alpha_1 \lor \alpha_2 & = a \\
\alpha_1 \land \alpha_2 & = b.
\end{align*}$$

(7.10)

When $a \geq b$, there are only two feasible solutions $(\alpha_1, \alpha_2) = (a, b)$ and $(\alpha_1, \alpha_2) = (b, a)$. If $a + b < 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \lor \alpha_2^* = 1 - a;$$

if $a + b = 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \lor \alpha_2^* = a \text{ or } b;$$

if $a + b > 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \lor \alpha_2^* = b.$$

When $a < b$, there is no feasible solution and the truth values are ill-assigned. In summary, from $T(A \lor B) = a$ and $T(A \land B) = b$ we entail

$$T(A \rightarrow B) = \begin{cases} 
1 - a, & \text{if } a \geq b \text{ and } a + b < 1 \\
a \text{ or } b, & \text{if } a \geq b \text{ and } a + b = 1 \\
b, & \text{if } a \geq b \text{ and } a + b > 1 \\
\text{illness}, & \text{if } a < b.
\end{cases}$$

(7.11)

**Exercise 7.1:** Let $A, B, C$ be independent uncertain propositions. It is known that

$$T(A \rightarrow C) = a, \quad T(B \rightarrow C) = b, \quad T(A \lor B) = c.$$ 

(7.12)
What is the truth value of $C$?

**Exercise 7.2:** Let $A, B, C, D$ be independent uncertain propositions. It is known that

$$T(A \rightarrow C) = a, \quad T(B \rightarrow D) = b, \quad T(A \lor B) = c.$$  \hspace{1cm} (7.13)

What is the truth value of $C \lor D$?

**Exercise 7.3:** Let $A, B, C$ be independent uncertain propositions. It is known that

$$T(A \lor B) = a, \quad T(\neg A \lor C) = b.$$ \hspace{1cm} (7.14)

What is the truth value of $B \lor C$?

### 7.2 Uncertain Modus Ponens

Uncertain modus ponens was presented by Liu [92]. Let $A$ and $B$ be independent uncertain propositions. Assume $A$ and $A \rightarrow B$ have truth values $a$ and $b$, respectively. What is the truth value of $B$? Denote the truth values of $A$ and $B$ by $\alpha_1$ and $\alpha_2$, respectively, and write

$$Y_1 = A, \quad Y_2 = A \rightarrow B, \quad Z = B.$$  

It is clear that

$$T(Y_1) = \alpha_1 = a,$$
$$T(Y_2) = (1 - \alpha_1) \lor \alpha_2 = b,$$
$$T(Z) = \alpha_2.$$  

In this case, the uncertain entailment model (7.8) becomes

$$\begin{align*}
\min |\alpha_2 - 0.5| \\
\text{subject to:} \\
0 \leq \alpha_1 \leq 1 \\
0 \leq \alpha_2 \leq 1 \\
\alpha_1 = a \\
(1 - \alpha_1) \lor \alpha_2 = b.
\end{align*}$$ \hspace{1cm} (7.15)

When $a + b > 1$, there is a unique feasible solution and then the optimal solution is

$$\alpha_1^* = a, \quad \alpha_2^* = b.$$  

Thus $T(B) = \alpha_2^* = b$. When $a + b = 1$, the feasible set is $\{a\} \times [0, b]$ and the optimal solution is

$$\alpha_1^* = a, \quad \alpha_2^* = 0.5 \land b.$$
Thus $T(B) = \alpha_2^* = 0.5 \land b$. When $a + b < 1$, there is no feasible solution and the truth values are ill-assigned. In summary, from
\[ T(A) = a, \quad T(A \rightarrow B) = b \quad (7.16) \]
we entail
\[ T(B) = \begin{cases} 
  b, & \text{if } a + b > 1 \\
  0.5 \land b, & \text{if } a + b = 1 \\
  \text{illness}, & \text{if } a + b < 1.
\end{cases} \quad (7.17) \]
This result coincides with the classical modus ponens that if both $A$ and $A \rightarrow B$ are true, then $B$ is true.

## 7.3 Uncertain Modus Tollens

Uncertain modus tollens was presented by Liu [92]. Let $A$ and $B$ be independent uncertain propositions. Assume $A \rightarrow B$ and $B$ have truth values $a$ and $b$, respectively. What is the truth value of $A$? Denote the truth values of $A$ and $B$ by $\alpha_1$ and $\alpha_2$, respectively, and write
\[ Y_1 = A \rightarrow B, \quad Y_2 = B, \quad Z = A. \]
It is clear that
\[ T(Y_1) = (1 - \alpha_1) \lor \alpha_2 = a, \]
\[ T(Y_2) = \alpha_2 = b, \]
\[ T(Z) = \alpha_1. \]
In this case, the uncertain entailment model (7.8) becomes
\[ \begin{align*}
  \min |\alpha_1 - 0.5| \\
  \text{subject to:} \\
  0 \leq \alpha_1 \leq 1 \\
  0 \leq \alpha_2 \leq 1 \\
  (1 - \alpha_1) \lor \alpha_2 = a \\
  \alpha_2 = b.
\end{align*} \quad (7.18) \]
When $a > b$, there is a unique feasible solution and then the optimal solution is
\[ \alpha_1^* = 1 - a, \quad \alpha_2^* = b. \]
Thus $T(A) = \alpha_1^* = 1 - a$. When $a = b$, the feasible set is $[1 - a, 1] \times \{b\}$ and the optimal solution is
\[ \alpha_1^* = (1 - a) \lor 0.5, \quad \alpha_2^* = b. \]
Thus $T(A) = \alpha_1^* = (1 - a) \lor 0.5$. When $a < b$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A \rightarrow B) = a, \quad T(B) = b$$  \hspace{1cm} (7.19)

we entail

$$T(A) = \begin{cases} 
1 - a, & \text{if } a > b \\
(1 - a) \lor 0.5, & \text{if } a = b \\
\text{illness,} & \text{if } a < b.
\end{cases}$$  \hspace{1cm} (7.20)

This result coincides with the classical modus tollens that if $A \rightarrow B$ is true and $B$ is false, then $A$ is false.

### 7.4 Uncertain Hypothetical Syllogism

Uncertain hypothetical syllogism was presented by Liu [92]. Let $A, B, C$ be independent uncertain propositions. Assume $A \rightarrow B$ and $B \rightarrow C$ have truth values $a$ and $b$, respectively. What is the truth value of $A \rightarrow C$? Denote the truth values of $A, B, C$ by $\alpha_1, \alpha_2, \alpha_3$, respectively, and write

$$Y_1 = A \rightarrow B, \quad Y_2 = B \rightarrow C, \quad Z = A \rightarrow C.$$

It is clear that

$$T(Y_1) = (1 - \alpha_1) \lor \alpha_2 = a,$$
$$T(Y_2) = (1 - \alpha_2) \lor \alpha_3 = b,$$
$$T(Z) = (1 - \alpha_1) \lor \alpha_3.$$

In this case, the uncertain entailment model (7.8) becomes

$$\begin{cases} 
\min |(1 - \alpha_1) \lor \alpha_3 - 0.5| \\
\text{subject to:} \\
0 \leq \alpha_1 \leq 1 \\
0 \leq \alpha_2 \leq 1 \\
0 \leq \alpha_3 \leq 1 \\
(1 - \alpha_1) \lor \alpha_2 = a \\
(1 - \alpha_2) \lor \alpha_3 = b.
\end{cases}$$  \hspace{1cm} (7.21)

Write the optimal solution by $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. When $a \land b \geq 0.5$, we have

$$T(A \rightarrow C) = (1 - \alpha_1^*) \lor \alpha_3^* = a \land b.$$  

When $a + b \geq 1$ and $a \land b < 0.5$, we have

$$T(A \rightarrow C) = (1 - \alpha_1^*) \lor \alpha_3^* = 0.5.$$
When \( a + b < 1 \), there is no feasible solution and the truth values are ill-assigned. In summary, from

\[
T(A \rightarrow B) = a, \quad T(B \rightarrow C) = b
\]  

we entail

\[
T(A \rightarrow C) = \begin{cases} 
  a \land b, & \text{if } a \geq 0.5 \text{ and } b \geq 0.5 \\
  0.5, & \text{if } a + b \geq 1 \text{ and } a \land b < 0.5 \\
  \text{illness}, & \text{if } a + b < 1.
\end{cases}
\]  

(7.23)

This result coincides with the classical hypothetical syllogism that if both \( A \rightarrow B \) and \( B \rightarrow C \) are true, then \( A \rightarrow C \) is true.

### 7.5 Bibliographic Notes

Uncertain entailment was proposed by Liu [92] for determining the truth value of an uncertain proposition via the maximum uncertainty principle when the truth values of other uncertain propositions are given.

From the uncertain entailment model, Liu [92] deduced uncertain modus ponens, uncertain modus tollens, and uncertain hypothetical syllogism. After that, Yang-Gao-Ni [180] investigated the uncertain resolution principle.
Chapter 8

Uncertain Set

Uncertain set was first proposed by Liu [93] in 2010 for modelling unsharp concepts. This chapter will introduce the concepts of uncertain set, membership function, independence, expected value, variance, distance, and entropy. This chapter will also introduce the operational law for uncertain sets via membership functions or inverse membership functions. Finally, conditional uncertain set and conditional membership function are documented.

8.1 Uncertain Set

Roughly speaking, an uncertain set is a set-valued function on an uncertainty space, and attempts to model “unsharp concepts” that are essentially sets but their boundaries are not sharply described (because of the ambiguity of human language). Some typical examples include “young”, “tall”, “warm”, and “most”. A formal definition is given as follows.

Definition 8.1 (Liu [93]) An uncertain set is a function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to a collection of sets of real numbers such that both \( \{ B \subset \xi \} \) and \( \{ \xi \subset B \} \) are events for any Borel set \( B \) of real numbers.

Remark 8.1: Note that the events \( \{ B \subset \xi \} \) and \( \{ \xi \subset B \} \) are subsets of the universal set \( \Gamma \), i.e.,

\[
\{ B \subset \xi \} = \{ \gamma \in \Gamma | B \subset \xi(\gamma) \},
\]

\[
\{ \xi \subset B \} = \{ \gamma \in \Gamma | \xi(\gamma) \subset B \}. \tag{8.1}
\]

Remark 8.2: It is clear that uncertain set (Liu [93]) is very different from random set (Robbins [145] and Matheron [126]) and fuzzy set (Zadeh [209]). The essential difference among them is that different measures are used, i.e., random set uses probability measure, fuzzy set uses possibility measure, and uncertain set uses uncertain measure.
Remark 8.3: What is the difference between uncertain variable and uncertain set? Both of them belong to the same broad category of uncertain concepts. However, they are differentiated by their mathematical definitions: the former refers to one value, while the latter to a collection of values. Essentially, the difference between uncertain variable and uncertain set focuses on the property of exclusivity. If the concept has exclusivity, then it is an uncertain variable. Otherwise, it is an uncertain set. Consider the statement “John is a young man”. If we are interested in John’s real age, then “young” is an uncertain variable because it is an exclusive concept (John’s age cannot be more than one value). For example, if John is 20 years old, then it is impossible that John is 25 years old. In other words, “John is 20 years old” does exclude the possibility that “John is 25 years old”. By contrast, if we are interested in what ages can be regarded “young”, then “young” is an uncertain set because the concept now has no exclusivity. For example, both 20-year-old and 25-year-old men can be considered “young”. In other words, “a 20-year-old man is young” does not exclude the possibility that “a 25-year-old man is young”.

Example 8.1: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2, \gamma_3\}\) with power set and \(\mathcal{M}\{\gamma_1\} = 0.6, \mathcal{M}\{\gamma_2\} = 0.3, \mathcal{M}\{\gamma_3\} = 0.2\). Then

\[
\xi(\gamma) = \begin{cases} 
[1, 3], & \text{if } \gamma = \gamma_1 \\
[2, 4], & \text{if } \gamma = \gamma_2 \\
[3, 5], & \text{if } \gamma = \gamma_3
\end{cases}
\]  

(8.3)

is an uncertain set. See Figure 8.1. Furthermore, we have

\[
\mathcal{M}\{2 \in \xi\} = \mathcal{M}\{\gamma \mid 2 \in \xi(\gamma)\} = \mathcal{M}\{\gamma_1, \gamma_2\} = 0.8, 
\]

(8.4)

\[
\mathcal{M}\{[3, 4] \subset \xi\} = \mathcal{M}\{\gamma \mid [3, 4] \subset \xi(\gamma)\} = \mathcal{M}\{\gamma_2, \gamma_3\} = 0.4, 
\]

(8.5)

\[
\mathcal{M}\{\xi \subset [1, 5]\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \subset [1, 5]\} = \mathcal{M}\{\gamma_1, \gamma_2, \gamma_3\} = 1. 
\]

(8.6)

![Figure 8.1: An Uncertain Set](image)
Example 8.2: Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then
\[
\xi(\gamma) = [0, 3\gamma], \quad \forall \gamma \in \Gamma
\] (8.7)
is an uncertain set. Furthermore, we have
\[
\mathcal{M}\{2 \in \xi\} = \mathcal{M}\{\gamma \mid 2 \in \xi(\gamma)\} = \mathcal{M}\{[2/3, 1]\} = 1/3,
\] (8.8)
\[
\mathcal{M}\{[0, 1] \subset \xi\} = \mathcal{M}\{\gamma \mid [0, 1] \subset \xi(\gamma)\} = \mathcal{M}\{[1/3, 1]\} = 2/3,
\] (8.9)
\[
\mathcal{M}\{\xi \subset [0, 3]\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \subset [0, 3]\} = \mathcal{M}\{[0, 1]\} = 1.
\] (8.10)

Example 8.3: A crisp set \(A\) of real numbers is a special uncertain set on an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) defined by
\[
\xi(\gamma) \equiv A, \quad \forall \gamma \in \Gamma.
\] (8.11)
Furthermore, for any Borel set \(B\) of real numbers, we have
\[
\mathcal{M}\{B \subset \xi\} = \mathcal{M}\{\gamma \mid B \subset \xi(\gamma)\} = \mathcal{M}\{\Gamma\} = 1, \quad \text{if } B \subset A,
\] (8.12)
\[
\mathcal{M}\{B \subset \xi\} = \mathcal{M}\{\gamma \mid B \subset \xi(\gamma)\} = \mathcal{M}\{\emptyset\} = 0, \quad \text{if } B \not\subset A,
\] (8.13)
\[
\mathcal{M}\{\xi \subset B\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \subset B\} = \mathcal{M}\{\Gamma\} = 1, \quad \text{if } A \subset B,
\] (8.14)
\[
\mathcal{M}\{\xi \subset B\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \subset B\} = \mathcal{M}\{\emptyset\} = 0, \quad \text{if } A \not\subset B.
\] (8.15)

Example 8.4: Let \(\xi\) be an uncertain set and let \(x\) be a real number. Then
\[
\{x \in \xi\}^c = \{\gamma \mid x \in \xi(\gamma)\}^c = \{\gamma \mid x \not\in \xi(\gamma)\} = \{x \not\in \xi\}.
\]
Thus \(\{x \in \xi\}\) and \(\{x \not\in \xi\}\) are opposite events. Furthermore, by the duality axiom, we obtain
\[
\mathcal{M}\{x \in \xi\} + \mathcal{M}\{x \not\in \xi\} = 1.
\] (8.16)

Exercise 8.1: Let \(\xi\) be an uncertain set and let \(B\) be a Borel set of real numbers. Show that \(\{\xi \subset B\}\) and \(\{\xi \not\subset B\}\) are opposite events, and
\[
\mathcal{M}\{\xi \subset B\} + \mathcal{M}\{\xi \not\subset B\} = 1.
\] (8.17)

Exercise 8.2: Let \(\xi\) and \(\eta\) be two uncertain sets. Show that \(\{\xi \subset \eta\}\) and \(\{\xi \not\subset \eta\}\) are opposite events, and
\[
\mathcal{M}\{\xi \subset \eta\} + \mathcal{M}\{\xi \not\subset \eta\} = 1.
\] (8.18)

Exercise 8.3: Let \(\emptyset\) be the empty set, and let \(\xi\) be an uncertain set. Show that
\[
\mathcal{M}\{\emptyset \subset \xi\} = 1.
\] (8.19)
Exercise 8.4: Let \( \xi \) be an uncertain set, and let \( \mathbb{R} \) be the set of real numbers. Show that
\[
\mathcal{M}\{\xi \subset \mathbb{R}\} = 1.
\] (8.20)

Exercise 8.5: Let \( \xi \) be an uncertain set. Show that \( \xi \) is always included in itself, i.e.,
\[
\mathcal{M}\{\xi \subset \xi\} = 1.
\] (8.21)

Theorem 8.1 (Liu [109], Fundamental Relationship) Let \( \xi \) be an uncertain set, and let \( B \) be a crisp set of real numbers. Then
\[
\{B \subset \xi\} = \bigcap_{x \in B} \{x \in \xi\},
\] (8.22)
\[
\{\xi \subset B\} = \bigcap_{x \in B^c} \{x \notin \xi\}.
\] (8.23)

Proof: For any \( \gamma \in \{B \subset \xi\} \), we have \( B \subset \xi(\gamma) \). Thus \( x \in \xi(\gamma) \) whenever \( x \in B \). This means \( \gamma \in \{x \in \xi\} \) and then \( \{B \subset \xi\} \subset \{x \in \xi\} \) for any \( x \in B \). Hence
\[
\{B \subset \xi\} \subset \bigcap_{x \in B} \{x \in \xi\}.
\] (8.24)

On the other hand, for any \( \gamma \)
\[
\gamma \in \bigcap_{x \in B} \{x \in \xi\},
\]
we have \( x \in \xi(\gamma) \) whenever \( x \in B \). Thus \( B \subset \xi(\gamma) \), i.e., \( \gamma \in \{B \subset \xi\} \). This means
\[
\{B \subset \xi\} \supset \bigcap_{x \in B} \{x \in \xi\}.
\] (8.25)

It follows from (8.24) and (8.25) that (8.22) holds. The first equation is proved. Next we verify the second equation. For any \( \gamma \in \{\xi \subset B\} \), we have \( \xi(\gamma) \subset B \). Thus \( x \notin \xi(\gamma) \) whenever \( x \in B^c \). This means \( \gamma \in \{x \notin \xi\} \) and then \( \{\xi \subset B\} \subset \{x \notin \xi\} \) for any \( x \in B^c \). Hence
\[
\{\xi \subset B\} \subset \bigcap_{x \in B^c} \{x \notin \xi\}.
\] (8.26)

On the other hand, for any \( \gamma \)
\[
\gamma \in \bigcap_{x \in B^c} \{x \notin \xi\},
\]
we have \( x \notin \xi(\gamma) \) whenever \( x \in B^c \). Thus \( \xi(\gamma) \subset B \), i.e., \( \gamma \in \{\xi \subset B\} \). This means
\[
\{\xi \subset B\} \supset \bigcap_{x \in B^c} \{x \notin \xi\}.
\] (8.27)

It follows from (8.26) and (8.27) that (8.23) holds. The theorem is proved.
Definition 8.2 An uncertain set $\xi$ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be (i) nonempty if
\[ \xi(\gamma) \neq \emptyset \] (8.28) for almost all $\gamma \in \Gamma$, (ii) empty if
\[ \xi(\gamma) = \emptyset \] (8.29) for almost all $\gamma \in \Gamma$, and (iii) half-empty if otherwise.

Example 8.5: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then
\[ \xi(\gamma) = [0, \gamma], \quad \forall \gamma \in \Gamma \] (8.30)
is a nonempty uncertain set,
\[ \xi(\gamma) = \emptyset, \quad \forall \gamma \in \Gamma \] (8.31)
is an empty uncertain set, and
\[ \xi(\gamma) = \begin{cases} 0, & \text{if } \gamma > 0.8 \\ [0, \gamma], & \text{if } \gamma \leq 0.8 \end{cases} \] (8.32)
is a half-empty uncertain set.

Union, Intersection and Complement

Definition 8.3 Let $\xi$ and $\eta$ be two uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then (i) the union $\xi \cup \eta$ of the uncertain sets $\xi$ and $\eta$ is
\[ (\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma), \quad \forall \gamma \in \Gamma; \] (8.33)
(ii) the intersection $\xi \cap \eta$ of the uncertain sets $\xi$ and $\eta$ is
\[ (\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma), \quad \forall \gamma \in \Gamma; \] (8.34)
(iii) the complement $\xi^c$ of the uncertain set $\xi$ is
\[ \xi^c(\gamma) = \xi(\gamma)^c, \quad \forall \gamma \in \Gamma. \] (8.35)

Example 8.6: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Let $\xi$ and $\eta$ be two uncertain sets,
\[ \xi(\gamma) = \begin{cases} [1, 2], & \text{if } \gamma = \gamma_1 \\ [1, 3], & \text{if } \gamma = \gamma_2 \\ [1, 4], & \text{if } \gamma = \gamma_3, \end{cases} \]
\[ \eta(\gamma) = \begin{cases} (2, 3), & \text{if } \gamma = \gamma_1 \\ (2, 4), & \text{if } \gamma = \gamma_2 \\ (2, 5), & \text{if } \gamma = \gamma_3. \end{cases} \]
Then their union is
\[(\xi \cup \eta)(\gamma) = \begin{cases} [1, 3), & \text{if } \gamma = \gamma_1 \\ [1, 4), & \text{if } \gamma = \gamma_2 \\ [1, 5), & \text{if } \gamma = \gamma_3, \end{cases} \]
their intersection is
\[(\xi \cap \eta)(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma = \gamma_1 \\ (2, 3], & \text{if } \gamma = \gamma_2 \\ (2, 4], & \text{if } \gamma = \gamma_3, \end{cases} \]
and their complement sets are
\[
\xi^c(\gamma) = \begin{cases} (-\infty, 1) \cup (2, +\infty), & \text{if } \gamma = \gamma_1 \\ (-\infty, 1) \cup (3, +\infty), & \text{if } \gamma = \gamma_2 \\ (-\infty, 1) \cup (4, +\infty), & \text{if } \gamma = \gamma_3, \end{cases}
\]
\[
\eta^c(\gamma) = \begin{cases} (-\infty, 2] \cup [3, +\infty), & \text{if } \gamma = \gamma_1 \\ (-\infty, 2] \cup [4, +\infty), & \text{if } \gamma = \gamma_2 \\ (-\infty, 2] \cup [5, +\infty), & \text{if } \gamma = \gamma_3. \end{cases}
\]

**Theorem 8.2** (Law of Excluded Middle and Law of Contradiction) Let \(\xi\) be an uncertain set and let \(\xi^c\) be its complement. Then
\[
\xi \cup \xi^c \equiv \mathbb{R}, \quad \xi \cap \xi^c \equiv \emptyset. \tag{8.36}
\]

**Proof:** For each \(\gamma \in \Gamma\), it follows from the definition of uncertain set that the union is
\[(\xi \cup \xi^c)(\gamma) = \xi(\gamma) \cup \xi^c(\gamma) = \xi(\gamma) \cup (\xi(\gamma))^c \equiv \mathbb{R}.\]
Thus we have \(\xi \cup \xi^c \equiv \mathbb{R}\). In addition, the intersection is
\[(\xi \cap \xi^c)(\gamma) = \xi(\gamma) \cap \xi^c(\gamma) = \xi(\gamma) \cap (\xi(\gamma))^c \equiv \emptyset.\]
Thus we have \(\xi \cap \xi^c \equiv \emptyset\).

**Theorem 8.3** (Double-Negation Law) Let \(\xi\) be an uncertain set. Then we have
\[(\xi^c)^c = \xi. \tag{8.37}\]

**Proof:** For each \(\gamma \in \Gamma\), it follows from the definition of complement that
\[(\xi^c)(\gamma) = (\xi^c(\gamma))^c = (\xi(\gamma))^c = \xi(\gamma).\]
Thus we have \((\xi^c)^c = \xi\).
Theorem 8.4 (De Morgan’s Law) Let $\xi$ and $\eta$ be uncertain sets. Then we have
\[(\xi \cup \eta)^c = \xi^c \cap \eta^c, \quad (\xi \cap \eta)^c = \xi^c \cup \eta^c.\] (8.38)

Proof: For each $\gamma \in \Gamma$, it follows from the definition of complement that
\[(\xi \cup \eta)^c(\gamma) = ((\xi(\gamma) \cup \eta(\gamma))^c) = \xi(\gamma)^c \cap \eta(\gamma)^c = (\xi^c \cap \eta^c)(\gamma).\]
Thus we have $(\xi \cup \eta)^c = \xi^c \cap \eta^c$. In addition, since
\[(\xi \cap \eta)^c(\gamma) = ((\xi(\gamma) \cap \eta(\gamma))^c) = \xi(\gamma)^c \cup \eta(\gamma)^c = (\xi^c \cup \eta^c)(\gamma),\]
we get $(\xi \cap \eta)^c = \xi^c \cup \eta^c$.

Exercise 8.6: Let $\xi$ be an uncertain set and let $x$ be a real number. Show that
\[\{x \in \xi^c\} = \{x \not\in \xi\}\] (8.39) and
\[M\{x \in \xi^c\} = M\{x \not\in \xi\}.\] (8.40)

Exercise 8.7: Let $\xi$ be an uncertain set and let $x$ be a real number. Show that $\{x \in \xi\}$ and $\{x \in \xi^c\}$ are opposite events, and
\[M\{x \in \xi\} + M\{x \in \xi^c\} = 1.\] (8.41)

Exercise 8.8: Let $\xi$ and $\eta$ be two uncertain sets. Show that $\{\xi \subset \eta\}$ and $\{\eta^c \subset \xi^c\}$ are identical events, i.e.,
\[\{\xi \subset \eta\} = \{\eta^c \subset \xi^c\}.\] (8.42)

Exercise 8.9: Let $\xi$ and $\eta$ be two uncertain sets. Show that $\{\xi \subset \eta\}$ and $\{\xi \subset \eta^c\}$ are not necessarily opposite events.

Function of Uncertain Sets

Definition 8.4 Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, and let $f$ be a measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain set defined by
\[\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma.\] (8.43)

Example 8.7: Let $\xi$ be an uncertain set on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and let $A$ be a crisp set of real numbers. Then $\xi + A$ is also an uncertain set determined by
\[(\xi + A)(\gamma) = \xi(\gamma) + A, \quad \forall \gamma \in \Gamma.\] (8.44)
Example 8.8: Note that the empty set $\emptyset$ annihilates every other set. For example, $A + \emptyset = \emptyset$ and $A \times \emptyset = \emptyset$. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Define two uncertain sets,

$$
\xi(\gamma) = \begin{cases} 
\emptyset, & \text{if } \gamma = \gamma_1 \\
[1, 3], & \text{if } \gamma = \gamma_2 \\
[1, 4], & \text{if } \gamma = \gamma_3,
\end{cases}
$$

$$
\eta(\gamma) = \begin{cases} 
(2, 3), & \text{if } \gamma = \gamma_1 \\
(2, 4), & \text{if } \gamma = \gamma_2 \\
(2, 5), & \text{if } \gamma = \gamma_3.
\end{cases}
$$

Then their sum is

$$
(\xi + \eta)(\gamma) = \begin{cases} 
\emptyset, & \text{if } \gamma = \gamma_1 \\
(3, 7), & \text{if } \gamma = \gamma_2 \\
(3, 9), & \text{if } \gamma = \gamma_3,
\end{cases}
$$

and their multiplication is

$$
(\xi \times \eta)(\gamma) = \begin{cases} 
\emptyset, & \text{if } \gamma = \gamma_1 \\
(2, 12), & \text{if } \gamma = \gamma_2 \\
(2, 20), & \text{if } \gamma = \gamma_3.
\end{cases}
$$

Exercise 8.10: Let $\xi$ be an uncertain set. (i) Show that $\xi + \xi \not\equiv 2\xi$. (ii) Do you think the same of crisp set?

8.2 Membership Function

It is well-known that a crisp set can be described by its indicator function. As a generalization of indicator function, membership function will be used to describe an uncertain set.

Definition 8.5 (Liu [99]) An uncertain set $\xi$ is said to have a membership function $\mu$ if for any Borel set $B$ of real numbers, we have

$$
\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x), \quad (8.45)
$$

$$
\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x). \quad (8.46)
$$

The above equations will be called measure inversion formulas.

Theorem 8.5 Let $\xi$ be an uncertain set whose membership function $\mu$ exists. Then

$$
\mu(x) = \mathcal{M}\{x \in \xi\} \quad (8.47)
$$

for any number $x$. 

Theorem 8.6 Let \( \xi \) be an uncertain set with membership function \( \mu \). Then
\[
\mathcal{M}\{x \notin \xi\} = 1 - \mu(x)
\]
(8.48)
for any number \( x \).

Proof: Since \( \{x \notin \xi\} \) and \( \{x \in \xi\} \) are opposite events, it follows from the duality axiom of uncertain measure that
\[
\mathcal{M}\{x \notin \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).
\]
The theorem is proved.

Remark 8.5: Theorem 8.6 states that if an element \( x \) belongs to an uncertain set with membership degree \( \alpha \), then \( x \) does not belong to the uncertain set with membership degree \( 1 - \alpha \).

Theorem 8.7 Let \( \xi \) be an uncertain set with membership function \( \mu \). Then
\[
\mathcal{M}\{x \in \xi^c\} = 1 - \mu(x)
\]
(8.49)
for any number \( x \).
Proof: Since \( \{x \in \xi^c\} \) and \( \{x \in \xi\} \) are opposite events, it follows from the duality axiom of uncertain measure that
\[
M\{x \in \xi^c\} = 1 - M\{x \in \xi\} = 1 - \mu(x).
\]
The theorem is proved.

Remark 8.6: Theorem 8.7 states that if an element \( x \) belongs to an uncertain set with membership degree \( \alpha \), then \( x \) belongs to its complement set with membership degree \( 1 - \alpha \).

Remark 8.7: For any membership function \( \mu \), it is clear that \( 0 \leq \mu(x) \leq 1 \). We will always take
\[
\inf_{x \in \emptyset} \mu(x) = 1, \quad \sup_{x \in \emptyset} \mu(x) = 0.
\] (8.50)
Thus
\[
M\{\emptyset \subset \xi\} = 1 = \inf_{x \in \emptyset} \mu(x).
\] (8.51)
That is, the first measure inversion formula always holds for \( B = \emptyset \). Furthermore, we have
\[
M\{\xi \subset \mathbb{R}\} = 1 = 1 - \sup_{x \in \emptyset} \mu(x).
\] (8.52)
That is, the second measure inversion formula always holds for \( B = \mathbb{R} \).

Example 8.9: The set \( \mathbb{R} \) of real numbers is a special uncertain set \( \xi(\gamma) \equiv \mathbb{R} \). Such an uncertain set has a membership function
\[
\mu(x) \equiv 1
\] (8.53)
that is just the indicator function of \( \mathbb{R} \). In order to prove it, we must verify that \( \mathbb{R} \) and \( \mu \) simultaneously satisfy the two measure inversion formulas (8.45) and (8.46). Let \( B \) be a Borel set of real numbers. Then
\[
M\{B \subset \xi\} = M\{\Gamma\} = 1 = \inf_{x \in B^c} \mu(x).
\]
The first measure inversion formula is verified. Next we prove the second measure inversion formula. When \( B = \mathbb{R} \), the second measure inversion formula has been verified by (8.52). When \( B \neq \mathbb{R} \), we have
\[
M\{\xi \subset B\} = M\{\emptyset\} = 0 = 1 - \sup_{x \in B^c} \mu(x).
\]
Thus the second measure inversion formula holds for any Borel set \( B \). Therefore, the uncertain set \( \xi(\gamma) \equiv \mathbb{R} \) has the membership function \( \mu(x) \equiv 1 \).

Example 8.10: The empty set \( \emptyset \) is a special uncertain set \( \xi(\gamma) \equiv \emptyset \). Such an uncertain set has a membership function
\[
\mu(x) \equiv 0
\] (8.54)
that is just the indicator function of $\emptyset$. In order to prove it, we must verify that $\emptyset$ and $\mu$ simultaneously satisfy the two measure inversion formulas (8.45) and (8.46). Let $B$ be a Borel set of real numbers. When $B = \emptyset$, the first measure inversion formula has been verified by (8.51). When $B \neq \emptyset$, we have

$$M\{B \subset \xi\} = M\{\emptyset\} = 0 = \inf_{x \in B} \mu(x).$$

Thus the first measure inversion formula holds for any Borel set $B$. Next we prove the second measure inversion formula. For any Borel set $B$, we have

$$M\{\xi \subset B\} = M\{\Gamma\} = 1 = 1 - \sup_{x \in B^c} \mu(x).$$

The second measure inversion formula is verified. Therefore, the uncertain set $\xi(\gamma) \equiv \emptyset$ has the membership function $\mu(x) \equiv 0$.

**Exercise 8.11:** A crisp set $A$ of real numbers is a special uncertain set $\xi(\gamma) \equiv A$. Show that such an uncertain set has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad (8.55)$$

that is just the indicator function of $A$.

**Exercise 8.12:** Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2\}$ with power set and $M\{\gamma_1\} = 0.4$, $M\{\gamma_2\} = 0.6$. Show that the uncertain set

$$\xi(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma = \gamma_1 \\ A, & \text{if } \gamma = \gamma_2 \end{cases}$$

has a membership function

$$\mu(x) = \begin{cases} 0.6, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad (8.56)$$

where $A$ is a crisp set of real numbers.

**Exercise 8.13:** Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. (i) Show that the uncertain set

$$\xi(\gamma) = [-\gamma, \gamma], \quad \forall \gamma \in [0, 1] \quad (8.57)$$

has a membership function

$$\mu(x) = \begin{cases} 1 - |x|, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad (8.58)$$
(ii) What is the membership function of $\xi(\gamma) = [\gamma - 1, 1 - \gamma]$? (iii) What do those two uncertain sets make you think about? (iv) Design a third uncertain set whose membership function is also (8.58).

**Exercise 8.14:** Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $M\{\gamma_1\} = 0.6$, $M\{\gamma_2\} = 0.3$, $M\{\gamma_3\} = 0.2$. Define an uncertain set

$$
\xi(\gamma) = \begin{cases} 
[2, 3], & \text{if } \gamma = \gamma_1 \\
[0, 5], & \text{if } \gamma = \gamma_2 \\
[1, 4], & \text{if } \gamma = \gamma_3.
\end{cases}
$$

(i) What is the membership function of $\xi$? (ii) Please justify your answer. (Hint: If $\xi$ does have a membership function, then $\mu(x) = M\{x \in \xi\}$.)

**Exercise 8.15:** Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Define an uncertain set

$$
\xi(\gamma) = (\gamma^2, +\infty).
$$

(8.59)

(i) What is the membership function of $\xi$? (ii) What is the membership function of the complement set $\xi^c$? (iii) What do those two uncertain sets make you think about?

**Exercise 8.16:** It is not true that every uncertain set has a membership function. Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $\{\gamma_1, \gamma_2\}$ with power set and $M\{\gamma_1\} = 0.4$, $M\{\gamma_2\} = 0.6$. Show that the uncertain set

$$
\xi(\gamma) = \begin{cases} 
[1, 3], & \text{if } \gamma = \gamma_1 \\
[2, 4], & \text{if } \gamma = \gamma_2
\end{cases}
$$

(8.60)

has no membership function. (Hint: If $\xi$ does have a membership function, then by using $\mu(x) = M\{x \in \xi\}$, we get

$$
\mu(x) = \begin{cases} 
0.4, & \text{if } 1 \leq x < 2 \\
1, & \text{if } 2 \leq x \leq 3 \\
0.6, & \text{if } 3 < x \leq 4 \\
0, & \text{otherwise}.
\end{cases}
$$

(8.61)

Verify that $\xi$ and $\mu$ cannot simultaneously satisfy the two measure inversion formulas (8.45) and (8.46).)

**Exercise 8.17:** Take an uncertainty space $(\Gamma, \mathcal{L}, M)$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Show that the uncertain set

$$
\xi(\gamma) = [\gamma, \gamma + 1], \quad \forall \gamma \in \Gamma
$$

(8.62)

has no membership function.
**Definition 8.6** An uncertain set $\xi$ is called triangular if it has a membership function

$$
\mu(x) = \begin{cases} 
& \frac{x - a}{b - a}, \quad \text{if } a \leq x \leq b \\
& \frac{x - c}{b - c}, \quad \text{if } b < x \leq c 
\end{cases}
$$

(8.63)

denoted by $(a,b,c)$ where $a,b,c$ are real numbers with $a < b < c$.

**Definition 8.7** An uncertain set $\xi$ is called trapezoidal if it has a membership function

$$
\mu(x) = \begin{cases} 
& \frac{x - a}{b - a}, \quad \text{if } a \leq x \leq b \\
& 1, \quad \text{if } b < x \leq c \\
& \frac{x - d}{c - d}, \quad \text{if } c < x \leq d 
\end{cases}
$$

(8.64)

denoted by $(a,b,c,d)$ where $a,b,c,d$ are real numbers with $a < b < c < d$.

![Figure 8.3: Triangular and Trapezoidal Membership Functions](image)

**What is “young”?**

Sometimes we say “those students are young”. What ages can be considered “young”? In this case, “young” may be regarded as an uncertain set whose membership function is

$$
\mu(x) = \begin{cases} 
& 0, \quad \text{if } x \leq 15 \\
& \frac{(x - 15)}{5}, \quad \text{if } 15 < x \leq 20 \\
& 1, \quad \text{if } 20 < x \leq 35 \\
& \frac{(45 - x)}{10}, \quad \text{if } 35 < x \leq 45 \\
& 0, \quad \text{if } x > 45.
\end{cases}
$$

(8.65)

Note that we do not say “young” if the age is below 15.
What is “tall”?  

Sometimes we say “those sportsmen are tall”. What heights (centimeters) can be considered “tall”? In this case, “tall” may be regarded as an uncertain set whose membership function is

\[
\mu(x) = \begin{cases} 
0, & \text{if } x \leq 180 \\
(x - 180)/5, & \text{if } 180 < x \leq 185 \\
1, & \text{if } 185 < x \leq 195 \\
(200 - x)/5, & \text{if } 195 < x \leq 200 \\
0, & \text{if } x > 200.
\end{cases} \tag{8.66}
\]

Note that we do not say “tall” if the height is over 200cm.

What is “warm”?  

Sometimes we say “those days are warm”. What temperatures can be considered “warm”? In this case, “warm” may be regarded as an uncertain set
whose membership function is

\[
\mu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 < x \leq 18 \\
1, & \text{if } 18 < x \leq 24 \\
(28 - x)/4, & \text{if } 24 < x \leq 28 \\
0, & \text{if } 28 < x.
\end{cases}
\] (8.67)

Note that we do not say “warm” if the temperature is above 28 degrees Celsius.

![Membership Function of “warm”](image)

Figure 8.6: Membership Function of “warm”

**What is “most”?**

Sometimes we say “most students are boys”. What percentages can be considered “most”? In this case, “most” may be regarded as an uncertain set whose membership function is

\[
\mu(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.7 \\
20(x - 0.7), & \text{if } 0.7 < x \leq 0.75 \\
1, & \text{if } 0.75 < x \leq 0.85 \\
20(0.9 - x), & \text{if } 0.85 < x \leq 0.9 \\
0, & \text{if } 0.9 < x \leq 1.
\end{cases}
\] (8.68)

**What uncertain sets have membership functions?**

It is known that some uncertain sets do not have membership functions. This subsection shows that totally ordered uncertain sets defined on a continuous uncertainty space always have membership functions.

**Definition 8.8 (Liu [109])** An uncertain set \( \xi \) defined on the uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) is called totally ordered if \( \{\xi(\gamma) | \gamma \in \Gamma\} \) is a totally ordered
Chapter 8 - Uncertain Set

set, i.e., for any given $\gamma_1$ and $\gamma_2 \in \Gamma$, either $\xi(\gamma_1) \subset \xi(\gamma_2)$ or $\xi(\gamma_2) \subset \xi(\gamma_1)$ holds.

Example 8.11: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and let $A$ be a crisp set of real numbers. The uncertain set $\xi(\gamma) \equiv A$ is of total order.

Example 8.12: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. The uncertain set

$$\xi(\gamma) = \begin{cases} [2, 3], & \text{if } \gamma = \gamma_1 \\ [0, 5], & \text{if } \gamma = \gamma_2 \\ [1, 4], & \text{if } \gamma = \gamma_3 \end{cases} \quad (8.69)$$

is of total order.

Example 8.13: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. The uncertain set

$$\xi(\gamma) = [-\gamma, \gamma], \quad \forall \gamma \in \Gamma \quad (8.70)$$

is of total order.

Example 8.14: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. The uncertain set

$$\xi(\gamma) = [\gamma, \gamma + 1], \quad \forall \gamma \in \Gamma \quad (8.71)$$

is not of total order.

Exercise 8.18: Let $\xi$ be a totally ordered uncertain set. Show that its complement $\xi^c$ is also of total order.

Exercise 8.19: Let $\xi$ be a totally ordered uncertain set, and let $f$ be a real-valued function. Show that $f(\xi)$ is also of total order.
**Theorem 8.8** (Liu [109]) Let $\xi$ be a totally ordered uncertain set, and let $B$ be a crisp set of real numbers. Then (i) the collection $\{x \in \xi\}$ indexed by $x \in B$ is of total order, and (ii) the collection $\{x \notin \xi\}$ indexed by $x \in B$ is also of total order.

**Proof:** If $\{x \in \xi\}$ indexed by $x \in B$ is not of total order, then there exist two numbers $x_1$ and $x_2$ in $B$ such that neither $\{x_1 \in \xi\} \subset \{x_2 \in \xi\}$ nor $\{x_2 \in \xi\} \subset \{x_1 \in \xi\}$ holds. This means there exist $\gamma_1$ and $\gamma_2$ in $\Gamma$ such that

$$\gamma_1 \in \{x_1 \in \xi\}, \quad \gamma_1 \notin \{x_2 \in \xi\},$$

$$\gamma_2 \in \{x_2 \in \xi\}, \quad \gamma_2 \notin \{x_1 \in \xi\}.$$ 

That is,

$$x_1 \in \xi(\gamma_1), \quad x_1 \notin \xi(\gamma_2),$$

$$x_2 \in \xi(\gamma_2), \quad x_2 \notin \xi(\gamma_1).$$

Thus neither $\xi(\gamma_1) \subset \xi(\gamma_2)$ nor $\xi(\gamma_2) \subset \xi(\gamma_1)$ holds. This result is in contradiction with that $\xi$ is a totally ordered uncertain set. Therefore, $\{x \in \xi\}$ indexed by $x \in B$ is of total order. The first part is proved. It follows from

$$\{x \notin \xi\} = \{x \in \xi\}^c$$

that $\{x \notin \xi\}$ indexed by $x \in B$ is also of total order. The second part is verified.

**Theorem 8.9** (Liu [109], Existence Theorem) Let $\xi$ be a totally ordered uncertain set on a continuous uncertainty space. Then its membership function always exists, and

$$\mu(x) = M\{x \in \xi\}. \quad (8.72)$$

**Proof:** In order to prove that $\mu$ is the membership function of $\xi$, we must verify the two measure inversion formulas. Let $B$ be any Borel set of real numbers. Theorem 8.1 states that

$$\{B \subset \xi\} = \bigcap_{x \in B} \{x \in \xi\}.$$ 

Since the uncertain measure is assumed to be continuous, and $\{x \in \xi\}$ indexed by $x \in B$ is of total order, we obtain

$$M\{B \subset \xi\} = M\left\{\bigcap_{x \in B} (x \in \xi)\right\} = \inf_{x \in B} M\{x \in \xi\} = \inf_{x \in B} \mu(x).$$

The first measure inversion formula is verified. Next, Theorem 8.1 states that

$$\{\xi \subset B\} = \bigcap_{x \in B^c} \{x \notin \xi\}.$$
Since the uncertain measure is assumed to be continuous, and \( \{x \notin \xi\} \) indexed by \( x \in B^c \) is of total order, we obtain
\[
M\{\xi \subset B\} = M\left(\bigcap_{x \in B^c} (x \notin \xi)\right) = \inf_{x \in B^c} M\{x \notin \xi\} = 1 - \sup_{x \in B^c} \mu(x).
\]
The second measure inversion formula is verified. Therefore, \( \mu \) is the membership function of \( \xi \).

**Example 8.15:** The continuity condition in Theorem 8.9 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \((0, 1)\) with power set and
\[
M\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
0.5, & \text{otherwise.}
\end{cases}
\]  
(8.73)

Then
\[
\xi(\gamma) = (-\gamma, \gamma), \quad \forall \gamma \in (0, 1)
\]
(8.74)
is a totally ordered uncertain set on a discontinuous uncertainty space. If it indeed has a membership function, then
\[
\mu(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0.5, & \text{if } -1 < x < 0 \text{ or } 0 < x < 1 \\
0, & \text{otherwise.}
\end{cases}
\]  
(8.75)

However,
\[
M\{(-1, 1) \subset \xi\} = M\{\emptyset\} = 0 \neq 0.5 = \inf_{x \in (-1, 1)} \mu(x).
\]
(8.76)

That is, the first measure inversion formula is not valid and then \( \xi \) has no membership function. Therefore, the continuity condition cannot be removed.

**Example 8.16:** Some non-totally ordered uncertain sets may have membership functions. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}\) with power set and
\[
M\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
0.5, & \text{otherwise.}
\end{cases}
\]  
(8.77)

Then
\[
\xi(\gamma) = \begin{cases} 
\{1\}, & \text{if } \gamma = \gamma_1 \\
\{1, 2\}, & \text{if } \gamma = \gamma_2 \\
\{1, 3\}, & \text{if } \gamma = \gamma_3 \\
\{1, 2, 3\}, & \text{if } \gamma = \gamma_4
\end{cases}
\]  
(8.78)
is a non-totally ordered uncertain set. However, it has a membership function

$$\mu(x) = \begin{cases} 
1, & \text{if } x = 1 \\
0.5, & \text{if } x = 2 \text{ or } 3 \\
0, & \text{otherwise}
\end{cases} \quad (8.79)$$

because $\xi$ and $\mu$ can simultaneously satisfy the two measure inversion formulas (8.45) and (8.46).

**Remark 8.8:** In practice, the unsharp concepts like “young”, “tall”, “warm”, and “most” can be regarded as totally ordered uncertain sets on a continuous uncertainty space.

**Sufficient and Necessary Condition**

**Theorem 8.10** *(Liu [96])* A real-valued function $\mu$ is a membership function of uncertain set if and only if

$$0 \leq \mu(x) \leq 1. \quad (8.80)$$

**Proof:** If $\mu$ is a membership function of some uncertain set $\xi$, then $\mu(x) = \mathcal{M}\{x \in \xi\}$ and $0 \leq \mu(x) \leq 1$. Conversely, suppose $\mu$ is a function such that $0 \leq \mu(x) \leq 1$. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = \{x \in \mathbb{R} | \mu(x) \geq \gamma\} \quad (8.81)$$

is a totally ordered uncertain set defined on the continuous uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. See Figure 8.8. By using Theorem 8.9, it is easy to verify that $\xi$ has the membership function $\mu$.

![Figure 8.8](image)

Figure 8.8: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then $\xi(\gamma) = \{x \in \mathbb{R} | \mu(x) \geq \gamma\}$ has the membership function $\mu$. Keep in mind that $\xi$ is not the unique uncertain set whose membership function is $\mu$.

**Example 8.17:** Let $c$ be a number between 0 and 1. It follows from the sufficient and necessary condition that

$$\mu(x) \equiv c \quad (8.82)$$
is a membership function. Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Define
\[
\xi(\gamma) = \begin{cases} 
\mathbb{R}, & \text{if } 0 \leq \gamma \leq c \\
\emptyset, & \text{if } c < \gamma \leq 1.
\end{cases} \tag{8.83}
\]
It is easy to verify that \(\xi\) is a totally ordered uncertain set on a continuous uncertainty space, and has the membership function \(\mu\).

**Example 8.18:** Let us design an uncertain set whose membership function is
\[
\mu(x) = \exp(-x^2) \tag{8.84}
\]
for any real number \(x\). Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Define
\[
\xi(\gamma) = (-\sqrt{-\ln \gamma}, \sqrt{-\ln \gamma}), \quad \forall \gamma \in [0, 1]. \tag{8.85}
\]
It is easy to verify that \(\xi\) is a totally ordered uncertain set on a continuous uncertainty space, and has the membership function \(\mu\).

**Exercise 8.20:** Design an uncertain set whose membership function is just
\[
\mu(x) = \frac{1}{2} \exp(-x^2) \tag{8.86}
\]
for any real number \(x\).

**Exercise 8.21:** Design an uncertain set whose membership function is just
\[
\mu(x) = \frac{1}{2} \exp(-x^2) + \frac{1}{2} \tag{8.87}
\]
for any real number \(x\).

**Theorem 8.11** Let \(\xi\) be an uncertain set whose membership function \(\mu\) exists. Then \(\xi\) is (i) nonempty if and only if
\[
\sup_{x \in \mathbb{R}} \mu(x) = 1, \tag{8.88}
\]
(ii) empty if and only if
\[
\mu(x) \equiv 0, \tag{8.89}
\]
and (iii) half-empty if and only if otherwise.

**Proof:** Since the membership function \(\mu\) exists, it follows from the second measure inversion formula that
\[
\mathcal{M}\{\xi = \emptyset\} = \mathcal{M}\{\xi \subset \emptyset\} = 1 - \sup_{x \in \mathbb{R}^c} \mu(x) = 1 - \sup_{x \in \mathbb{R}^c} \mu(x).
\]
Thus $\xi$ is (i) nonempty if and only if $M\{\xi = \emptyset\} = 0$, i.e., (8.88) holds, (ii) empty if and only if $M\{\xi = \emptyset\} = 1$, i.e., (8.89) holds, and (iii) half-empty if and only if otherwise.

**Exercise 8.22:** Some people prefer the uncertain set whose height (i.e., the supremum of the membership function) achieves 1. When the height is below 1, they divide all its membership values by the height and obtain a “normalized” membership function. Why is this idea wrong and harmful?

### Regular Membership Function

**Definition 8.9** (Liu [99]) A membership function $\mu$ is said to be regular if there exists a point $x_0$ such that $\mu(x_0) = 1$ and $\mu(x)$ is unimodal about the mode $x_0$. That is, $\mu(x)$ is increasing on $(-\infty, x_0]$ and decreasing on $[x_0, +\infty)$.

For example, both triangular and trapezoidal membership functions are regular. In addition, the membership function $\mu(x) \equiv 1$ is regular, but $\mu(x) \equiv 0$ is not.

**Exercise 8.23:** Show that an uncertain set is nonempty if it has a regular membership function.

### 8.3 Inverse Membership Function

**Definition 8.10** (Liu [99]) Let $\xi$ be an uncertain set with membership function $\mu$. Then the set-valued function

$$
\mu^{-1}(\alpha) = \{x \in \mathbb{R} \mid \mu(x) \geq \alpha\}, \quad \forall \alpha \in [0, 1]
$$

is called the inverse membership function of $\xi$. For each given $\alpha$, the set $\mu^{-1}(\alpha)$ is also called the $\alpha$-cut of $\mu$.

![Figure 8.9: Inverse Membership Function $\mu^{-1}(\alpha)$](image-url)
Remark 8.9: Let $\xi$ be an uncertain set with inverse membership function $\mu^{-1}(\alpha)$. Then the membership function of $\xi$ is determined by

$$
\mu(x) = \sup \{ \alpha \in [0,1] \mid x \in \mu^{-1}(\alpha) \}.
$$

(8.91)

Example 8.19: Note that an inverse membership function may take value of the empty set $\emptyset$. Let $\xi$ be an uncertain set with membership function

$$
\mu(x) = \begin{cases} 
0.8, & \text{if } 1 \leq x \leq 2 \\
0, & \text{otherwise.}
\end{cases}
$$

(8.92)

Then its inverse membership function is

$$
\mu^{-1}(\alpha) = \begin{cases} 
\emptyset, & \text{if } \alpha > 0.8 \\
[1,2], & \text{otherwise.}
\end{cases}
$$

(8.93)

Example 8.20: The triangular uncertain set $\xi = (a,b,c)$ has an inverse membership function

$$
\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha b + (1-\alpha)c].
$$

(8.94)

Example 8.21: The trapezoidal uncertain set $\xi = (a,b,c,d)$ has an inverse membership function

$$
\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha c + (1-\alpha)d].
$$

(8.95)

Theorem 8.12 (Liu [99], Sufficient and Necessary Condition) A function $\mu^{-1}(\alpha)$ is an inverse membership function if and only if it is a monotone decreasing set-valued function with respect to $\alpha \in [0,1]$. That is,

$$
\mu^{-1}(\alpha) \subset \mu^{-1}(\beta), \text{ if } \alpha > \beta.
$$

(8.96)

Proof: Suppose $\mu^{-1}(\alpha)$ is an inverse membership function of some uncertain set. For any $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \geq \alpha$. Since $\alpha > \beta$, we have $\mu(x) > \beta$ and then $x \in \mu^{-1}(\beta)$. Hence $\mu^{-1}(\alpha) \subset \mu^{-1}(\beta)$. Conversely, suppose $\mu^{-1}(\alpha)$ is a monotone decreasing set-valued function. Then

$$
\mu(x) = \sup \{ \alpha \in [0,1] \mid x \in \mu^{-1}(\alpha) \}
$$

is a membership function of some uncertain set. It is easy to verify that $\mu^{-1}(\alpha)$ is the inverse membership function of the uncertain set. The theorem is proved.
Uncertain set does not necessarily take values of its $\alpha$-cut!

Please keep in mind that uncertain set does not necessarily take values of its $\alpha$-cuts. In fact, an $\alpha$-cut is included in the uncertain set with uncertain measure $\alpha$. Conversely, the uncertain set is included in its $\alpha$-cut with uncertain measure $1 - \alpha$. More precisely, we have the following theorem.

**Theorem 8.13** (Liu [99]) Let $\xi$ be an uncertain set with inverse membership function $\mu^{-1}(\alpha)$. Then for each $\alpha \in [0, 1]$, we have

$$M\{\mu^{-1}(\alpha) \subset \xi\} \geq \alpha,$$

(8.97)

$$M\{\xi \subset \mu^{-1}(\alpha)\} \geq 1 - \alpha.$$  

(8.98)

**Proof:** For each $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \geq \alpha$. It follows from the first measure inversion formula that

$$M\{\mu^{-1}(\alpha) \subset \xi\} = \inf_{x \in \mu^{-1}(\alpha)} \mu(x) \geq \alpha.$$

For each $x \not\in \mu^{-1}(\alpha)$, we have $\mu(x) < \alpha$. It follows from the second measure inversion formula that

$$M\{\xi \subset \mu^{-1}(\alpha)\} = 1 - \sup_{x \not\in \mu^{-1}(\alpha)} \mu(x) \geq 1 - \alpha.$$

### 8.4 Independence

Note that an uncertain set is a measurable function from an uncertainty space to a collection of sets of real numbers. The independence of two functions means that knowing the value of one does not change our estimation of the value of another. Two uncertain sets meet this condition if they are defined on different uncertainty spaces. For example, let $\xi_1(\gamma_1)$ and $\xi_2(\gamma_2)$ be uncertain sets on the uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively. It is clear that they are also uncertain sets on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$. Then for any Borel sets $B_1$ and $B_2$ of real numbers, we have

$$M\{(\xi_1 \subset B_1) \cap (\xi_2 \subset B_2)\}$$

$$= M\{(\gamma_1, \gamma_2) \mid \xi_1(\gamma_1) \subset B_1, \xi_2(\gamma_2) \subset B_2\}$$

$$= M\{(\gamma_1 \mid \xi_1(\gamma_1) \subset B_1) \times (\gamma_2 \mid \xi_2(\gamma_2) \subset B_2)\}$$

$$= M_1\{\gamma_1 \mid \xi_1(\gamma_1) \subset B_1\} \land M_2\{\gamma_2 \mid \xi_2(\gamma_2) \subset B_2\}$$

$$= M\{\xi_1 \subset B_1\} \land M\{\xi_2 \subset B_2\}.$$

\(^1\) For example, in a rectangular coordinate system $(x, y, z)$, it is clear that $z = f(x)$ and $z = g(y)$ are always independent for any set-valued functions $f$ and $g$ of one variable. However, $z = [x, x + 1]$ and $z = \{x, y\}$ are not.
That is
\[ M\{\xi_1 \subset B_1 \} \cap M\{\xi_2 \subset B_2 \} = M\{\xi_1 \subset B_1 \} \wedge M\{\xi_2 \subset B_2 \}. \] (8.99)

Similarly, we may verify the following seven equations:
\[
\begin{align*}
M\{\xi_1 \subset B_1 \} \cap M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \wedge M\{\xi_2 \subset B_2 \}, \quad (8.100) \\
M\{\xi_1 \subset B_1 \} \cap M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \wedge M\{\xi_2 \subset B_2 \}, \quad (8.101) \\
M\{\xi_1 \subset B_1 \} \cap M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \wedge M\{\xi_2 \subset B_2 \}, \quad (8.102) \\
M\{\xi_1 \subset B_1 \} \cup M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \vee M\{\xi_2 \subset B_2 \}, \quad (8.103) \\
M\{\xi_1 \subset B_1 \} \cup M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \vee M\{\xi_2 \subset B_2 \}, \quad (8.104) \\
M\{\xi_1 \subset B_1 \} \cup M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \vee M\{\xi_2 \subset B_2 \}, \quad (8.105) \\
M\{\xi_1 \subset B_1 \} \cup M\{\xi_2 \subset B_2 \} &= M\{\xi_1 \subset B_1 \} \vee M\{\xi_2 \subset B_2 \}. \quad (8.106)
\]

Thus we say two uncertain sets are independent if the above eight equations hold. Generally, we may define independence in the following form.

**Definition 8.11** (Liu [102]) The uncertain sets \(\xi_1, \xi_2, \ldots, \xi_n\) are said to be independent if for any Borel sets \(B_1, B_2, \ldots, B_n\) of real numbers, we have
\[
M\left\{ \bigcap_{i=1}^{n} (\xi_i^* \subset B_i) \right\} = \bigwedge_{i=1}^{n} M\{\xi_i^* \subset B_i\} \quad (8.107)
\]
and
\[
M\left\{ \bigcup_{i=1}^{n} (\xi_i^* \subset B_i) \right\} = \bigvee_{i=1}^{n} M\{\xi_i^* \subset B_i\} \quad (8.108)
\]
where \(\xi_i^*\) are arbitrarily chosen from \(\{\xi_i, \xi_i^c\}\), \(i = 1, 2, \ldots, n\), respectively.

**Remark 8.10:** Note that (8.107) and (8.108) represent \(2^{n+1}\) equations. For example, when \(n = 2\), they represent the 8 equations from (8.99) to (8.106).

**Exercise 8.24:** Show that a crisp set of real numbers (a special uncertain set) is always independent of any uncertain set.

**Exercise 8.25:** Let \(\xi_1, \xi_2, \ldots, \xi_n\) be independent uncertain sets. Show that \(\xi_i\) and \(\xi_j\) are independent for any indexes \(i\) and \(j\) with \(1 \leq i < j \leq n\).

**Exercise 8.26:** Let \(\xi\) be an uncertain set. Are \(\xi\) and \(\xi^c\) independent? Please justify your answer.

**Exercise 8.27:** Let \(\xi\) be an uncertain set, and let \(A\) be a crisp set. Are \(\xi\) and \(\xi + A\) independent? Please justify your answer.
Exercise 8.28: Construct \( n \) independent uncertain sets. (Hint: Define them on the product uncertainty space \((\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2) \times \cdots \times (\Gamma_n, \mathcal{L}_n, \mathcal{M}_n)\).)

Exercise 8.29: Show that the following four statements are equivalent: (i) \( \xi_1 \) and \( \xi_2 \) are independent; (ii) \( \xi_1^c \) and \( \xi_2 \) are independent; (iii) \( \xi_1 \) and \( \xi_2^c \) are independent; and (iv) \( \xi_1^c \) and \( \xi_2^c \) are independent.

Theorem 8.14 (Liu [102]) The uncertain sets \( \xi_1, \xi_2, \ldots, \xi_n \) are independent if and only if for any Borel sets \( B_1, B_2, \ldots, B_n \) of real numbers, we have

\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} (B_i \subset \xi_i^*) \right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{ B_i \subset \xi_i^* \} \tag{8.109}
\]

and

\[
\mathcal{M}\left\{ \bigcup_{i=1}^{n} (B_i \subset \xi_i^*) \right\} = \bigvee_{i=1}^{n} \mathcal{M}\{ B_i \subset \xi_i^* \} \tag{8.110}
\]

where \( \xi_i^* \) are arbitrarily chosen from \( \{\xi_i, \xi_i^c\} \), \( i = 1, 2, \ldots, n \), respectively.

Proof: Since \( \{B_i \subset \xi_i^*\} = \{\xi_i^c \subset B_i^c\} \) for \( i = 1, 2, \ldots, n \), we immediately have

\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} (B_i \subset \xi_i^*) \right\} = \mathcal{M}\left\{ \bigcap_{i=1}^{n} (\xi_i^c \subset B_i^c) \right\}, \tag{8.111}
\]

\[
\bigwedge_{i=1}^{n} \mathcal{M}\{ B_i \subset \xi_i^* \} = \bigwedge_{i=1}^{n} \mathcal{M}\{ \xi_i^c \subset B_i^c \}, \tag{8.112}
\]

\[
\mathcal{M}\left\{ \bigcup_{i=1}^{n} (B_i \subset \xi_i^*) \right\} = \mathcal{M}\left\{ \bigcup_{i=1}^{n} (\xi_i^c \subset B_i^c) \right\}, \tag{8.113}
\]

\[
\bigvee_{i=1}^{n} \mathcal{M}\{ B_i \subset \xi_i^* \} = \bigvee_{i=1}^{n} \mathcal{M}\{ \xi_i^c \subset B_i^c \}. \tag{8.114}
\]

It follows from (8.111), (8.112), (8.113) and (8.114) that (8.109) and (8.110) are valid if and only if

\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} (\xi_i^c \subset B_i^c) \right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{ \xi_i^c \subset B_i^c \}, \tag{8.115}
\]

\[
\mathcal{M}\left\{ \bigcup_{i=1}^{n} (\xi_i^c \subset B_i^c) \right\} = \bigvee_{i=1}^{n} \mathcal{M}\{ \xi_i^c \subset B_i^c \}. \tag{8.116}
\]

The above two equations are also equivalent to the independence of the uncertain sets \( \xi_1, \xi_2, \ldots, \xi_n \). The theorem is thus proved.
8.5 Set Operational Law

This section will discuss the union, intersection and complement of uncertain sets via membership functions.

**Union of Uncertain Sets**

**Theorem 8.15** (Liu [99]) Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu$ and $\nu$, respectively. Then their union $\xi \cup \eta$ has a membership function

$$\lambda(x) = \mu(x) \lor \nu(x).$$

**(8.117)**

**Proof:** In order to prove $\mu \lor \nu$ is the membership function of $\xi \cup \eta$, we must verify the two measure inversion formulas. Let $B$ be any Borel set of real numbers, and write

$$\beta = \inf_{x \in B} \mu(x) \lor \nu(x).$$

Then $B \subset \mu^{-1}(\beta) \cup \nu^{-1}(\beta)$. By the independence of $\xi$ and $\eta$, we have

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \geq \mathcal{M}\{(\mu^{-1}(\beta) \cup \nu^{-1}(\beta)) \subset (\xi \cup \eta)\}$$

$$\geq \mathcal{M}\{(\mu^{-1}(\beta) \subset \xi) \cap (\nu^{-1}(\beta) \subset \eta)\}$$

$$= \mathcal{M}\{\mu^{-1}(\beta) \subset \xi\} \land \mathcal{M}\{\nu^{-1}(\beta) \subset \eta\}$$

$$\geq \beta \land \beta = \beta.$$

Thus

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \geq \inf_{x \in B} \mu(x) \lor \nu(x).$$

**(8.118)**

On the other hand, for any $x \in B$, we have

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \leq \mathcal{M}\{x \in (\xi \cup \eta)\} = \mathcal{M}\{(x \in \xi) \cup (x \in \eta)\}$$

$$= \mathcal{M}\{x \in \xi\} \lor \mathcal{M}\{x \in \eta\} = \mu(x) \lor \nu(x).$$

Thus

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \leq \inf_{x \in B} \mu(x) \lor \nu(x).$$

**(8.119)**

It follows from (8.118) and (8.119) that

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} = \inf_{x \in B} \mu(x) \lor \nu(x).$$

**(8.120)**

The first measure inversion formula is verified. Next we prove the second measure inversion formula. By the independence of $\xi$ and $\eta$, we have

$$\mathcal{M}\{(\xi \cup \eta) \subset B\} = \mathcal{M}\{(\xi \subset B) \cap (\eta \subset B)\} = \mathcal{M}\{\xi \subset B\} \land \mathcal{M}\{\eta \subset B\}$$

$$= \left(1 - \sup_{x \in B^c} \mu(x)\right) \land \left(1 - \sup_{x \in B^c} \nu(x)\right)$$

$$= 1 - \sup_{x \in B^c} \mu(x) \lor \nu(x).$$
That is,
\[ M\{(ξ ∪ η) ⊂ B\} = 1 - \sup_{x \in B^c} \mu(x) \lor \nu(x). \] (8.121)

The second measure inversion formula is verified. Therefore, the union ξ ∪ η is proved to have the membership function μ ∨ ν by the measure inversion formulas (8.120) and (8.121).

![Figure 8.10: Membership Function of Union of Uncertain Sets](image)

**Example 8.22:** The independence condition in Theorem 8.15 cannot be removed. For example, take an uncertainty space (Γ, ℳ, M) to be \{γ₁, γ₂\} with power set and M{γ₁} = M{γ₂} = 0.5. Then
\[ \xi(γ) = \begin{cases} [0, 1], & \text{if } γ = γ₁ \\ [0, 2], & \text{if } γ = γ₂ \end{cases} \]
is an uncertain set with membership function
\[ \mu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0.5, & \text{if } 1 < x \leq 2 \\ 0, & \text{otherwise,} \end{cases} \]
and
\[ \eta(γ) = \begin{cases} [0, 2], & \text{if } γ = γ₁ \\ [0, 1], & \text{if } γ = γ₂ \end{cases} \]
is also an uncertain set with membership function
\[ \nu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0.5, & \text{if } 1 < x \leq 2 \\ 0, & \text{otherwise.} \end{cases} \]

Note that ξ and η are not independent, and ξ ∪ η ≡ [0, 2] whose membership function is
\[ \lambda(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases} \]
Thus
\[ \lambda(x) \neq \mu(x) \lor \nu(x). \tag{8.122} \]
Therefore, the independence condition cannot be removed.

**Exercise 8.30:** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain sets with membership functions \( \mu_1, \mu_2, \cdots, \mu_n \), respectively. What is the membership function of \( \xi_1 \cup \xi_2 \cup \cdots \cup \xi_n \)?

**Exercise 8.31:** Some people suggest \( \lambda(x) = \mu(x) + \nu(x) - \mu(x) \cdot \nu(x) \) and \( \lambda(x) = \min\{1, \mu(x) + \nu(x)\} \) for the membership function of the union of uncertain sets. Why is this idea wrong and harmful?

**Exercise 8.32:** Why is \( \lambda(x) = \mu(x) \lor \nu(x) \) the only option for the membership function of the union of uncertain sets?

### Intersection of Uncertain Sets

**Theorem 8.16** (Liu [99]) Let \( \xi \) and \( \eta \) be independent uncertain sets with membership functions \( \mu \) and \( \nu \), respectively. Then their intersection \( \xi \cap \eta \) has a membership function
\[ \lambda(x) = \mu(x) \land \nu(x). \tag{8.123} \]

**Proof:** In order to prove \( \mu \land \nu \) is the membership function of \( \xi \cap \eta \), we must verify the two measure inversion formulas. Let \( B \) be any Borel set of real numbers. By the independence of \( \xi \) and \( \eta \), we have
\[
\mathcal{M}\{B \subseteq (\xi \cap \eta)\} = \mathcal{M}\{(B \subseteq \xi) \cap (B \subseteq \eta)\} = \mathcal{M}\{B \subseteq \xi\} \land \mathcal{M}\{B \subseteq \eta\}
= \inf_{x \in B} \mu(x) \land \inf_{x \in B} \nu(x) = \inf_{x \in B} \mu(x) \land \nu(x).
\]
That is,
\[ \mathcal{M}\{B \subseteq (\xi \cap \eta)\} = \inf_{x \in B} \mu(x) \land \nu(x). \tag{8.124} \]
The first measure inversion formula is verified. In order to prove the second measure inversion formula, we write
\[ \beta = \sup_{x \in B} \mu(x) \land \nu(x). \]
Then for any given number \( \varepsilon > 0 \), we have \( \mu^{-1}(\beta + \varepsilon) \cap \nu^{-1}(\beta + \varepsilon) \subseteq B \). By the independence of \( \xi \) and \( \eta \), we obtain
\[
\mathcal{M}\{(\xi \cap \eta) \subseteq B\} \geq \mathcal{M}\{(\xi \cap \eta) \subseteq (\mu^{-1}(\beta + \varepsilon) \cap \nu^{-1}(\beta + \varepsilon))\}
\geq \mathcal{M}\{(\xi \subseteq \mu^{-1}(\beta + \varepsilon)) \cap (\eta \subseteq \nu^{-1}(\beta + \varepsilon))\}
= \mathcal{M}\{\xi \subseteq \mu^{-1}(\beta + \varepsilon)\} \land \mathcal{M}\{\eta \subseteq \nu^{-1}(\beta + \varepsilon)\}
\geq (1 - \beta - \varepsilon) \land (1 - \beta - \varepsilon) = 1 - \beta - \varepsilon.
\]
Letting $\varepsilon \to 0$, we get
\[
\mathcal{M}\{(\xi \cap \eta) \subset B\} \geq 1 - \sup_{x \in B^c} \mu(x) \land \nu(x). \tag{8.125}
\]

On the other hand, for any $x \in B^c$, we have
\[
\mathcal{M}\{(\xi \cap \eta) \subset B\} \leq \mathcal{M}\{x \notin (\xi \cap \eta)\} = \mathcal{M}\{x \notin \xi \cup (x \notin \eta)\} = \mathcal{M}\{x \notin \xi\} \lor \mathcal{M}\{x \notin \eta\} = (1 - \mu(x)) \lor (1 - \nu(x))
\[
= 1 - \mu(x) \land \nu(x).
\]
Thus
\[
\mathcal{M}\{(\xi \cap \eta) \subset B\} \leq 1 - \sup_{x \in B^c} \mu(x) \land \nu(x). \tag{8.126}
\]
It follows from (8.125) and (8.126) that
\[
\mathcal{M}\{(\xi \cap \eta) \subset B\} = 1 - \sup_{x \in B^c} \mu(x) \land \nu(x). \tag{8.127}
\]
The second measure inversion formula is verified. Therefore, the intersection $\xi \cap \eta$ is proved to have the membership function $\mu \land \nu$ by the measure inversion formulas (8.124) and (8.127).

![Figure 8.11: Membership Function of Intersection of Uncertain Sets](image)

**Example 8.23:** The independence condition in Theorem 8.16 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Then
\[
\xi(\gamma) = \begin{cases} 
[0, 1], & \text{if } \gamma = \gamma_1 \\
[0, 2], & \text{if } \gamma = \gamma_2
\end{cases}
\]
is an uncertain set with membership function
\[
\mu(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 1 \\
0.5, & \text{if } 1 < x \leq 2 \\
0, & \text{otherwise},
\end{cases}
\]
and
\[ \eta(\gamma) = \begin{cases} [0, 2], & \text{if } \gamma = \gamma_1 \\ [0, 1], & \text{if } \gamma = \gamma_2 \end{cases} \]
is also an uncertain set with membership function
\[ \nu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0.5, & \text{if } 1 < x \leq 2 \\ 0, & \text{otherwise}. \end{cases} \]

Note that \( \xi \) and \( \eta \) are not independent, and \( \xi \cap \eta \equiv [0, 1] \) whose membership function is
\[ \lambda(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise}. \end{cases} \]
Thus
\[ \lambda(x) \neq \mu(x) \land \nu(x). \quad (8.128) \]

Therefore, the independence condition cannot be removed.

**Exercise 8.33:** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain sets with membership functions \( \mu_1, \mu_2, \cdots, \mu_n \), respectively. What is the membership function of \( \xi_1 \cap \xi_2 \cap \cdots \cap \xi_n \)?

**Exercise 8.34:** Some people suggest \( \lambda(x) = \max\{0, \mu(x) + \nu(x) - 1\} \) and \( \lambda(x) = \mu(x) \cdot \nu(x) \) for the membership function of the intersection of uncertain sets. Why is this idea wrong and harmful?

**Exercise 8.35:** Why is \( \lambda(x) = \mu(x) \land \nu(x) \) the only option for the membership function of the intersection of uncertain sets?

**Complement of Uncertain Set**

**Theorem 8.17** (Liu [99]) Let \( \xi \) be an uncertain set with membership function \( \mu \). Then its complement \( \xi^c \) has a membership function
\[ \lambda(x) = 1 - \mu(x). \quad (8.129) \]

**Proof:** In order to prove \( 1 - \mu \) is the membership function of \( \xi^c \), we must verify the two measure inversion formulas. Let \( B \) be a Borel set of real numbers. It follows from the definition of membership function that
\[ \mathcal{M}\{B \subset \xi^c\} = \mathcal{M}\{\xi \subset B^c\} = 1 - \sup_{x \in (B^c)^c} \mu(x) = \inf_{x \in B} (1 - \mu(x)), \]
\[ \mathcal{M}\{\xi^c \subset B\} = \mathcal{M}\{B^c \subset \xi\} = \inf_{x \in B^c} \mu(x) = 1 - \sup_{x \in B^c} (1 - \mu(x)). \]
Thus \( \xi^c \) has the membership function \( 1 - \mu \).
Exercise 8.36: Let $\xi$ be an uncertain set with membership function $\mu(x)$. Theorem 8.17 tells us that $\xi^c$ has a membership function $1 - \mu(x)$. (i) It is known that $\xi \cup \xi^c \equiv \mathbb{R}$ whose membership function is $\lambda(x) \equiv 1$, and

$$\lambda(x) \neq \mu(x) \lor (1 - \mu(x)).$$

(8.130)

Why is Theorem 8.15 not applicable to the union of $\xi$ and $\xi^c$? (ii) It is known that $\xi \cap \xi^c \equiv \emptyset$ whose membership function is $\lambda(x) \equiv 0$, and

$$\lambda(x) \neq \mu(x) \land (1 - \mu(x)).$$

(8.131)

Why is Theorem 8.16 not applicable to the intersection of $\xi$ and $\xi^c$?

Exercise 8.37: Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu$ and $\nu$, respectively. Then the set difference of $\xi$ and $\eta$, denoted by $\xi \setminus \eta$, is the set of all elements that are members of $\xi$ but not members of $\eta$. That is,

$$\xi \setminus \eta = \xi \cap \eta^c.$$

(8.132)

Show that $\xi \setminus \eta$ has a membership function

$$\lambda(x) = \mu(x) \land (1 - \nu(x)).$$

(8.133)

8.6 Arithmetic Operational Law

This section will present an arithmetic operational law of independent uncertain sets, including addition, subtraction, multiplication and division.

Arithmetic Operational Law via Inverse Membership Functions

Theorem 8.18 (Liu [99]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \cdots, \mu_n^{-1}$, respectively, and let $f$ be a measurable function. Then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

(8.134)
has an inverse membership function,
\[ \lambda^{-1}(\alpha) = f(\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)). \] (8.135)

**Proof:** For simplicity, we only prove the case \( n = 2 \). Let \( B \) be any Borel set of real numbers, and write \( \beta = \inf_{x \in B} \lambda(x) \).

Then \( B \subset \lambda^{-1}(\beta) \). Since \( \lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta)) \), by the independence of \( \xi_1 \) and \( \xi_2 \), we have
\[
M\{B \subset \xi\} \geq M\{\lambda^{-1}(\beta) \subset \xi\} = M\{f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta)) \subset \xi\}
\]
\[
\geq M\{(\mu_1^{-1}(\beta) \subset \xi_1) \cap (\mu_2^{-1}(\beta) \subset \xi_2)\}
\]
\[
= M\{\mu_1^{-1}(\beta) \subset \xi_1\} \land M\{\mu_2^{-1}(\beta) \subset \xi_2\}
\]
\[
\geq \beta \land \beta = \beta.
\]

Thus
\[
M\{B \subset \xi\} \geq \inf_{x \in B} \lambda(x). \tag{8.136}
\]

On the other hand, for any given number \( \varepsilon > 0 \), we have \( B \not\subset \lambda^{-1}(\beta + \varepsilon) \). Since \( \lambda^{-1}(\beta + \varepsilon) = f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon)) \), we obtain
\[
M\{B \not\subset \xi\} \geq M\{\xi \subset \lambda^{-1}(\beta + \varepsilon)\} = M\{\xi \subset f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon))\}
\]
\[
\geq M\{(\xi_1 \subset \mu_1^{-1}(\beta + \varepsilon)) \cap (\xi_2 \subset \mu_2^{-1}(\beta + \varepsilon))\}
\]
\[
= M\{\xi_1 \subset \mu_1^{-1}(\beta + \varepsilon)\} \land M\{\xi_2 \subset \mu_2^{-1}(\beta + \varepsilon)\}
\]
\[
\geq (1 - \beta - \varepsilon) \land (1 - \beta - \varepsilon) = 1 - \beta - \varepsilon
\]

and then
\[
M\{B \subset \xi\} = 1 - M\{B \not\subset \xi\} \leq \beta + \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we get
\[
M\{B \subset \xi\} \leq \beta = \inf_{x \in B} \lambda(x). \tag{8.137}
\]

It follows from (8.136) and (8.137) that
\[
M\{B \subset \xi\} = \inf_{x \in B} \lambda(x). \tag{8.138}
\]

The first measure inversion formula is verified. In order to prove the second measure inversion formula, we write
\[
\beta = \sup_{x \in B^c} \lambda(x).
\]
Then for any given number \( \varepsilon > 0 \), we have \( \lambda^{-1}(\beta + \varepsilon) \subseteq B \). Please note that \( \lambda^{-1}(\beta + \varepsilon) = f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon)) \). By the independence of \( \xi_1 \) and \( \xi_2 \), we obtain

\[
M\{\xi \subseteq B\} \geq M\{\xi \subseteq \lambda^{-1}(\beta + \varepsilon)\} = M\{\xi \subseteq f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon))\}
\]

\[
\geq M\{\{\xi_1 \subseteq \mu_1^{-1}(\beta + \varepsilon)\} \cap \{\xi_2 \subseteq \mu_2^{-1}(\beta + \varepsilon)\}\}
\]

\[
= M\{\xi_1 \subseteq \mu_1^{-1}(\beta + \varepsilon)\} \wedge M\{\xi_2 \subseteq \mu_2^{-1}(\beta + \varepsilon)\}
\]

\[
\geq (1 - \beta - \varepsilon) \wedge (1 - \beta - \varepsilon) = 1 - \beta - \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we get

\[
M\{\xi \subseteq B\} \geq 1 - \sup_{x \in B^c} \lambda(x). \tag{8.139}
\]

On the other hand, for any given number \( \varepsilon > 0 \), we have \( \lambda^{-1}(\beta - \varepsilon) \not\subseteq B \). Since \( \lambda^{-1}(\beta - \varepsilon) = f(\mu_1^{-1}(\beta - \varepsilon), \mu_2^{-1}(\beta - \varepsilon)) \), we obtain

\[
M\{\xi \not\subseteq B\} \geq M\{\lambda^{-1}(\beta - \varepsilon) \subseteq \xi\} = M\{f(\mu_1^{-1}(\beta - \varepsilon), \mu_2^{-1}(\beta - \varepsilon)) \subseteq \xi\}
\]

\[
\geq M\{\{\mu_1^{-1}(\beta - \varepsilon) \subseteq \xi_1\} \cap \{\mu_2^{-1}(\beta - \varepsilon) \subseteq \xi_2\}\}
\]

\[
= M\{\mu_1^{-1}(\beta - \varepsilon) \subseteq \xi_1\} \wedge M\{\mu_2^{-1}(\beta - \varepsilon) \subseteq \xi_2\}
\]

\[
\geq (\beta - \varepsilon) \wedge (\beta - \varepsilon) = \beta - \varepsilon
\]

and then

\[
M\{\xi \subseteq B\} = 1 - M\{\xi \not\subseteq B\} \leq 1 - \beta + \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we get

\[
M\{\xi \subseteq B\} \leq 1 - \beta = 1 - \sup_{x \in B^c} \lambda(x). \tag{8.140}
\]

It follows from (8.139) and (8.140) that

\[
M\{\xi \subseteq B\} = 1 - \sup_{x \in B^c} \lambda(x). \tag{8.141}
\]

The second measure inversion formula is verified. Therefore, \( \xi \) is proved to have the membership function \( \lambda \) by the measure inversion formulas (8.138) and (8.141).

**Example 8.24:** Let \( \xi = (a_1, a_2, a_3) \) and \( \eta = (b_1, b_2, b_3) \) be two independent triangular uncertain sets. At first, \( \xi \) has an inverse membership function,

\[
\mu^{-1}(\alpha) = [(1 - \alpha)a_1 + \alpha a_2, \alpha a_2 + (1 - \alpha)a_3], \tag{8.142}
\]

and \( \eta \) has an inverse membership function,

\[
\nu^{-1}(\alpha) = [(1 - \alpha)b_1 + \alpha b_2, \alpha b_2 + (1 - \alpha)b_3]. \tag{8.143}
\]
It follows from the operational law that the sum $\xi + \eta$ has an inverse membership function,
\[
\lambda^{-1}(\alpha) = [(1-\alpha)(a_1+b_1) + \alpha(a_2+b_2), \alpha(a_2+b_2) + (1-\alpha)(a_3+b_3)]. \tag{8.144}
\]
In other words, the sum $\xi + \eta$ is also a triangular uncertain set, and
\[
\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3). \tag{8.145}
\]

**Example 8.25:** Let $\xi = (a_1, a_2, a_3)$ and $\eta = (b_1, b_2, b_3)$ be two independent triangular uncertain sets. It follows from the operational law that the difference $\xi - \eta$ has an inverse membership function,
\[
\lambda^{-1}(\alpha) = [(1-\alpha)(a_1-b_3) + \alpha(a_2-b_2), \alpha(a_2-b_2) + (1-\alpha)(a_3-b_1)]. \tag{8.146}
\]
In other words, the difference $\xi - \eta$ is also a triangular uncertain set, and
\[
\xi - \eta = (a_1 - b_3, a_2 - b_2, a_3 - b_1). \tag{8.147}
\]

**Example 8.26:** Let $\xi = (a_1, a_2, a_3)$ be a triangular uncertain set, and $k$ a real number. When $k \geq 0$, the multiplication $k \cdot \xi$ has an inverse membership function,
\[
\lambda^{-1}(\alpha) = [(1-\alpha)(ka_1) + \alpha(ka_2), \alpha(ka_2) + (1-\alpha)(ka_3)]. \tag{8.148}
\]
That is, the multiplication $k \cdot \xi$ is a triangular uncertain set $(ka_1, ka_2, ka_3)$. When $k < 0$, the multiplication $k \cdot \xi$ has an inverse membership function,
\[
\lambda^{-1}(\alpha) = [(1-\alpha)(ka_3) + \alpha(ka_2), \alpha(ka_2) + (1-\alpha)(ka_1)]. \tag{8.149}
\]
That is, the multiplication $k \cdot \xi$ is a triangular uncertain set $(ka_3, ka_2, ka_1)$. In summary, we have
\[
k \cdot \xi = \begin{cases} 
(ka_1, ka_2, ka_3), & \text{if } k \geq 0 \\
(ka_3, ka_2, ka_1), & \text{if } k < 0.
\end{cases} \tag{8.150}
\]

**Exercise 8.38:** Show that the multiplication of triangular uncertain sets is no longer a triangular one even they are independent and positive.

**Exercise 8.39:** Let $\xi = (a_1, a_2, a_3, a_4)$ and $\eta = (b_1, b_2, b_3, b_4)$ be two independent trapezoidal uncertain sets, and $k$ a real number. Show that
\[
\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4), \tag{8.151}
\]
\[
\xi - \eta = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1), \tag{8.152}
\]
\[ k \cdot \xi = \begin{cases} (ka_1, ka_2, ka_3, ka_4), & \text{if } k \geq 0 \\ (ka_4, ka_3, ka_2, ka_1), & \text{if } k < 0. \end{cases} \] (8.153)

**Example 8.27:** The independence condition in Theorem 8.18 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then

\[ \xi_1(\gamma) = [-\gamma, \gamma] \] (8.154)

is a triangular uncertain set \((-1, 0, 1)\) with inverse membership function

\[ \mu_1^{-1}(\alpha) = [\alpha - 1, 1 - \alpha], \] (8.155)

and

\[ \xi_2(\gamma) = [\gamma - 1, 1 - \gamma] \] (8.156)

is also a triangular uncertain set \((-1, 0, 1)\) with inverse membership function

\[ \mu_2^{-1}(\alpha) = [\alpha - 1, 1 - \alpha]. \] (8.157)

Note that \(\xi_1\) and \(\xi_2\) are not independent, and \(\xi_1 + \xi_2 \equiv [-1, 1]\) whose inverse membership function is

\[ \lambda^{-1}(\alpha) = [-1, 1]. \] (8.158)

Thus

\[ \lambda^{-1}(\alpha) \neq \mu_1^{-1}(\alpha) + \mu_2^{-1}(\alpha). \] (8.159)

Therefore, the independence condition cannot be removed.

**Arithmetic Operational Law via Membership Functions**

**Theorem 8.19** Let \(\xi_1, \xi_2, \cdots, \xi_n\) be independent uncertain sets with membership functions \(\mu_1(x), \mu_2(x), \cdots, \mu_n(x)\), respectively, and let \(f\) be a measurable function. Then

\[ \xi = f(\xi_1, \xi_2, \cdots, \xi_n) \] (8.160)

has a membership function,

\[ \lambda(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \min_{1 \leq i \leq n} \mu_i(x_i). \] (8.161)

**Proof:** Let \(\lambda\) be the membership function of \(\xi\). For any given real number \(x\), write \(\lambda(x) = \beta\). By using Theorem 8.18, we get

\[ \lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta), \cdots, \mu_n^{-1}(\beta)). \]

Since \(x \in \lambda^{-1}(\beta)\), there exist real numbers \(x_i \in \mu_i^{-1}(\beta), i = 1, 2, \cdots, n\) such that \(f(x_1, x_2, \cdots, x_n) = x\). Noting that \(\mu_i(x_i) \geq \beta\) for \(i = 1, 2, \cdots, n\), we have

\[ \lambda(x) = \beta \leq \min_{1 \leq i \leq n} \mu_i(x_i) \]
and then
\[ \lambda(x) \leq \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1 \leq i \leq n} \mu_i(x_i). \] (8.162)

On the other hand, assume \( x_1, x_2, \ldots, x_n \) are any given real numbers with \( f(x_1, x_2, \ldots, x_n) = x \). Write
\[ \min_{1 \leq i \leq n} \mu_i(x_i) = \beta. \]

By using Theorem 8.18, we get
\[ \lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta), \cdots, \mu_n^{-1}(\beta)). \]

Noting that \( x_i \in \mu_i^{-1}(\beta) \) for \( i = 1, 2, \ldots, n \), we have
\[ x = f(x_1, x_2, \ldots, x_n) \in f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta), \cdots, \mu_n^{-1}(\beta)) = \lambda^{-1}(\beta). \]

Hence
\[ \lambda(x) \geq \beta = \min_{1 \leq i \leq n} \mu_i(x_i) \]
and then
\[ \lambda(x) \geq \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1 \leq i \leq n} \mu_i(x_i). \] (8.163)

It follows from (8.162) and (8.163) that (8.161) holds.

**Remark 8.11:** It is possible that the equation \( f(x_1, x_2, \ldots, x_n) = x \) does not have a root for some values of \( x \). In this case, we set \( \lambda(x) = 0 \).

**Example 8.28:** The independence condition in Theorem 8.19 cannot be removed. For example, take an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then
\[ \xi_1(\gamma) = [-\gamma, \gamma] \] (8.164)
is a triangular uncertain set \((-1, 0, 1)\) with membership function
\[ \mu_1(x) = \begin{cases} 1 - |x|, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \] (8.165)
and
\[ \xi_2(\gamma) = [\gamma - 1, 1 - \gamma] \] (8.166)
is also a triangular uncertain set \((-1, 0, 1)\) with membership function
\[ \mu_2(x) = \begin{cases} 1 - |x|, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \] (8.167)
Note that $\xi_1$ and $\xi_2$ are not independent, and $\xi_1 + \xi_2 \equiv [-1, 1]$ whose membership function is

$$
\lambda(x) = \begin{cases} 
1, & \text{if } -1 \leq x \leq 1 \\
0, & \text{otherwise.}
\end{cases} 
$$

(8.168)

Thus

$$
\lambda(x) \neq \sup_{x_1 + x_2 = x} \mu_1(x_1) \land \mu_2(x_2).
$$

(8.169)

Therefore, the independence condition cannot be removed.

**Exercise 8.40:** Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu(x)$ and $\nu(x)$, respectively. Show that $\xi + \eta$ has a membership function,

$$
\lambda(x) = \sup_{y \in \mathbb{R}} \mu(x - y) \land \nu(y).
$$

(8.170)

**Exercise 8.41:** Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu(x)$ and $\nu(x)$, respectively. Show that $\xi - \eta$ has a membership function,

$$
\lambda(x) = \sup_{y \in \mathbb{R}} \mu(x + y) \land \nu(y).
$$

(8.171)

### 8.7 Inclusion Relation

Let $\xi$ be an uncertain set with membership function $\mu$, and let $B$ be a Borel set of real numbers. By using the definition of membership function, Liu [99] presented two measure inversion formulas for calculating the uncertain measure of inclusion relation,

$$
\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x),
$$

(8.172)

$$
\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x).
$$

(8.173)

Especially, for any point $x$, Liu [99] also gave a formula for calculating the uncertain measure of containment relation,

$$
\mathcal{M}\{x \in \xi\} = \mu(x).
$$

(8.174)

A general formula was derived by Yao [195] for calculating the uncertain measure of inclusion relation between uncertain sets.

**Theorem 8.20** (Yao [195]) Let $\xi$ and $\eta$ be independent uncertain sets with membership functions $\mu$ and $\nu$, respectively. Then

$$
\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \mathbb{R}} (1 - \mu(x)) \lor \nu(x).
$$

(8.175)
Proof: Note that $\xi \cap \eta^c$ has a membership function $\lambda(x) = \mu(x) \wedge (1 - \nu(x))$. It follows from $\{\xi \subset \eta\} \equiv \{\xi \cap \eta^c = \emptyset\}$ and the second measure inversion formula that

$$\mathcal{M}\{\xi \subset \eta\} = \mathcal{M}\{\xi \cap \eta^c = \emptyset\}$$

$$= \mathcal{M}\{\xi \cap \eta^c \subset \emptyset\}$$

$$= 1 - \sup_{x \in \emptyset} \mu(x) \wedge (1 - \nu(x))$$

$$= \inf_{x \in \mathbb{R}} (1 - \mu(x)) \vee \nu(x).$$

The theorem is proved.

Example 8.29: Consider two special uncertain sets $\xi = [1, 2]$ and $\eta = [0, 3]$ that are essentially crisp intervals whose membership functions are

$$\mu(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq 2 \\ 0, & \text{otherwise}, \end{cases}$$

$$\nu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 3 \\ 0, & \text{otherwise}, \end{cases}$$

respectively. Mention that $\xi \subset \eta$ is a completely true relation since $[1, 2]$ is indeed included in $[0, 3]$. By using (8.175), we also obtain

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \mathbb{R}} (1 - \mu(x)) \vee \nu(x) = 1.$$

Example 8.30: Consider two special uncertain sets $\xi = [0, 2]$ and $\eta = [1, 3]$ that are essentially crisp intervals whose membership functions are

$$\mu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise}, \end{cases}$$

$$\nu(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq 3 \\ 0, & \text{otherwise}, \end{cases}$$

respectively. Mention that $\xi \subset \eta$ is a completely false relation since $[0, 2]$ is not a subset of $[1, 3]$. By using (8.175), we also obtain

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \mathbb{R}} (1 - \mu(x)) \vee \nu(x) = 0.$$

Example 8.31: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.8, & \text{if } \gamma_1 \in \Lambda \neq \Gamma \\ 0.2, & \text{if } \gamma_1 \notin \Lambda \neq \emptyset. \end{cases}$$

(8.176)
Section 8.7 - Inclusion Relation

Define two uncertain sets,
\[\xi(\gamma) = \begin{cases} [0, 3], & \text{if } \gamma = \gamma_1 \text{ or } \gamma_2 \\ [1, 2], & \text{if } \gamma = \gamma_3 \text{ or } \gamma_4, \end{cases}\]  
(8.177)

\[\eta(\gamma) = \begin{cases} [0, 3], & \text{if } \gamma = \gamma_1 \text{ or } \gamma_3 \\ [1, 2], & \text{if } \gamma = \gamma_2 \text{ or } \gamma_4. \end{cases}\]  
(8.178)

We may verify that \(\xi\) and \(\eta\) are independent, and share a common membership function,
\[\mu(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq 2 \\ 0.8, & \text{if } 0 \leq x < 1 \text{ or } 2 < x \leq 3 \\ 0, & \text{otherwise}. \end{cases}\]  
(8.179)

Note that \(M\{\xi \subset \eta\} = M\{\gamma_1, \gamma_3, \gamma_4\} = 0.8.\)  
(8.180)

By using (8.175), we also obtain
\[M\{\xi \subset \eta\} = \inf_{x \in \mathbb{R}} (1 - \mu(x)) \lor \mu(x) = 0.8.\]  
(8.181)

**Exercise 8.42:** Let \(\xi\) and \(\eta\) be independent uncertain sets with membership functions \(\mu\) and \(\nu\), respectively. Show that if \(\mu \leq \nu\), then
\[M\{\xi \subset \eta\} \geq 0.5.\]  
(8.182)

**Exercise 8.43:** Let \(\xi\) and \(\eta\) be independent uncertain sets with membership functions \(\mu\) and \(\nu\), respectively, and let \(c\) be a number between 0.5 and 1. (i) Construct \(\xi\) and \(\eta\) such that \(\mu \equiv \nu\) and 
\[M\{\xi \subset \eta\} = c.\]  
(8.183)

(ii) Is it possible to construct \(\xi\) and \(\eta\) such that \(\mu \equiv \nu\) and 
\[M\{\xi \subset \eta\} = c\]  
when \(c\) is below 0.5? (iii) Is it stupid to think that \(\xi \subset \eta\) if and only if \(\mu(x) \leq \nu(x)\) for all \(x\)? (iv) Is it stupid to think that \(\xi = \eta\) if and only if \(\mu(x) = \nu(x)\) for all \(x\)? (Hint: Use (8.176), (8.177) and (8.178) as a reference.)

**Example 8.32:** The independence condition in Theorem 8.20 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Then \[\xi(\gamma) = [-\gamma, \gamma]\]  
(8.184)
is a triangular uncertain set \((-1, 0, 1)\) with membership function
\[\mu(x) = \begin{cases} 1 - |x|, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{otherwise}, \end{cases}\]  
(8.185)
and
\[ \eta(\gamma) = [-\gamma, \gamma] \] (8.186)
is also a triangular uncertain set \((-1, 0, 1)\) with membership function
\[ \nu(x) = \begin{cases} 
1 - |x|, & \text{if } -1 \leq x \leq 1 \\
0, & \text{otherwise}.
\end{cases} \] (8.187)

Note that \(\xi\) and \(\eta\) are not independent (in fact, they are the same one), and \(\mathbb{M}\{\xi \subset \eta\} = 1\). However, by using (8.175), we obtain
\[ \mathbb{M}\{\xi \subset \eta\} = \inf_{x \in \mathbb{R}} (1 - \mu(x)) \lor \nu(x) = 0.5 \neq 1. \] (8.188)
Thus the independence condition cannot be removed.

### 8.8 Expected Value

This section will introduce a concept of expected value for nonempty uncertain set (Empty set and half-empty uncertain set have no expected value).

**Definition 8.12** (Liu [93]) Let \(\xi\) be a nonempty uncertain set. Then the expected value of \(\xi\) is defined by
\[
E[\xi] = \int_{0}^{+\infty} \mathbb{M}\{\xi \geq x\} dx - \int_{-\infty}^{0} \mathbb{M}\{\xi \leq x\} dx
\] (8.189)
provided that at least one of the two integrals is finite.

Please note that \(\xi \geq x\) represents “\(\xi\) is imaginarily included in \([x, +\infty)\)”, and \(\xi \leq x\) represents “\(\xi\) is imaginarily included in \((-\infty, x]\)”. What are the appropriate values of \(\mathbb{M}\{\xi \geq x\}\) and \(\mathbb{M}\{\xi \leq x\}\)? Unfortunately, this problem is not as simple as you think.

![Figure 8.13: \{\xi \geq x\} \subset \{\xi \geq x\} \subset \{\xi \not< x\}](image)

It is clear that the imaginary event \(\{\xi \geq x\}\) is one between \(\{\xi \geq x\}\) and \(\{\xi \not< x\}\). See Figure 8.13. Intuitively, for the value of \(\mathbb{M}\{\xi \geq x\}\), it is too conservative if we take \(\mathbb{M}\{\xi \geq x\}\), and it is too adventurous if we take
\[ M\{\xi < x\} = 1 - M\{\xi < x\}. \] Thus we assign \( M\{\xi \geq x\} \) the middle value between \( M\{\xi \geq x\} \) and \( 1 - M\{\xi < x\} \). That is,

\[ M\{\xi \geq x\} = \frac{1}{2} (M\{\xi \geq x\} + 1 - M\{\xi < x\}). \quad (8.190) \]

Similarly, we also define

\[ M\{\xi \leq x\} = \frac{1}{2} (M\{\xi \leq x\} + 1 - M\{\xi > x\}). \quad (8.191) \]

**Example 8.33:** Let \([a, b]\) be a crisp interval and assume \( a > 0 \) for simplicity. Then

\[ \xi(\gamma) \equiv [a, b], \quad \forall \gamma \in \Gamma \]

is a special uncertain set. It follows from the definition of \( M\{\xi \geq x\} \) and \( M\{\xi \leq x\} \) that

\[ M\{\xi \geq x\} = \begin{cases} 1, & \text{if } x \leq a \\ 0.5, & \text{if } a < x \leq b \\ 0, & \text{if } x > b, \end{cases} \]

\[ M\{\xi \leq x\} \equiv 0, \quad \forall x \leq 0. \]

Thus

\[ E[\xi] = \int_0^a 1 \, dx + \int_a^b 0.5 \, dx = \frac{a + b}{2}. \]

**Example 8.34:** In order to further illustrate the expected value operator, let us consider an uncertain set,

\[ \xi = \begin{cases} [1, 2] \text{ with uncertain measure 0.6} \\ [2, 3] \text{ with uncertain measure 0.3} \\ [3, 4] \text{ with uncertain measure 0.2}. \end{cases} \]

It follows from the definition of \( M\{\xi \geq x\} \) and \( M\{\xi \leq x\} \) that

\[ M\{\xi \geq x\} = \begin{cases} 1, & \text{if } x \leq 1 \\ 0.7, & \text{if } 1 < x \leq 2 \\ 0.3, & \text{if } 2 < x \leq 3 \\ 0.1, & \text{if } 3 < x \leq 4 \\ 0, & \text{if } x > 4, \end{cases} \]

\[ M\{\xi \leq x\} \equiv 0, \quad \forall x \leq 0. \]

Thus

\[ E[\xi] = \int_0^1 1 \, dx + \int_1^2 0.7 \, dx + \int_2^3 0.3 \, dx + \int_3^4 0.1 \, dx = 2.1. \]
How to Obtain Expected Value from Membership Function?

Let \( \xi \) be an uncertain set with membership function \( \mu \). In order to calculate its expected value via (8.189), we must determine the values of \( M\{\xi \geq x\} \) and \( M\{\xi \leq x\} \) from the membership function \( \mu \).

**Theorem 8.21** (Liu [95]) Let \( \xi \) be a nonempty uncertain set with membership function \( \mu \). Then for any real number \( x \), we have

\[
M\{\xi \geq x\} = \frac{1}{2} \left( \sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y) \right), \quad (8.192)
\]

\[
M\{\xi \leq x\} = \frac{1}{2} \left( \sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y) \right). \quad (8.193)
\]

**Proof:** Since the uncertain set \( \xi \) has a membership function \( \mu \), the second measure inversion formula tells us that

\[
M\{\xi \geq x\} = 1 - \sup_{y < x} \mu(y),
\]

\[
M\{\xi < x\} = 1 - \sup_{y > x} \mu(y).
\]

Thus (8.192) follows from (8.190) immediately. We may also prove (8.193) similarly.

**Theorem 8.22** (Liu [95]) Let \( \xi \) be a nonempty uncertain set with membership function \( \mu \). Then

\[
E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y) \, dx - \frac{1}{2} \int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y) \, dx \quad (8.194)
\]

where \( x_0 \) is a point such that \( \mu(x_0) = 1 \).

**Proof:** Since \( \mu \) achieves 1 at \( x_0 \), it follows from Theorem 8.21 that for almost all \( x \), we have

\[
M\{\xi \geq x\} = \begin{cases} 
1 - \sup_{y < x} \mu(x) / 2, & \text{if } x \leq x_0 \\
\sup_{y \geq x} \mu(x) / 2, & \text{if } x > x_0
\end{cases} \quad (8.195)
\]

and

\[
M\{\xi \leq x\} = \begin{cases} 
\sup_{y \leq x} \mu(x) / 2, & \text{if } x < x_0 \\
1 - \sup_{y > x} \mu(x) / 2, & \text{if } x \geq x_0
\end{cases} \quad (8.196)
\]
If \( x_0 \geq 0 \), then

\[
E[\xi] = \int_0^{\infty} M\{\xi \geq x\} \, dx - \int_{-\infty}^0 M\{\xi \leq x\} \, dx
\]

\[
= \int_0^{x_0} \left(1 - \sup_{y \leq x} \frac{\mu(x)}{2}\right) \, dx + \int_{x_0}^{\infty} \sup_{y \geq x} \frac{\mu(x)}{2} \, dx - \int_{-\infty}^0 \sup_{y \leq x} \frac{\mu(x)}{2} \, dx
\]

\[
= x_0 + \frac{1}{2} \int_{x_0}^{\infty} \sup_{y \geq x} \mu(y) \, dx - \frac{1}{2} \int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y) \, dx.
\]

If \( x_0 < 0 \), then

\[
E[\xi] = \int_0^{\infty} M\{\xi \geq x\} \, dx - \int_{-\infty}^0 M\{\xi \leq x\} \, dx
\]

\[
= \int_{-\infty}^x \sup_{y \geq x} \frac{\mu(x)}{2} \, dx - \int_{x_0}^{\infty} \sup_{y \leq x} \frac{\mu(x)}{2} \, dx - \int_{-\infty}^{x_0} \left(1 - \sup_{y \geq x} \frac{\mu(x)}{2}\right) \, dx
\]

\[
= x_0 + \frac{1}{2} \int_{x_0}^{\infty} \sup_{y \geq x} \mu(y) \, dx - \frac{1}{2} \int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y) \, dx.
\]

The theorem is thus proved.

**Theorem 8.23** (Liu [95]) Let \( \xi \) be an uncertain set with regular membership function \( \mu \). Then

\[
E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{\infty} \mu(x) \, dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x) \, dx
\]  

(8.197)

where \( x_0 \) is a point such that \( \mu(x_0) = 1 \).

**Proof:** Since \( \mu \) is increasing on \( (-\infty, x_0] \) and decreasing on \( [x_0, +\infty) \), for almost all \( x \geq x_0 \), we have

\[
\sup_{y \geq x} \mu(y) = \mu(x); \tag{8.198}
\]

and for almost all \( x \leq x_0 \), we have

\[
\sup_{y \leq x} \mu(y) = \mu(x). \tag{8.199}
\]

Thus the theorem follows from (8.194) immediately.

**Exercise 8.44:** Show that the triangular uncertain set \( \xi = (a, b, c) \) has an expected value

\[
E[\xi] = a + \frac{2b + c}{4}. \tag{8.200}
\]
**Exercise 8.45:** Show that the trapezoidal uncertain set $\xi = (a, b, c, d)$ has an expected value

$$E[\xi] = \frac{a + b + c + d}{4}.$$  \hspace{1cm} (8.201)

**Theorem 8.24** (Liu [99]) Let $\xi$ be a nonempty uncertain set with membership function $\mu$. Then

$$E[\xi] = \frac{1}{2} \int_0^1 \left( \inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha) \right) d\alpha$$  \hspace{1cm} (8.202)

where $\inf \mu^{-1}(\alpha)$ and $\sup \mu^{-1}(\alpha)$ are the infimum and supremum of the $\alpha$-cut, respectively.

**Proof:** Since $\xi$ is a nonempty uncertain set and has a finite expected value, we may assume that there exists a point $x_0$ such that $\mu(x_0) = 1$ (perhaps after a small perturbation). It is clear that the two integrals

$$\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y) dx \quad \text{and} \quad \int_0^1 (\sup \mu^{-1}(\alpha) - x_0) d\alpha$$

make an identical acreage. Thus

$$\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y) dx = \int_0^1 (\sup \mu^{-1}(\alpha) - x_0) d\alpha = \int_0^1 \sup \mu^{-1}(\alpha) d\alpha - x_0.$$

Similarly, we may prove

$$\int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y) dx = \int_0^1 (x_0 - \inf \mu^{-1}(\alpha)) d\alpha = x_0 - \int_0^1 \inf \mu^{-1}(\alpha) d\alpha.$$

It follows from (8.194) that

$$E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y) dx - \frac{1}{2} \int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y) dx$$

$$= x_0 + \frac{1}{2} \left( \int_0^1 \sup \mu^{-1}(\alpha) d\alpha - x_0 \right) - \frac{1}{2} \left( x_0 - \int_0^1 \inf \mu^{-1}(\alpha) d\alpha \right)$$

$$= \frac{1}{2} \int_0^1 (\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)) d\alpha.$$

The theorem is thus verified.

**Linearity of Expected Value Operator**

**Theorem 8.25** (Liu [99]) Let $\xi$ and $\eta$ be independent uncertain sets with finite expected values. Then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$  \hspace{1cm} (8.203)
Proof: Denote the membership functions of $\xi$ and $\eta$ by $\mu$ and $\nu$, respectively. Then
\[
E[\xi] = \frac{1}{2} \int_0^1 (\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)) \, d\alpha,
\]
\[
E[\eta] = \frac{1}{2} \int_0^1 (\inf \nu^{-1}(\alpha) + \sup \nu^{-1}(\alpha)) \, d\alpha.
\]

Step 1: We first prove $E[a\xi] = aE[\xi]$. The multiplication $a\xi$ has an inverse membership function,
\[
\lambda^{-1}(\alpha) = a\mu^{-1}(\alpha).
\]
It follows from Theorem 8.24 that
\[
E[a\xi] = \frac{1}{2} \int_0^1 (\inf \lambda^{-1}(\alpha) + \sup \lambda^{-1}(\alpha)) \, d\alpha
= a \frac{1}{2} \int_0^1 (\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)) \, d\alpha
= aE[\xi].
\]

Step 2: We then prove $E[\xi + \eta] = E[\xi] + E[\eta]$. The sum $\xi + \eta$ has an inverse membership function,
\[
\lambda^{-1}(\alpha) = \mu^{-1}(\alpha) + \nu^{-1}(\alpha).
\]
It follows from Theorem 8.24 that
\[
E[\xi + \eta] = \frac{1}{2} \int_0^1 (\inf \lambda^{-1}(\alpha) + \sup \lambda^{-1}(\alpha)) \, d\alpha
= \frac{1}{2} \int_0^1 (\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)) \, d\alpha
+ \frac{1}{2} \int_0^1 (\inf \nu^{-1}(\alpha) + \sup \nu^{-1}(\alpha)) \, d\alpha
= E[\xi] + E[\eta].
\]

Step 3: Finally, for any real numbers $a$ and $b$, it follows from Steps 1 and 2 that
\[
E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta].
\]
The theorem is proved.

Example 8.35: Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed. For example, take an
uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{\gamma_1, \gamma_2, \gamma_3\}\) with power set and \(M\{\gamma_1\} = 0.6, M\{\gamma_2\} = 0.3, M\{\gamma_3\} = 0.2\). Define two uncertain sets as follows,

\[
\xi(\gamma) = \begin{cases} 
[1, 4], & \text{if } \gamma = \gamma_1 \\
[1, 3], & \text{if } \gamma = \gamma_2 \\
[1, 2], & \text{if } \gamma = \gamma_3,
\end{cases}
\eta(\gamma) = \begin{cases} 
[1, 5], & \text{if } \gamma = \gamma_1 \\
[1, 2], & \text{if } \gamma = \gamma_2 \\
[1, 4], & \text{if } \gamma = \gamma_3.
\end{cases}
\]

Note that \(\xi\) and \(\eta\) are not independent, and their sum is

\[
(\xi + \eta)(\gamma) = \begin{cases} 
[2, 9], & \text{if } \gamma = \gamma_1 \\
[2, 5], & \text{if } \gamma = \gamma_2 \\
[2, 6], & \text{if } \gamma = \gamma_3.
\end{cases}
\]

It is easy to verify that \(E[\xi] = 2.2, E[\eta] = 2.5\) and \(E[\xi + \eta] = 4.75\). Thus we have

\[E[\xi + \eta] > E[\xi] + E[\eta].\]

If the uncertain sets are defined by

\[
\xi(\gamma) = \begin{cases} 
[1, 4], & \text{if } \gamma = \gamma_1 \\
[1, 3], & \text{if } \gamma = \gamma_2 \\
[1, 2], & \text{if } \gamma = \gamma_3,
\end{cases}
\eta(\gamma) = \begin{cases} 
[1, 4], & \text{if } \gamma = \gamma_1 \\
[1, 6], & \text{if } \gamma = \gamma_2 \\
[1, 2], & \text{if } \gamma = \gamma_3,
\end{cases}
\]

then

\[
(\xi + \eta)(\gamma) = \begin{cases} 
[2, 8], & \text{if } \gamma = \gamma_1 \\
[2, 9], & \text{if } \gamma = \gamma_2 \\
[2, 4], & \text{if } \gamma = \gamma_3.
\end{cases}
\]

It is easy to verify that \(E[\xi] = 2.2, E[\eta] = 2.6\) and \(E[\xi + \eta] = 4.75\). Thus we have

\[E[\xi + \eta] < E[\xi] + E[\eta].\]

Therefore, the independence condition cannot be removed.

### 8.9 Variance

The variance of uncertain set provides a degree of the spread of the membership function around its expected value.

**Definition 8.13** *(Liu [96])* Let \(\xi\) be an uncertain set with finite expected value \(e\). Then the variance of \(\xi\) is defined by

\[
V[\xi] = E[(\xi - e)^2].
\]
This definition says that the variance is just the expected value of \((\xi - e)^2\). Since \((\xi - e)^2\) is a nonnegative uncertain set, we also have

\[ V[\xi] = \int_0^{+\infty} M\{(\xi - e)^2 \geq x\} \, dx. \tag{8.205} \]

Please note that \((\xi - e)^2 \geq x\) represents “\((\xi - e)^2\) is imaginarily included in \([x, +\infty)\)”. What is the appropriate value of \(M\{(\xi - e)^2 \geq x\}\)? Intuitively, it is too conservative if we take the value \(M\{(\xi - e)^2 \geq x\}\), and it is too adventurous if we take the value \(1 - M\{(\xi - e)^2 < x\}\). Thus we assign \(M\{(\xi - e)^2 \geq x\}\) the middle value between them. That is,

\[ M\{(\xi - e)^2 \geq x\} = \frac{1}{2} (M\{(\xi - e)^2 \geq x\} + 1 - M\{(\xi - e)^2 < x\}). \tag{8.206} \]

**Theorem 8.26** If \(\xi\) is an uncertain set with finite expected value, \(a\) and \(b\) are real numbers, then

\[ V[a\xi + b] = a^2 V[\xi]. \tag{8.207} \]

**Proof:** If \(\xi\) has an expected value \(e\), then \(a\xi + b\) has an expected value \(ae + b\). It follows from the definition of variance that

\[ V[a\xi + b] = E[(a\xi + b - ae - b)^2] = a^2 E[(\xi - e)^2] = a^2 V[\xi]. \]

**Theorem 8.27** Let \(\xi\) be an uncertain set with expected value \(e\). Then \(V[\xi] = 0\) if and only if \(\xi = \{e\}\) almost surely.

**Proof:** We first assume \(V[\xi] = 0\). It follows from the equation (8.205) that

\[ \int_0^{+\infty} M\{(\xi - e)^2 \geq x\} \, dx = 0 \]

which implies \(M\{(\xi - e)^2 \geq x\} = 0\) for any \(x > 0\). Hence \(M\{\xi = \{e\}\} = 1\). Conversely, assume \(M\{\xi = \{e\}\} = 1\). Then we have \(M\{(\xi - e)^2 \geq x\} = 0\) for any \(x > 0\). Thus

\[ V[\xi] = \int_0^{+\infty} M\{(\xi - e)^2 \geq x\} \, dx = 0. \]

The theorem is proved.

**How to Obtain Variance from Membership Function?**

Let \(\xi\) be an uncertain set with membership function \(\mu\). In order to calculate its variance by (8.205), we must determine the value of \(M\{(\xi - e)^2 \geq x\}\) from the membership function \(\mu\).
Theorem 8.28 (Liu [106]) Let $\xi$ be a nonempty uncertain set with membership function $\mu$. Then for any real numbers $e$ and $x$, we have

$$M\{ (\xi - e)^2 \geq x \} = \frac{1}{2} \left( \sup_{(y-e)^2 \geq x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y) \right). \quad (8.208)$$

Proof: Since $\xi$ is an uncertain set with membership function $\mu$, it follows from the measure inversion formula that for any real numbers $e$ and $x$, we have

$$M\{ (\xi - e)^2 \geq x \} = 1 - \sup_{(y-e)^2 < x} \mu(y),$$
$$M\{ (\xi - e)^2 < x \} = 1 - \sup_{(y-e)^2 \geq x} \mu(y).$$

The equation (8.208) is thus proved by (8.206).

Theorem 8.29 (Liu [106]) Let $\xi$ be an uncertain set with membership function $\mu$ and finite expected value $e$. Then

$$V[\xi] = \frac{1}{2} \int_0^{+\infty} \left( \sup_{(y-e)^2 \geq x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y) \right) dx. \quad (8.209)$$

Proof: This theorem follows from (8.205) and (8.208) immediately.

8.10 Distance

Definition 8.14 (Liu [96]) The distance between nonempty uncertain sets $\xi$ and $\eta$ is defined as

$$d(\xi, \eta) = E[|\xi - \eta|]. \quad (8.210)$$

That is, the distance between $\xi$ and $\eta$ is just the expected value of $|\xi - \eta|$. Since $|\xi - \eta|$ is a nonnegative uncertain set, we have

$$d(\xi, \eta) = \int_0^{+\infty} M\{|\xi - \eta| \geq x\} dx. \quad (8.211)$$

Please note that $|\xi - \eta| \geq x$ represents “$|\xi - \eta|$ is imaginarily included in $[x, +\infty]$”. What is the appropriate value of $M\{|\xi - \eta| \geq x\}$? Intuitively, it is too conservative if we take the value $M\{|\xi - \eta| \geq x\}$, and it is too adventurous if we take the value $1 - M\{|\xi - \eta| < x\}$. Thus we assign $M\{|\xi - \eta| \geq x\}$ the middle value between them. That is,

$$M\{|\xi - \eta| \geq x\} = \frac{1}{2} \left( M\{|\xi - \eta| \geq x\} + 1 - M\{|\xi - \eta| < x\} \right). \quad (8.212)$$
Theorem 8.30 (Liu [106]) Let $\xi$ and $\eta$ be nonempty uncertain sets. Then for any real number $x$, we have

$$M\{\|\xi - \eta\| \geq x\} = \frac{1}{2} \left( \sup_{|y| \geq x} \lambda(y) + 1 - \sup_{|y| < x} \lambda(y) \right)$$

(8.213)

where $\lambda$ is the membership function of $\xi - \eta$.

Proof: Since $\xi - \eta$ is an uncertain set with membership function $\lambda$, it follows from the measure inversion formula that for any real number $x$, we have

$$M\{\|\xi - \eta\| \geq x\} = 1 - \sup_{|y| < x} \mu(y),$$

$$M\{\|\xi - \eta\| < x\} = 1 - \sup_{|y| \geq x} \mu(y).$$

The equation (8.213) is thus proved by (8.212).

Theorem 8.31 (Liu [106]) Let $\xi$ and $\eta$ be nonempty uncertain sets. Then the distance between $\xi$ and $\eta$ is

$$d(\xi, \eta) = \frac{1}{2} \int_0^{+\infty} \left( \sup_{|y| \geq x} \lambda(y) + 1 - \sup_{|y| < x} \lambda(y) \right) dx$$

(8.214)

where $\lambda$ is the membership function of $\xi - \eta$.

Proof: The theorem follows from (8.211) and (8.213) immediately.

Exercise 8.46: Let $\xi$ be a nonempty uncertain set with membership function $\mu$, and let $b$ be a real number. Show that the distance between $\xi$ and $b$ is

$$d(\xi, b) = \frac{1}{2} \int_0^{+\infty} \left( \sup_{|y-b| \geq x} \mu(y) + 1 - \sup_{|y-b| < x} \mu(y) \right) dx.$$  

(8.215)

Exercise 8.47: Let $\xi$ and $\eta$ be independent nonempty uncertain sets with membership functions $\mu$ and $\nu$, respectively. What is the distance between $\xi$ and $\eta$?

8.11 Entropy

This section defines an entropy as the degree of difficulty of predicting the realization of an uncertain set.

Definition 8.15 (Liu [96]) Suppose that $\xi$ is an uncertain set with membership function $\mu$. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x)) dx$$

(8.216)

where $S(t) = -t \ln t - (1 - t) \ln(1 - t)$. 

Remark 8.12: Note that the entropy (8.216) has the same form with de Luca and Termini’s entropy for fuzzy set [25].

Remark 8.13: If $\xi$ is a discrete uncertain set taking values in $\{x_1, x_2, \ldots \}$, then the entropy becomes

$$H[\xi] = \sum_{i=1}^{\infty} S(\mu(x_i)). \quad (8.217)$$

Example 8.36: A crisp set $A$ of real numbers is a special uncertain set $\xi(\gamma) \equiv A$. Its membership function is

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

and entropy is

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))dx = \int_{-\infty}^{+\infty} 0dx = 0.$$ 

This means a crisp set has entropy 0.

Exercise 8.48: Let $\xi = (a, b, c)$ be a triangular uncertain set. Show that its entropy is

$$H[\xi] = \frac{c-a}{2}. \quad (8.218)$$

Exercise 8.49: Let $\xi = (a, b, c, d)$ be a trapezoidal uncertain set. Show that its entropy is

$$H[\xi] = \frac{b-a + d - c}{2}. \quad (8.219)$$

Theorem 8.32 Let $\xi$ be an uncertain set. Then $H[\xi] \geq 0$ and equality holds if $\xi$ is essentially a crisp set.

Proof: The nonnegativity is clear. In addition, when an uncertain set tends to a crisp set, its entropy tends to the minimum value 0.

Theorem 8.33 Let $\xi$ be an uncertain set on the interval $[a, b]$. Then

$$H[\xi] \leq (b-a) \ln 2 \quad (8.220)$$

and equality holds if $\xi$ has a membership function $\mu(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum value $\ln 2$ at $t = 0.5$. 
Theorem 8.34 Let $\xi$ be an uncertain set, and let $\xi^c$ be its complement. Then

$$H[\xi^c] = H[\xi]. \quad (8.221)$$

Proof: Write the membership function of $\xi$ by $\mu$. Then its complement $\xi^c$ has a membership function $1 - \mu(x)$. It follows from the definition of entropy that

$$H[\xi^c] = \int_{-\infty}^{+\infty} S(1 - \mu(x)) \, dx = \int_{-\infty}^{+\infty} S(\mu(x)) \, dx = H[\xi].$$

The theorem is proved.

8.12 Conditional Membership Function

What is the conditional membership function of an uncertain set $\xi$ after it has been learned that some event $A$ has occurred? This section will answer this question. At first, it follows from the definition of conditional uncertain measure that

$$M\{B \subset \xi | A\} = \begin{cases} 
\frac{M\{(B \subset \xi) \cap A\}}{M\{A\}}, & \text{if } \frac{M\{(B \subset \xi) \cap A\}}{M\{A\}} < 0.5 \\
1 - \frac{M\{(B \not\subset \xi) \cap A\}}{M\{A\}}, & \text{if } \frac{M\{(B \not\subset \xi) \cap A\}}{M\{A\}} < 0.5 \\
0.5, & \text{otherwise,}
\end{cases}$$

$$M\{\xi \subset B | A\} = \begin{cases} 
\frac{M\{(\xi \subset B) \cap A\}}{M\{A\}}, & \text{if } \frac{M\{(\xi \subset B) \cap A\}}{M\{A\}} < 0.5 \\
1 - \frac{M\{(\xi \not\subset B) \cap A\}}{M\{A\}}, & \text{if } \frac{M\{(\xi \not\subset B) \cap A\}}{M\{A\}} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}$$

Definition 8.16 (Liu [106]) Let $\xi$ be an uncertain set, and let $A$ be an event with $M\{A\} > 0$. Then the conditional uncertain set $\xi$ given $A$ is said to have a membership function $\mu(x|A)$ if for any Borel set $B$ of real numbers, we have

$$M\{B \subset \xi | A\} = \inf_{x \in B} \mu(x|A), \quad (8.222)$$

$$M\{\xi \subset B | A\} = 1 - \sup_{x \in B^c} \mu(x|A). \quad (8.223)$$

Theorem 8.35 (Yao [202]) Let $\xi$ be a totally ordered uncertain set on a continuous uncertainty space, and let $A$ be an event with $M\{A\} > 0$. Then the conditional membership function of $\xi$ given $A$ exists, and

$$\mu(x|A) = M\{x \in \xi | A\}. \quad (8.224)$$
Proof: Since the original uncertain measure $\mathcal{M}$ is continuous, the conditional uncertain measure $\mathcal{M}\{\cdot|A\}$ is also continuous. Thus the conditional uncertain set $\xi$ given $A$ is a totally ordered uncertain set on a continuous uncertainty space. It follows from Theorem 8.9 that the conditional membership function exists, and $\mu(x|A) = \mathcal{M}\{x \in \xi|A\}$. The proof is complete.

Example 8.37: The total order condition in Theorem 8.35 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with power set and

$$\mathcal{M}\{A\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \quad (8.225)$$

Then

$$\xi(\gamma) = \begin{cases} [1, 4], & \text{if } \gamma = \gamma_1 \\ [1, 3], & \text{if } \gamma = \gamma_2 \\ [2, 4], & \text{if } \gamma = \gamma_3 \\ [2, 3], & \text{if } \gamma = \gamma_4 \end{cases} \quad (8.226)$$

is a non-totally ordered uncertain set on a continuous uncertainty space, but has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } 2 \leq x \leq 3 \\ 0.5, & \text{if } 1 \leq x < 2 \text{ or } 3 < x \leq 4 \\ 0, & \text{otherwise.} \end{cases} \quad (8.227)$$

However, the conditional uncertain measure given $A = \{\gamma_1, \gamma_2, \gamma_3\}$ is

$$\mathcal{M}\{\Lambda|A\} = \begin{cases} 0, & \text{if } \Lambda \cap A = \emptyset \\ 1, & \text{if } \Lambda \supset A \\ 0.5, & \text{otherwise.} \end{cases} \quad (8.228)$$

If the conditional uncertain set $\xi$ given $A$ has a membership function, then

$$\mu(x|A) = \begin{cases} 1, & \text{if } 2 \leq x \leq 3 \\ 0.5, & \text{if } 1 \leq x < 2 \text{ or } 3 < x \leq 4 \\ 0, & \text{otherwise.} \end{cases} \quad (8.229)$$

Taking $B = [1.5, 3.5]$, we obtain

$$\mathcal{M}\{\xi \subset B|A\} = \mathcal{M}\{\gamma_4|A\} = 0 \neq 0.5 = 1 - \sup_{x \in B^c} \mu(x|A). \quad (8.230)$$

That is, the second measure inversion formula is not valid and then the conditional membership function does not exist. Thus the total order condition cannot be removed.
Example 8.38: The continuity condition in Theorem 8.35 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with power set and

\[
\mathcal{M}\{\Lambda\} = \begin{cases} 
0, & \text{if } \Lambda = \emptyset \\
1, & \text{if } \Lambda = \Gamma \\
0.5, & \text{otherwise.}
\end{cases} \tag{8.231}
\]

Then

\[
\xi(\gamma) = (-\gamma, \gamma), \quad \forall \gamma \in [0, 1] \tag{8.232}
\]

is a totally ordered uncertain set on a discontinuous uncertainty space, but has a membership function

\[
\mu(x) = \begin{cases} 
0.5, & \text{if } -1 < x < 1 \\
0, & \text{otherwise.}
\end{cases} \tag{8.233}
\]

However, the conditional uncertain measure given \(A = (0, 1)\) is

\[
\mathcal{M}\{\Lambda | A\} = \begin{cases} 
0, & \text{if } \Lambda \cap A = \emptyset \\
1, & \text{if } \Lambda \supset A \\
0.5, & \text{otherwise.}
\end{cases} \tag{8.234}
\]

If the conditional uncertain set \(\xi\) given \(A\) has a membership function, then

\[
\mu(x | A) = \begin{cases} 
1, & \text{if } x = 0 \\
0.5, & \text{if } -1 < x < 0 \text{ or } 0 < x < 1 \\
0, & \text{otherwise.}
\end{cases} \tag{8.235}
\]

Taking \(B = (-1, 1)\), we obtain

\[
\mathcal{M}\{B \subset \xi | A\} = \mathcal{M}\{1 | A\} = 0 \neq 0.5 = \inf_{x \in B} \mu(x | A). \tag{8.236}
\]

That is, the first measure inversion formula is not valid and then the conditional membership function does not exist. Thus the continuity condition cannot be removed.

Theorem 8.36 (Liu [93] and Yao [202]) Let \(\xi\) and \(\eta\) be independent uncertain sets with membership functions \(\mu\) and \(\nu\), respectively. Then for any real number \(a\), the conditional uncertain set \(\eta\) given \(a \in \xi\) has a membership function

\[
\nu(y | a \in \xi) = \begin{cases} 
\frac{\nu(y)}{\mu(a)}, & \text{if } \nu(y) < \mu(a)/2 \\
\frac{\nu(y) + \mu(a) - 1}{\mu(a)}, & \text{if } \nu(y) > 1 - \mu(a)/2 \\
0.5, & \text{otherwise.}
\end{cases} \tag{8.237}
\]
Proof: In order prove that \( \nu(y|a \in \xi) \) is the membership function of the conditional uncertain set \( \eta \) given \( a \in \xi \), we must verify the two measure inversion formulas,

\[
M\{B \subset \eta|a \in \xi\} = \inf_{y \in B} \nu(y|a \in \xi), \quad (8.238)
\]

\[
M\{\eta \subset B|a \in \xi\} = 1 - \sup_{y \in B^c} \nu(y|a \in \xi). \quad (8.239)
\]

First, for any Borel set \( B \) of real numbers, by using the definition of conditional uncertainty and independence of \( \xi \) and \( \eta \), we have

\[
M\{B \subset \eta|a \in \xi\} = \begin{cases} 
\frac{M\{B \subset \eta\}}{M\{a \in \xi\}}, & \text{if } \frac{M\{B \subset \eta\}}{M\{a \in \xi\}} < 0.5 \\
1 - \frac{M\{B \not\subset \eta\}}{M\{a \in \xi\}}, & \text{if } \frac{M\{B \not\subset \eta\}}{M\{a \in \xi\}} < 0.5 \\
0.5, & \text{otherwise}
\end{cases}
\]

Since \( M\{B \subset \eta\} = \inf_{y \in B} \nu(y), \quad M\{B \not\subset \eta\} = 1 - \inf_{y \in B} \nu(y), \quad M\{a \in \xi\} = \mu(a) \), we get

\[
M\{B \subset \eta|a \in \xi\} = \begin{cases} 
\frac{\inf_{y \in B} \nu(y)}{\mu(a)}, & \text{if } \frac{\inf_{y \in B} \nu(y)}{\mu(a)} < \frac{\mu(a)}{2} \\
\frac{\inf_{y \in B} \nu(y) + \mu(a) - 1}{\mu(a)}, & \text{if } \frac{\inf_{y \in B} \nu(y)}{\mu(a)} > 1 - \frac{\mu(a)}{2} \\
0.5, & \text{otherwise}
\end{cases}
\]

That is,

\[
M\{B \subset \eta|a \in \xi\} = \inf_{y \in B} \nu(y|a \in \xi).
\]

The first measure inversion formula is verified. Next, by using the definition of conditional uncertainty and independence of \( \xi \) and \( \eta \), we have

\[
M\{\eta \subset B|a \in \xi\} = \begin{cases} 
\frac{M\{\eta \subset B\}}{M\{a \in \xi\}}, & \text{if } \frac{M\{\eta \subset B\}}{M\{a \in \xi\}} < 0.5 \\
1 - \frac{M\{\eta \not\subset B\}}{M\{a \in \xi\}}, & \text{if } \frac{M\{\eta \not\subset B\}}{M\{a \in \xi\}} < 0.5 \\
0.5, & \text{otherwise}
\end{cases}
\]

Since \( M\{\eta \subset B\} = 1 - \sup_{y \in B^c} \nu(y), \quad M\{\eta \not\subset B\} = \sup_{y \in B^c} \nu(y), \quad M\{a \in \xi\} = \mu(a) \),
we get

\[ M\{\eta \subset B|a \in \xi\} = \begin{cases} 
1 - \sup_{y \in B^c} \nu(y), & \text{if } \sup_{y \in B^c} \nu(y) > 1 - \mu(a)/2 \\
\mu(a) - \sup_{y \in B^c} \nu(y), & \text{if } \sup_{y \in B^c} \nu(y) < \mu(a)/2 \\
0.5, & \text{otherwise.}
\end{cases} \]

That is,

\[ M\{\eta \subset B|a \in \xi\} = 1 - \sup_{y \in B^c} \nu(y|a \in \xi). \]

The second measure inversion formula is verified. Hence \( \nu(y|a \in \xi) \) is the membership function of the conditional uncertain set \( \eta \) given \( a \in \xi \).

![Figure 8.14: Membership Functions \( \nu(y) \) and \( \nu(y|a \in \xi) \)](image)

**Exercise 8.50:** Let \( \xi_1, \xi_2, \ldots, \xi_m, \eta \) be independent uncertain sets with membership functions \( \mu_1, \mu_2, \ldots, \mu_m, \nu \), respectively. For any real numbers \( a_1, a_2, \ldots, a_m \), show that the conditional uncertain set \( \eta \) given \( a_1 \in \xi_1, a_2 \in \xi_2, \ldots, a_m \in \xi_m \) has a membership function

\[ \nu^*(y) = \begin{cases} 
\nu(y) - \min_{1 \leq i \leq m} \mu_i(a_i), & \text{if } \nu(y) < \min_{1 \leq i \leq m} \mu_i(a_i)/2 \\
\nu(y) + \min_{1 \leq i \leq m} \mu_i(a_i) - 1, & \text{if } \nu(y) > 1 - \min_{1 \leq i \leq m} \mu_i(a_i)/2 \\
0.5, & \text{otherwise.}
\end{cases} \]

**8.13 Bibliographic Notes**

In order to model unsharp concepts like “young”, “tall” and “most”, uncertain set was proposed by Liu [93] in 2010. After that, membership function
was presented by Liu [99] in 2012 to describe uncertain sets. However, not all uncertain sets have membership functions. Liu [109] proved that totally ordered uncertain sets on a continuous uncertainty space always have membership functions. In addition, Liu [102] defined the independence of uncertain sets, and provided the operational law through membership functions. Yao [195] derived a formula for calculating the uncertain measure of inclusion relation between uncertain sets.

The expected value of uncertain set was defined by Liu [93]. Next, Liu [95] gave a formula for calculating the expected value by membership function, and Liu [99] provided a formula by inverse membership function. Based on the expected value operator, Liu [96] presented the variance and distance between uncertain sets, and Yang-Gao [174] investigated the moments of uncertain set.

The entropy was presented by Liu [96] as the degree of difficulty of predicting the realization of an uncertain set. Some formulas were also provided by Yao-Ke [190] for calculating the value of entropy.

Conditional uncertain set was first investigated by Liu [93] and conditional membership function was formally defined by Liu [106]. Furthermore, Yao [202] presented some criteria for judging the existence of conditional membership function.
Chapter 9
Uncertain Logic

Uncertain logic is a methodology for calculating the truth values of uncertain propositions via uncertain set theory. This chapter will introduce individual feature data, uncertain quantifier, uncertain subject, uncertain predicate, uncertain proposition, and truth value. Uncertain logic may provide a flexible means for extracting linguistic summary from a collection of raw data.

9.1 Individual Feature Data

At first, we should have a universe $A$ of individuals we are talking about. Without loss of generality, we may assume that $A$ consists of $n$ individuals and is represented by

$$A = \{ a_1, a_2, \cdots, a_n \}. \quad (9.1)$$

In order to deal with the universe $A$, we should have feature data of all individuals $a_1, a_2, \cdots, a_n$. When we talk about “those days are warm”, we should know the individual feature data of all days, for example,

$$A = \{ 22, 23, 25, 28, 30, 32, 36 \} \quad (9.2)$$

whose elements are temperatures in centigrades. When we talk about “those students are young”, we should know the individual feature data of all students, for example,

$$A = \{ 21, 22, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 38, 40 \} \quad (9.3)$$

whose elements are ages in years. When we talk about “those sportsmen are tall”, we should know the individual feature data of all sportsmen, for example,

$$A = \{ 175, 178, 178, 180, 183, 184, 186, 186 \} \quad 178, 190, 192, 192, 193, 194, 195, 196 \} \quad (9.4)$$

whose elements are heights in centimeters.
Sometimes the individual feature data are represented by vectors rather than scalar numbers. When we talk about “those young students are tall”, we should know the individual feature data of all students, for example,

\[ A = \left\{ (24, 185), (25, 190), (26, 184), (26, 170), (27, 187), (27, 188), \\
(28, 160), (30, 190), (32, 185), (33, 176), (35, 185), (36, 188), \\
(38, 164), (38, 178), (39, 182), (40, 186), (42, 165), (44, 170) \right\} \]

whose elements are ages and heights in years and centimeters, respectively.

### 9.2 Uncertain Quantifier

If we want to represent all individuals in the universe \( A \), we use the universal quantifier (\( \forall \)) and

\[ \forall = \text{“for all”}. \]  \hspace{1cm} (9.6)

If we want to represent some (at least one) individuals, we use the existential quantifier (\( \exists \)) and

\[ \exists = \text{“there exists at least one”}. \]  \hspace{1cm} (9.7)

In addition to the two quantifiers, there are numerous imprecise quantifiers in human language, for example, *almost all, almost none, many, several, some, most, a few, about half*. This section will model them by the tool of uncertain quantifier.

**Definition 9.1 (Liu [96])** Uncertain quantifier is an uncertain set representing the number of individuals.

**Example 9.1:** The universal quantifier (\( \forall \)) on the universe \( A \) is a special uncertain quantifier,

\[ \forall \equiv \{ n \} \]  \hspace{1cm} (9.8)

whose membership function is

\[ \lambda(x) = \begin{cases} 
1, & \text{if } x = n \\
0, & \text{otherwise}.
\end{cases} \]  \hspace{1cm} (9.9)

**Example 9.2:** The existential quantifier (\( \exists \)) on the universe \( A \) is a special uncertain quantifier,

\[ \exists \equiv \{ 1, 2, \cdots, n \} \]  \hspace{1cm} (9.10)

whose membership function is

\[ \lambda(x) = \begin{cases} 
0, & \text{if } x = 0 \\
1, & \text{otherwise}.
\end{cases} \]  \hspace{1cm} (9.11)
Example 9.3: The quantifier “there does not exist one” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{0\}$$ (9.12)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0, & \text{otherwise}.
\end{cases}$$ (9.13)

Example 9.4: The quantifier “there exist exactly $m$” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{m\}$$ (9.14)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } x = m \\
0, & \text{otherwise}.
\end{cases}$$ (9.15)

Example 9.5: The quantifier “there exist at least $m$” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{m, m+1, \ldots, n\}$$ (9.16)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } m \leq x \leq n \\
0, & \text{if } 0 \leq x < m.
\end{cases}$$ (9.17)

Example 9.6: The quantifier “there exist at most $m$” on the universe $A$ is a special uncertain quantifier

$$Q \equiv \{0, 1, 2, \ldots, m\}$$ (9.18)

whose membership function is

$$\lambda(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq m \\
0, & \text{if } m < x \leq n.
\end{cases}$$ (9.19)

Example 9.7: The uncertain quantifier $Q$ of “almost all” on the universe $A$ may have a membership function

$$\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq n - 5 \\
(x - n + 5)/3, & \text{if } n - 5 \leq x \leq n - 2 \\
1, & \text{if } n - 2 \leq x \leq n.
\end{cases}$$ (9.20)
Example 9.8: The uncertain quantifier $Q$ of “almost none” on the universe $A$ may have a membership function

$$\lambda(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 2 \\
(5-x)/3, & \text{if } 2 \leq x \leq 5 \\
0, & \text{if } 5 \leq x \leq n.
\end{cases} \quad (9.21)$$

Example 9.9: The uncertain quantifier $Q$ of “about 10” on the universe $A$ may have a membership function

$$\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 7 \\
(x-7)/2, & \text{if } 7 \leq x \leq 9 \\
1, & \text{if } 9 \leq x \leq 11 \\
(13-x)/2, & \text{if } 11 \leq x \leq 13 \\
0, & \text{if } 13 \leq x \leq n.
\end{cases} \quad (9.22)$$

Example 9.10: In many cases, it is more convenient for us to use a percentage than an absolute quantity. For example, we may use the uncertain
quantifier $Q$ of “about 70%”. In this case, a possible membership function of $Q$ is

$$
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.6 \\
20(x - 0.6), & \text{if } 0.6 \leq x \leq 0.65 \\
1, & \text{if } 0.65 \leq x \leq 0.75 \\
20(0.8 - x), & \text{if } 0.75 \leq x \leq 0.8 \\
0, & \text{if } 0.8 \leq x \leq 1.
\end{cases}
$$

(9.23)

Figure 9.3: Membership Function of Quantifier “about 10”

Figure 9.4: Membership Function of Quantifier “about 70%”

**Definition 9.2** An uncertain quantifier is said to be unimodal if its membership function is unimodal.

**Example 9.11:** The uncertain quantifiers “almost all”, “almost none”, “about 10” and “about 70%” are unimodal.

**Definition 9.3** An uncertain quantifier is said to be monotone if its membership function is monotone. Especially, an uncertain quantifier is said to be increasing if its membership function is increasing; and an uncertain quantifier is said to be decreasing if its membership function is decreasing.
The uncertain quantifiers “almost all” and “almost none” are monotone, but “about 10” and “about 70%” are not monotone. Note that both increasing uncertain quantifiers and decreasing uncertain quantifiers are monotone. In addition, any monotone uncertain quantifiers are unimodal.

**Negated Quantifier**

What is the negation of an uncertain quantifier? The following definition gives a formal answer.

**Definition 9.4** (Liu [96]) Let $Q$ be an uncertain quantifier. Then the negated quantifier $\neg Q$ is the complement of $Q$ in the sense of uncertain set, i.e.,

$$\neg Q = Q^c.$$  \hfill (9.24)

**Example 9.12:** Let $\forall = \{n\}$ be the universal quantifier. Then its negated quantifier is

$$\neg \forall \equiv \{0, 1, 2, \ldots, n - 1\}.$$ \hfill (9.25)

**Example 9.13:** Let $\exists = \{1, 2, \ldots, n\}$ be the existential quantifier. Then its negated quantifier is

$$\neg \exists \equiv \{0\}.$$ \hfill (9.26)

**Theorem 9.1** Let $Q$ be an uncertain quantifier whose membership function is $\lambda$. Then the negated quantifier $\neg Q$ has a membership function

$$\neg \lambda(x) = 1 - \lambda(x).$$ \hfill (9.27)

**Proof:** This theorem follows from the operational law of uncertain set immediately.

**Example 9.14:** Let $Q$ be the uncertain quantifier “almost all” defined by (9.20). Then its negated quantifier $\neg Q$ has a membership function

$$\neg \lambda(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq n - 5 \\ (n - x - 2)/3, & \text{if } n - 5 \leq x \leq n - 2 \\ 0, & \text{if } n - 2 \leq x \leq n. \end{cases}$$ \hfill (9.28)

**Example 9.15:** Let $Q$ be the uncertain quantifier “about 70%” defined by (9.23). Then its negated quantifier $\neg Q$ has a membership function

$$\neg \lambda(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 0.6 \\ 20(0.65 - x), & \text{if } 0.6 \leq x \leq 0.65 \\ 0, & \text{if } 0.65 \leq x \leq 0.75 \\ 20(x - 0.75), & \text{if } 0.75 \leq x \leq 0.8 \\ 1, & \text{if } 0.8 \leq x \leq 1. \end{cases}$$ \hfill (9.29)
Theorem 9.2  Let $Q$ be an uncertain quantifier. Then we have $\neg \neg Q = Q$.

Proof: This theorem follows from $\neg \neg Q = \neg Q^c = (Q^c)^c = Q$.

Theorem 9.3  If $Q$ is a monotone uncertain quantifier, then $\neg Q$ is also monotone. Especially, if $Q$ is increasing, then $\neg Q$ is decreasing; if $Q$ is decreasing, then $\neg Q$ is increasing.

Proof: Assume $\lambda$ is the membership function of $Q$. Then $\neg Q$ has a membership function $\neg \lambda(x) = 1 - \lambda(x)$. The theorem follows from this fact immediately.

Dual Quantifier

Definition 9.5  (Liu [96]) Let $Q$ be an uncertain quantifier. Then the dual quantifier of $Q$ is

$$Q^* = \forall - Q.$$  \hspace{1cm} (9.30)

Remark 9.1: Note that $Q$ and $Q^*$ are dependent uncertain sets such that $Q + Q^* \equiv \forall$. Since the cardinality of the universe $A$ is $n$, we also have

$$Q^* = \{n\} - Q.$$ \hspace{1cm} (9.31)
Example 9.16: Since $\forall \equiv \{n\}$, we immediately have $\forall^* = \{0\} = \neg \exists$. That is

$$\forall^* \equiv \neg \exists. \quad (9.32)$$

Example 9.17: Since $\neg \forall = \{0, 1, 2, \ldots, n-1\}$, we immediately have $(\neg \forall)^* = \{1, 2, \ldots, n\} = \exists$. That is,

$$(\neg \forall)^* \equiv \exists. \quad (9.33)$$

Example 9.18: Since $\exists \equiv \{1, 2, \ldots, n\}$, we have $\exists^* = \{0, 1, 2, \ldots, n-1\} = \neg \forall$. That is,

$$\exists^* \equiv \neg \forall. \quad (9.34)$$

Example 9.19: Since $\neg \exists = \{0\}$, we immediately have $(\neg \exists)^* = \{n\} = \forall$. That is,

$$(\neg \exists)^* = \forall. \quad (9.35)$$

**Theorem 9.4** Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is $\lambda$. Then the dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \lambda(n - x) \quad (9.36)$$

where $n$ is the cardinality of the universe $\mathcal{A}$.

**Proof:** This theorem follows from the operational law of uncertain set immediately.

Example 9.20: Let $\mathcal{Q}$ be the uncertain quantifier “almost all” defined by (9.20). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 2 \\
(5 - x)/3, & \text{if } 2 \leq x \leq 5 \\
0, & \text{if } 5 \leq x \leq n. 
\end{cases} \quad (9.37)$$

![Figure 9.7: Membership Function of Dual Quantifier of “almost all”](image_url)
Example 9.21: Let $Q$ be the uncertain quantifier “about 70%” defined by (9.23). Then its dual quantifier $Q^*$ has a membership function

$$
\lambda^*(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.2 \\
20(x - 0.2), & \text{if } 0.2 \leq x \leq 0.25 \\
1, & \text{if } 0.25 \leq x \leq 0.35 \\
20(0.4 - x), & \text{if } 0.35 \leq x \leq 0.4 \\
0, & \text{if } 0.4 \leq x \leq 1.
\end{cases}
$$

(9.38)

Figure 9.8: Membership Function of Dual Quantifier of “about 70%”

Theorem 9.5 Let $Q$ be an uncertain quantifier. Then we have $Q^{**} = Q$.

Proof: The theorem follows from $Q^{**} = \forall - Q^* = \forall - (\forall - Q) = \Omega$.

Theorem 9.6 If $Q$ is a unimodal uncertain quantifier, then $Q^*$ is also unimodal. Especially, if $Q$ is a monotone, then $Q^*$ is monotone; if $Q$ is increasing, then $Q^*$ is decreasing; if $Q$ is decreasing, then $Q^*$ is increasing.

Proof: Assume $\lambda$ is the membership function of $Q$. Then $Q^*$ has a membership function $\lambda^*(x) = \lambda(n - x)$. The theorem follows from this fact immediately.

9.3 Uncertain Subject

Sometimes, we are interested in a subset of the universe of individuals, for example, “warm days”, “young students” and “tall sportsmen”. This section will model them by the concept of uncertain subject.

Definition 9.6 (Liu [96]) Uncertain subject is an uncertain set containing some specified individuals in the universe.
Example 9.22: “Warm days are here again” is a statement in which “warm days” is an uncertain subject that is an uncertain set on the universe of “all days”, whose membership function may be defined by

\[
\nu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 \leq x \leq 18 \\
1, & \text{if } 18 \leq x \leq 24 \\
(28 - x)/4, & \text{if } 24 \leq x \leq 28 \\
0, & \text{if } 28 \leq x.
\end{cases}
\] (9.39)

Example 9.23: “Young students are tall” is a statement in which “young students” is an uncertain subject that is an uncertain set on the universe of “all students”, whose membership function may be defined by

\[
\nu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/5, & \text{if } 15 \leq x \leq 20 \\
1, & \text{if } 20 \leq x \leq 35 \\
(45 - x)/10, & \text{if } 35 \leq x \leq 45 \\
0, & \text{if } x \geq 45.
\end{cases}
\] (9.40)

Example 9.24: “Tall students are heavy” is a statement in which “tall students” is an uncertain subject that is an uncertain set on the universe of “all students”, whose membership function may be defined by

\[
\nu(x) = \begin{cases} 
0, & \text{if } x \leq 180 \\
(x - 180)/5, & \text{if } 180 \leq x \leq 185 \\
1, & \text{if } 185 \leq x \leq 195 \\
(200 - x)/5, & \text{if } 195 \leq x \leq 200 \\
0, & \text{if } x \geq 200.
\end{cases}
\] (9.41)

Let \( S \) be an uncertain subject with membership function \( \nu \) on the universe \( A = \{a_1, a_2, \ldots, a_n\} \) of individuals. Then \( S \) is an uncertain set of \( A \) such that

\[
\mathcal{M}\{a_i \in S\} = \nu(a_i), \quad i = 1, 2, \ldots, n.
\] (9.42)

In many cases, we are interested in some individuals \( a \)'s with \( \nu(a) \geq \omega \), where \( \omega \) is a confidence level. Thus we have a subuniverse,

\[
S_\omega = \{a \in A \mid \nu(a) \geq \omega\}
\] (9.43)

that will play a new universe of individuals we are talking about, and the individuals out of \( S_\omega \) will be ignored at the confidence level \( \omega \).
**Theorem 9.7** Let $\omega_1$ and $\omega_2$ be confidence levels with $\omega_1 > \omega_2$, and let $S_{\omega_1}$ and $S_{\omega_2}$ be subuniverses with confidence levels $\omega_1$ and $\omega_2$, respectively. Then

$$S_{\omega_1} \subset S_{\omega_2}. \quad (9.44)$$

That is, $S_\omega$ is a decreasing sequence of sets with respect to $\omega$.

**Proof:** If $a \in S_{\omega_1}$, then $\nu(a) \geq \omega_1 > \omega_2$. Thus $a \in S_{\omega_2}$. It follows that $S_{\omega_1} \subset S_{\omega_2}$. Note that $S_{\omega_1}$ and $S_{\omega_2}$ may be empty.

### 9.4 Uncertain Predicate

There are numerous imprecise predicates in human language, for example, *warm*, *cold*, *hot*, *young*, *old*, *tall*, *small*, and *big*. This section will model them by the concept of uncertain predicate.

**Definition 9.7** (Liu [96]) Uncertain predicate is an uncertain set representing a property that the individuals have in common.

**Example 9.25:** “Today is warm” is a statement in which “warm” is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 \leq x \leq 18 \\
1, & \text{if } 18 \leq x \leq 24 \\
(28 - x)/4, & \text{if } 24 \leq x \leq 28 \\
0, & \text{if } 28 \leq x.
\end{cases} \quad (9.45)$$

**Example 9.26:** “John is young” is a statement in which “young” is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 
0, & \text{if } x \leq 15 \\
(x - 15)/5, & \text{if } 15 \leq x \leq 20 \\
1, & \text{if } 20 \leq x \leq 35 \\
(45 - x)/10, & \text{if } 35 \leq x \leq 45 \\
0, & \text{if } x \geq 45.
\end{cases} \quad (9.46)$$

**Example 9.27:** “Tom is tall” is a statement in which “tall” is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 
0, & \text{if } x \leq 180 \\
(x - 180)/5, & \text{if } 180 \leq x \leq 185 \\
1, & \text{if } 185 \leq x \leq 195 \\
(200 - x)/5, & \text{if } 195 \leq x \leq 200 \\
0, & \text{if } x \geq 200.
\end{cases} \quad (9.47)$$
Negated Predicate

Definition 9.8 (Liu [96]) Let $P$ be an uncertain predicate. Then its negated predicate $\neg P$ is the complement of $P$ in the sense of uncertain set, i.e.,

$$\neg P = P^c.$$  \hspace{1cm} (9.48)

Theorem 9.8 Let $P$ be an uncertain predicate with membership function $\mu$. Then its negated predicate $\neg P$ has a membership function

$$\neg \mu(x) = 1 - \mu(x).$$  \hspace{1cm} (9.49)

Proof: The theorem follows from the definition of negated predicate and the operational law of uncertain set immediately.

Example 9.28: Let $P$ be the uncertain predicate “warm” defined by (9.45). Then its negated predicate $\neg P$ has a membership function

$$\neg \mu(x) = \begin{cases} 
1, & \text{if } x \leq 15 \\
(18 - x)/3, & \text{if } 15 \leq x \leq 18 \\
0, & \text{if } 18 \leq x \leq 24 \\
(x - 24)/4, & \text{if } 24 \leq x \leq 28 \\
1, & \text{if } 28 \leq x.
\end{cases}$$  \hspace{1cm} (9.50)

![Figure 9.9: Membership Function of Negated Predicate of “warm”](image)

Example 9.29: Let $P$ be the uncertain predicate “young” defined by (9.46). Then its negated predicate $\neg P$ has a membership function

$$\neg \mu(x) = \begin{cases} 
1, & \text{if } x \leq 15 \\
(20 - x)/5, & \text{if } 15 \leq x \leq 20 \\
0, & \text{if } 20 \leq x \leq 35 \\
(x - 35)/10, & \text{if } 35 \leq x \leq 45 \\
1, & \text{if } x \geq 45.
\end{cases}$$  \hspace{1cm} (9.51)
Example 9.30: Let $P$ be the uncertain predicate "tall" defined by (9.47). Then its negated predicate $\neg P$ has a membership function

$$\neg \mu(x) = \begin{cases} 
1, & \text{if } x \leq 180 \\
(185 - x)/5, & \text{if } 180 \leq x \leq 185 \\
0, & \text{if } 185 \leq x \leq 195 \\
(x - 195)/5, & \text{if } 195 \leq x \leq 200 \\
1, & \text{if } x \geq 200.
\end{cases}$$

(9.52)

Theorem 9.9 Let $P$ be an uncertain predicate. Then we have $\neg \neg P = P$.

Proof: The theorem follows from $\neg \neg P = \neg P^c = (P^c)^c = P$.

9.5 Uncertain Proposition

Definition 9.9 (Liu [96]) Assume that $\mathcal{Q}$ is an uncertain quantifier, $S$ is an uncertain subject, and $P$ is an uncertain predicate. Then the triplet

$$(\mathcal{Q}, S, P) = "\mathcal{Q} \text{ of } S \text{ are } P"$$

(9.53)

is called an uncertain proposition.
**Remark 9.2:** Let $A$ be the universe of individuals. Then $(\mathcal{Q}, A, P)$ is a special uncertain proposition because $A$ itself is a special uncertain subject.

**Remark 9.3:** Let $\forall$ be the universal quantifier. Then $(\forall, A, P)$ is an uncertain proposition representing “all of $A$ are $P$”.

**Remark 9.4:** Let $\exists$ be the existential quantifier. Then $(\exists, A, P)$ is an uncertain proposition representing “at least one of $A$ is $P$”.

**Example 9.31:** “Almost all students are young” is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is “almost all”, the uncertain subject $S$ is “students” (the universe itself) and the uncertain predicate $P$ is “young”.

**Example 9.32:** “Most young students are tall” is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is “most”, the uncertain subject $S$ is “young students” and the uncertain predicate $P$ is “tall”.

**Theorem 9.10** (Liu [96], Logical Equivalence Theorem) Let $(\mathcal{Q}, S, P)$ be an uncertain proposition. Then

$$(\mathcal{Q}^*, S, P) = (\mathcal{Q}, S, \neg P)$$

(9.54)

where $\mathcal{Q}^*$ is the dual quantifier of $\mathcal{Q}$ and $\neg P$ is the negated predicate of $P$.

**Proof:** Note that $(\mathcal{Q}^*, S, P)$ represents “$\mathcal{Q}^*$ of $S$ are $P$”. In fact, the statement “$\mathcal{Q}^*$ of $S$ are $P$” implies “$\mathcal{Q}^{**}$ of $S$ are not $P$”. Since $\mathcal{Q}^{**} = \mathcal{Q}$, we obtain $(\mathcal{Q}, S, \neg P)$. Conversely, the statement “$\mathcal{Q}$ of $S$ are not $P$” implies “$\mathcal{Q}^*$ of $S$ are $P$”, i.e., $(\mathcal{Q}^*, S, P)$. Thus (9.54) is verified.

**Example 9.33:** When $\mathcal{Q}^* = \neg\forall$, we have $\mathcal{Q} = \exists$. If $S = A$, then (9.54) becomes the classical equivalence

$$(-\forall, A, P) = (\exists, A, \neg P).$$

(9.55)

**Example 9.34:** When $\mathcal{Q}^* = \neg\exists$, we have $\mathcal{Q} = \forall$. If $S = A$, then (9.54) becomes the classical equivalence

$$(-\exists, A, P) = (\forall, A, \neg P).$$

(9.56)

**9.6 Truth Value**

Let $(\mathcal{Q}, S, P)$ be an uncertain proposition. The truth value of $(\mathcal{Q}, S, P)$ should be the uncertain measure that “$\mathcal{Q}$ of $S$ are $P$”. That is,

$$T(\mathcal{Q}, S, P) = M\{\mathcal{Q} \text{ of } S \text{ are } P\}.$$  

(9.57)

However, it is impossible for us to deduce the value of $M\{\mathcal{Q} \text{ of } S \text{ are } P\}$ from the information of $\mathcal{Q}$, $S$ and $P$ within the framework of uncertain set theory. Thus we need an additional formula to compose $\mathcal{Q}$, $S$ and $P$. 
Definition 9.10 (Liu [96]) Let \((Q, S, P)\) be an uncertain proposition in which \(Q\) is a unimodal uncertain quantifier with membership function \(\lambda\), \(S\) is an uncertain subject with membership function \(\nu\), and \(P\) is an uncertain predicate with membership function \(\mu\). Then the truth value of \((Q, S, P)\) with respect to the universe \(A\) is

\[
T(Q, S, P) = \sup_{0 \leq \omega \leq 1} \left( \omega \land \sup_{K \in K_\omega} \inf_{a \in K} \mu(a) \land \sup_{K \in K^*_\omega} \inf_{a \in K} \neg\mu(a) \right)
\]

where

\[
K_\omega = \left\{ K \subset S_\omega \mid \lambda(|K|) \geq \omega \right\},
\]

\[
K^*_\omega = \left\{ K \subset S_\omega \mid \lambda(|S_\omega| - |K|) \geq \omega \right\},
\]

\[
S_\omega = \{ a \in A \mid \nu(a) \geq \omega \}.
\]

Remark 9.5: Keep in mind that the truth value formula (9.58) is vacuous if the individual feature data of the universe \(A\) are not available.

Remark 9.6: The symbol \(|K|\) represents the cardinality of the set \(K\). For example, \(|\emptyset| = 0\) and \(|\{2, 5, 6\}| = 3\).

Remark 9.7: Note that \(\neg\mu\) is the membership function of the negated predicate of \(P\), and

\[
\neg\mu(a) = 1 - \mu(a).
\]

Remark 9.8: When the subset \(K\) of individuals becomes an empty set \(\emptyset\), we set

\[
\inf_{a \in \emptyset} \mu(a) = \inf_{a \in \emptyset} \neg\mu(a) = 1.
\]

Remark 9.9: If \(Q\) is an uncertain percentage rather than an absolute quantity, then

\[
K_\omega = \left\{ K \subset S_\omega \mid \lambda\left(\frac{|K|}{|S_\omega|}\right) \geq \omega \right\},
\]

\[
K^*_\omega = \left\{ K \subset S_\omega \mid \lambda\left(1 - \frac{|K|}{|S_\omega|}\right) \geq \omega \right\}.
\]

Remark 9.10: If the uncertain subject \(S\) is identical to the universe \(A\) itself (i.e., \(S = A\)), then

\[
K_\omega = \left\{ K \subset A \mid \lambda(|K|) \geq \omega \right\},
\]

\[
K^*_\omega = \left\{ K \subset A \mid \lambda(|A| - |K|) \geq \omega \right\}.
\]

Exercise 9.1: If the uncertain quantifier \(Q = \forall\) and the uncertain subject \(S = A\), then for any \(\omega > 0\), we have

\[
K_\omega = \{ A \}, \quad K^*_\omega = \{ \emptyset \}.
\]
Show that
\[ T(\forall, A, P) = \inf_{a \in A} \mu(a). \]  
(9.69)

**Exercise 9.2:** If the uncertain quantifier \( Q = \exists \) and the uncertain subject \( S = A \), then for any \( \omega > 0 \), we have
\[ K_\omega = \{ \text{any nonempty subsets of } A \}, \]  
(9.70)
\[ K^*_\omega = \{ \text{any proper subsets of } A \}. \]  
(9.71)
Show that
\[ T(\exists, A, P) = \sup_{a \in A} \mu(a). \]  
(9.72)

**Exercise 9.3:** If the uncertain quantifier \( Q = \neg \forall \) and the uncertain subject \( S = A \), then for any \( \omega > 0 \), we have
\[ K_\omega = \{ \text{any proper subsets of } A \}, \]  
(9.73)
\[ K^*_\omega = \{ \text{any nonempty subsets of } A \}. \]  
(9.74)
Show that
\[ T(\neg \forall, A, P) = 1 - \inf_{a \in A} \mu(a). \]  
(9.75)

**Exercise 9.4:** If the uncertain quantifier \( Q = \neg \exists \) and the uncertain subject \( S = A \), then for any \( \omega > 0 \), we have
\[ K_\omega = \{ \emptyset \}, \quad K^*_\omega = \{ A \}. \]  
(9.76)
Show that
\[ T(\neg \exists, A, P) = 1 - \sup_{a \in A} \mu(a). \]  
(9.77)

**Theorem 9.11** (Liu [96], Truth Value Theorem) Let \( (Q, S, P) \) be an uncertain proposition in which \( Q \) is a unimodal uncertain quantifier with membership function \( \lambda \), \( S \) is an uncertain subject with membership function \( \nu \), and \( P \) is an uncertain predicate with membership function \( \mu \). Then the truth value of \( (Q, S, P) \) is
\[ T(Q, S, P) = \sup_{0 \leq \omega \leq 1} (\omega \land \Delta(k_\omega) \land \Delta^*(k^*_\omega)) \]  
(9.78)
where
\[ k_\omega = \min \{ x \mid \lambda(x) \geq \omega \}, \]  
(9.79)
\[ \Delta(k_\omega) = k_\omega \cdot \max \{ \mu(a_i) \mid a_i \in S_\omega \}, \]  
(9.80)
\[ k^*_\omega = |S_\omega| - \max \{ x \mid \lambda(x) \geq \omega \}, \]  
(9.81)
\[ \Delta^*(k^*_\omega) = k^*_\omega \cdot \max \{ 1 - \mu(a_i) \mid a_i \in S_\omega \}. \]  
(9.82)
Proof: Since the supremum is achieved at the subset with minimum cardinality, we have

\[
\sup_{K \in \mathcal{K}} \inf_{a \in K} \mu(a) = \sup_{K \subset S, |K| = \kappa_\omega} \inf_{a \in K} \mu(a) = \Delta(\kappa_\omega),
\]

\[
\sup_{K \in \mathcal{K}} \inf_{a \in K} -\mu(a) = \sup_{K \subset S, |K| = \kappa_\omega^*} \inf_{a \in K} -\mu(a) = \Delta^*(\kappa_\omega^*).
\]

The theorem is thus verified. Please note that \(\Delta(0) = \Delta^*(0) = 1\).

Remark 9.11: If \(Q\) is an uncertain percentage rather than an absolute quantity, then

\[
k_\omega = \min \left\{ x \mid \lambda \left( \frac{x}{|S_\omega|} \right) \geq \omega \right\},
\]

(9.83)

\[
k_\omega^* = |S_\omega| - \max \left\{ x \mid \lambda \left( \frac{x}{|S_\omega|} \right) \geq \omega \right\}.
\]

(9.84)

Remark 9.12: If the uncertain subject \(S\) is identical to the universe \(A\) itself (i.e., \(S = A\)), then

\[
k_\omega = \min \left\{ x \mid \lambda(x) \geq \omega \right\},
\]

(9.85)

\[
\Delta(k_\omega) = k_\omega - \max\left\{ \mu(a_1), \mu(a_2), \cdots, \mu(a_n) \right\},
\]

(9.86)

\[
k_\omega^* = n - \max\left\{ x \mid \lambda(x) \geq \omega \right\},
\]

(9.87)

\[
\Delta^*(k_\omega^*) = k_\omega^* - \max\left\{ 1 - \mu(a_1), 1 - \mu(a_2), \cdots, 1 - \mu(a_n) \right\}.
\]

(9.88)

Exercise 9.5: If the uncertain quantifier \(Q = \{m, m+1, \cdots, n\}\) (i.e., “there exist at least \(m\)”) with \(m \geq 1\), then we have \(k_\omega = m\) and \(k_\omega^* = 0\). Show that

\[
T(Q, A, P) = m - \max\left\{ \mu(a_1), \mu(a_2), \cdots, \mu(a_n) \right\}.
\]

(9.89)

Exercise 9.6: If the uncertain quantifier \(Q = \{0, 1, 2, \ldots, m\}\) (i.e., “there exist at most \(m\)”) with \(m < n\), then we have \(k_\omega = 0\) and \(k_\omega^* = n - m\). Show that

\[
T(Q, A, P) = (n - m) - \max\left\{ 1 - \mu(a_1), 1 - \mu(a_2), \cdots, 1 - \mu(a_n) \right\}.
\]

(9.90)

Example 9.35: Assume that the daily temperatures of some week from Monday to Sunday are

\[
22, 23, 25, 28, 30, 32, 36
\]

(9.91)
in centigrades. Consider an uncertain proposition

\[
(Q, A, P) = “\text{two or three days are warm}”.
\]

(9.92)
Note that the uncertain quantifier is $Q = \{2, 3\}$. We also suppose that the uncertain predicate $P = \text{“warm”}$ has a membership function

$$
\mu(x) = \begin{cases}
0, & \text{if } x \leq 15 \\
(x - 15)/3, & \text{if } 15 \leq x \leq 18 \\
1, & \text{if } 18 \leq x \leq 24 \\
(28 - x)/4, & \text{if } 24 \leq x \leq 28 \\
0, & \text{if } 28 \leq x.
\end{cases}
$$

(9.93)

It is clear that Monday and Tuesday are warm with truth value 1, and Wednesday is warm with truth value 0.75. But Thursday to Sunday are not “warm” at all (in fact, they are “hot”). Intuitively, the uncertain proposition “two or three days are warm” should be completely true. The truth value formula (9.58) yields that the truth value is

$$
T(\text{“two or three days are warm”}) = 1.
$$

(9.94)

This is an intuitively expected result. In addition, we also have

$$
T(\text{“two days are warm”}) = 0.25, \quad (9.95)
$$

$$
T(\text{“three days are warm”}) = 0.75. \quad (9.96)
$$

**Example 9.36:** Assume that in a class there are 15 students whose ages are

21, 22, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 38, 40

in years. Consider an uncertain proposition

$$(Q, A, P) = \text{“almost all students are young”}. \quad (9.97)
$$

Suppose the uncertain quantifier $Q = \text{“almost all”}$ has a membership function

$$
\lambda(x) = \begin{cases}
0, & \text{if } 0 \leq x \leq 10 \\
(x - 10)/3, & \text{if } 10 \leq x \leq 13 \\
1, & \text{if } 13 \leq x \leq 15,
\end{cases}
$$

(9.99)

and the uncertain predicate $P = \text{“young”}$ has a membership function

$$
\mu(x) = \begin{cases}
0, & \text{if } x \leq 15 \\
(x - 15)/5, & \text{if } 15 \leq x \leq 20 \\
1, & \text{if } 20 \leq x \leq 35 \\
(45 - x)/10, & \text{if } 35 \leq x \leq 45 \\
0, & \text{if } x \geq 45.
\end{cases}
$$

(9.100)
The truth value formula (9.58) yields that the uncertain proposition has a truth value
\[ T(\text{“almost all students are young”}) = 0.9. \] (9.101)

**Example 9.37:** Assume that in a team there are 16 sportsmen whose heights are
\[ 175, 178, 178, 180, 183, 184, 186, 186 \]
\[ 188, 190, 192, 192, 193, 194, 195, 196 \] (9.102)
in centimeters. Consider an uncertain proposition
\[ (Q, A, P) = \text{“about 70% of sportsmen are tall”}. \] (9.103)

Suppose the uncertain quantifier \( Q = \text{“about 70%”} \) has a membership function
\[ \lambda(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 0.6 \\ 20(x - 0.6), & \text{if } 0.6 \leq x \leq 0.65 \\ 1, & \text{if } 0.65 \leq x \leq 0.75 \\ 20(0.8 - x), & \text{if } 0.75 \leq x \leq 0.8 \\ 0, & \text{if } 0.8 \leq x \leq 1 \end{cases} \] (9.104)

and the uncertain predicate \( P = \text{“tall”} \) has a membership function
\[ \mu(x) = \begin{cases} 0, & \text{if } x \leq 180 \\ (x - 180)/5, & \text{if } 180 \leq x \leq 185 \\ 1, & \text{if } 185 \leq x \leq 195 \\ (200 - x)/5, & \text{if } 195 \leq x \leq 200 \\ 0, & \text{if } x \geq 200. \end{cases} \] (9.105)

The truth value formula (9.58) yields that the uncertain proposition has a truth value
\[ T(\text{“about 70% of sportsmen are tall”}) = 0.8. \] (9.106)

**Example 9.38:** Assume that in a class there are 18 students whose ages and heights are
\[ (24, 185), (25, 190), (26, 184), (26, 170), (27, 187), (27, 188) \]
\[ (28, 160), (30, 190), (32, 185), (33, 176), (35, 185), (36, 188) \]
\[ (38, 164), (38, 178), (39, 182), (40, 186), (42, 165), (44, 170) \] (9.107)
in years and centimeters. Consider an uncertain proposition
\[ (Q, S, P) = \text{“most young students are tall”}. \] (9.108)
Suppose the uncertain quantifier (percentage) $Q = \text{“most”}$ has a membership function
\[
\lambda(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.7 \\
20(x - 0.7), & \text{if } 0.7 \leq x \leq 0.75 \\
1, & \text{if } 0.75 \leq x \leq 0.85 \\
20(0.9 - x), & \text{if } 0.85 \leq x \leq 0.9 \\
0, & \text{if } 0.9 \leq x \leq 1.
\end{cases} \quad (9.109)
\]

Note that each individual is described by a feature data $(y, z)$, where $y$ represents ages and $z$ represents heights. In this case, the uncertain subject $S = \text{“young students”}$ has a membership function
\[
\nu(y) = \begin{cases} 
0, & \text{if } y \leq 15 \\
(y - 15)/5, & \text{if } 15 \leq y \leq 20 \\
1, & \text{if } 20 \leq y \leq 35 \\
(45 - y)/10, & \text{if } 35 \leq y \leq 45 \\
0, & \text{if } y \geq 45
\end{cases} \quad (9.110)
\]

and the uncertain predicate $P = \text{“tall”}$ has a membership function
\[
\mu(z) = \begin{cases} 
0, & \text{if } z \leq 180 \\
(z - 180)/5, & \text{if } 180 \leq z \leq 185 \\
1, & \text{if } 185 \leq z \leq 195 \\
(200 - z)/5, & \text{if } 195 \leq z \leq 200 \\
0, & \text{if } z \geq 200.
\end{cases} \quad (9.111)
\]

The truth value formula (9.58) yields that the uncertain proposition has a truth value
\[
T(\text{“most young students are tall”}) = 0.8. \quad (9.112)
\]

### 9.7 Linguistic Summarizer

Linguistic summary is a human language statement that is concise and easy-to-understand by humans. For example, “most young students are tall” is a linguistic summary of students’ ages and heights. Thus a linguistic summary is a special uncertain proposition whose uncertain quantifier, uncertain subject and uncertain predicate are linguistic terms. Uncertain logic provides a flexible means that is capable of extracting linguistic summary from a collection of raw data.

What inputs does the uncertain logic need? First, we should have some raw data (i.e., the individual feature data),
\[
A = \{a_1, a_2, \ldots, a_n\}. \quad (9.113)
\]
Next, we should have some linguistic terms to represent quantifiers, for example, “most” and “all”. Denote them by a collection of uncertain quantifiers,

\[ Q = \{Q_1, Q_2, \ldots, Q_m\}. \]  

Then, we should have some linguistic terms to represent subjects, for example, “young students” and “old students”. Denote them by a collection of uncertain subjects,

\[ S = \{S_1, S_2, \ldots, S_n\}. \]  

Last, we should have some linguistic terms to represent predicates, for example, “short” and “tall”. Denote them by a collection of uncertain predicates,

\[ P = \{P_1, P_2, \ldots, P_k\}. \]

One problem of data mining is to choose an uncertain quantifier \( Q \in Q \), an uncertain subject \( S \in S \) and an uncertain predicate \( P \in P \) such that the truth value of the linguistic summary “\( Q \) of \( S \) are \( P \)” to be extracted is at least \( \beta \), i.e.,

\[ T(Q, S, P) \geq \beta \]  

for the universe \( A = \{a_1, a_2, \ldots, a_n\} \), where \( \beta \) is a confidence level. In order to solve this problem, Liu [96] proposed the following linguistic summarizer,

\[
\begin{cases}
    \text{Find } Q, S \text{ and } P \\
    \text{subject to:} \\
    Q \in Q \\
    S \in S \\
    P \in P \\
    T(Q, S, P) \geq \beta.
\end{cases}
\]  

(9.118)

Each solution \((Q, S, P)\) of the linguistic summarizer (9.118) produces a linguistic summary “\( \overline{Q} \) of \( \overline{S} \) are \( \overline{P} \).”

**Example 9.39:** Assume that in a class there are 18 students whose ages and heights are

(24, 185), (25, 190), (26, 184), (26, 170), (27, 187), (27, 188) 
(28, 160), (30, 190), (32, 185), (33, 176), (35, 185), (36, 188) 
(38, 164), (38, 178), (39, 182), (40, 186), (42, 165), (44, 170)  

(9.119)

in years and centimeters. Suppose we have three linguistic terms “about half”, “most” and “all” as uncertain quantifiers whose membership functions are

\[
\lambda_{\text{half}}(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.4 \\
20(x - 0.4), & \text{if } 0.4 \leq x \leq 0.45 \\
1, & \text{if } 0.45 \leq x \leq 0.55 \\
20(0.6 - x), & \text{if } 0.55 \leq x \leq 0.6 \\
0, & \text{if } 0.6 \leq x \leq 1,
\end{cases}
\]  

(9.120)
\( \lambda_{\text{most}}(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 0.7 \\
20(x - 0.7), & \text{if } 0.7 \leq x \leq 0.75 \\
1, & \text{if } 0.75 \leq x \leq 0.85 \\
20(0.9 - x), & \text{if } 0.85 \leq x \leq 0.9 \\
0, & \text{if } 0.9 \leq x \leq 1, 
\end{cases} \quad (9.121) \)

\( \lambda_{\text{all}}(x) = \begin{cases} 
1, & \text{if } x = 1 \\
0, & \text{if } 0 \leq x < 1, 
\end{cases} \quad (9.122) \)

respectively. Denote the collection of uncertain quantifiers by

\( \mathcal{Q} = \{\text{“about half”, “most”, “all”}\}. \quad (9.123) \)

We also have three linguistic terms “young students”, “middle-aged students” and “old students” as uncertain subjects whose membership functions are

\( \nu_{\text{young}}(y) = \begin{cases} 
0, & \text{if } y \leq 15 \\
(y - 15)/5, & \text{if } 15 \leq y \leq 20 \\
1, & \text{if } 20 \leq y \leq 35 \\
(45 - y)/10, & \text{if } 35 \leq y \leq 45 \\
0, & \text{if } y \geq 45, 
\end{cases} \quad (9.124) \)

\( \nu_{\text{middle}}(y) = \begin{cases} 
0, & \text{if } y \leq 40 \\
(y - 40)/5, & \text{if } 40 \leq y \leq 45 \\
1, & \text{if } 45 \leq y \leq 55 \\
(60 - y)/5, & \text{if } 55 \leq y \leq 60 \\
0, & \text{if } y \geq 60, 
\end{cases} \quad (9.125) \)

\( \nu_{\text{old}}(y) = \begin{cases} 
0, & \text{if } y \leq 55 \\
(y - 55)/5, & \text{if } 55 \leq y \leq 60 \\
1, & \text{if } 60 \leq y \leq 80 \\
(85 - y)/5, & \text{if } 80 \leq y \leq 85 \\
1, & \text{if } y \geq 85, 
\end{cases} \quad (9.126) \)

respectively. Denote the collection of uncertain subjects by

\( \mathcal{S} = \{\text{“young students”, “middle-aged students”, “old students”}\}. \quad (9.127) \)

Finally, we suppose that there are two linguistic terms “short” and “tall” as uncertain predicates whose membership functions are

\( \mu_{\text{short}}(z) = \begin{cases} 
0, & \text{if } z \leq 145 \\
(z - 145)/5, & \text{if } 145 \leq z \leq 150 \\
1, & \text{if } 150 \leq z \leq 155 \\
(160 - z)/5, & \text{if } 155 \leq z \leq 160 \\
0, & \text{if } z \geq 200, 
\end{cases} \quad (9.128) \)
\[ \mu_{\text{tall}}(z) = \begin{cases} 
0, & \text{if } z \leq 180 \\
(z - 180)/5, & \text{if } 180 \leq z \leq 185 \\
1, & \text{if } 185 \leq z \leq 195 \\
(200 - z)/5, & \text{if } 195 \leq z \leq 200 \\
0, & \text{if } z \geq 200, 
\end{cases} \] (9.129)

respectively. Denote the collection of uncertain predicates by
\[ \mathcal{P} = \{ \text{“short”}, \text{“tall”} \}. \] (9.130)

We would like to extract an uncertain quantifier \( \mathcal{Q} \in \mathcal{Q} \), an uncertain subject \( S \in \mathcal{S} \) and an uncertain predicate \( P \in \mathcal{P} \) such that the truth value of the linguistic summary \( \mathcal{Q} \) of \( S \) are \( P \) to be extracted is at least 0.8, i.e.,
\[ T(\mathcal{Q}, S, P) \geq 0.8 \] (9.131)

where 0.8 is a predetermined confidence level. The linguistic summarizer (9.118) yields
\[ \overline{\mathcal{Q}} = \text{“most”}, \quad \overline{S} = \text{“young students”}, \quad \overline{P} = \text{“tall”} \]
and then extracts a linguistic summary “most young students are tall”.

\section*{9.8 Bibliographic Notes}

Based on uncertain set theory, uncertain logic was designed by Liu [96] in 2011 for dealing with human language by using the truth value formula for uncertain propositions. As an application of uncertain logic, Liu [96] also proposed a linguistic summarizer that provides a means for extracting linguistic summary from a collection of raw data.
Chapter 10

Uncertain Inference

Uncertain inference is a process of deriving consequences from human knowledge via uncertain set theory. This chapter will introduce a family of uncertain inference rules, uncertain system, and uncertain control with application to an inverted pendulum system.

10.1 Uncertain Inference Rule

Let $X$ and $Y$ be two concepts. It is assumed that we only have a single if-then rule,

$$\text{“if } X \text{ is } \xi \text{ then } Y \text{ is } \eta\"$$  \hspace{1cm} (10.1)

where $\xi$ and $\eta$ are two uncertain sets. We first introduce the following inference rule.

**Inference Rule 10.1** (Liu [93]) Let $X$ and $Y$ be two concepts. Assume a rule “if $X$ is an uncertain set $\xi$ then $Y$ is an uncertain set $\eta$”. From $X$ is a constant $a$ we infer that $Y$ is an uncertain set $\eta^\ast = \eta|_{a \in \xi}$ \hspace{1cm} (10.2)

which is the conditional uncertain set $\eta$ given $a \in \xi$. The inference rule is represented by

$$\text{Rule: If } X \text{ is } \xi \text{ then } Y \text{ is } \eta\"$$

$$\text{From: } X \text{ is a constant } a$$

$$\text{Infer: } Y \text{ is } \eta^\ast = \eta|_{a \in \xi} \hspace{1cm} (10.3)$$

**Theorem 10.1** (Liu [93]) In Inference Rule 10.1, if $\xi$ and $\eta$ are independent uncertain sets with membership functions $\mu$ and $\nu$, respectively, then $\eta^\ast$ has
a membership function

\[
\nu^*(y) = \begin{cases} 
    \frac{\nu(y)}{\mu(a)}, & \text{if } \nu(y) < \mu(a)/2 \\
    \frac{\nu(y) + \mu(a) - 1}{\mu(a)}, & \text{if } \nu(y) > 1 - \mu(a)/2 \\
    0.5, & \text{otherwise.}
\end{cases} \quad (10.4)
\]

**Proof:** It follows from Inference Rule 10.1 that \( \eta^* \) is the conditional uncertain set \( \eta \) given \( a \in \xi \). By applying Theorem 8.36, the membership function of \( \eta^* \) is just \( \nu^* \).

**Inference Rule 10.2** (Gao-Gao-Ralescu [47]) Let \( X, Y \) and \( Z \) be three concepts. Assume a rule “if \( X \) is an uncertain set \( \xi \) and \( Y \) is an uncertain set \( \eta \) then \( Z \) is an uncertain set \( \tau \).” From \( X \) is a constant \( a \) and \( Y \) is a constant \( b \) we infer that \( Z \) is an uncertain set \( \tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \) which is the conditional uncertain set \( \tau \) given \( a \in \xi \) and \( b \in \eta \). The inference rule is represented by

**Rule:** If \( X \) is \( \xi \) and \( Y \) is \( \eta \) then \( Z \) is \( \tau \)  
**From:** \( X \) is \( a \) and \( Y \) is \( b \)  
**Infer:** \( Z \) is \( \tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \) \( (10.5) \)

**Theorem 10.2** (Gao-Gao-Ralescu [47]) In Inference Rule 10.2, if \( \xi, \eta, \tau \) are independent uncertain sets with membership functions \( \mu, \nu, \lambda \), respectively, then \( \tau^* \) has a membership function

\[
\lambda^*(z) = \begin{cases} 
    \frac{\lambda(z)}{\mu(a) \vee \nu(b)}, & \text{if } \lambda(z) < \frac{\mu(a) \wedge \nu(b)}{2} \\
    \frac{\lambda(z) + \mu(a) \wedge \nu(b) - 1}{\mu(a) \wedge \nu(b)}, & \text{if } \lambda(z) > 1 - \frac{\mu(a) \wedge \nu(b)}{2} \\
    0.5, & \text{otherwise.}
\end{cases} \quad (10.7)
\]

**Proof:** It follows from Inference Rule 10.2 that \( \tau^* \) is the conditional uncertain set \( \tau \) given \( a \in \xi \) and \( b \in \eta \). By applying Theorem 8.36, the membership function of \( \tau^* \) is just \( \lambda^* \).

**Inference Rule 10.3** (Gao-Gao-Ralescu [47]) Let \( X \) and \( Y \) be two concepts. Assume two rules “if \( X \) is an uncertain set \( \xi_1 \) then \( Y \) is an uncertain set \( \eta_1 \)” and “if \( X \) is an uncertain set \( \xi_2 \) then \( Y \) is an uncertain set \( \eta_2 \).” From \( X \) is a constant \( a \) we infer that \( Y \) is an uncertain set

\[
\eta^* = \frac{\mathcal{M}\{a \in \xi_1\} \cdot \eta_1|_{a \in \xi_1}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}} + \frac{\mathcal{M}\{a \in \xi_2\} \cdot \eta_2|_{a \in \xi_2}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}}. \quad (10.8)
\]
The inference rule is represented by

Rule 1: If $X$ is $\xi_1$ then $Y$ is $\eta_1$
Rule 2: If $X$ is $\xi_2$ then $Y$ is $\eta_2$

From: $X$ is a constant $a$
Infer: $Y$ is $\eta^*$ determined by (10.8)

Theorem 10.3 (Gao-Gao-Ralescu [47]) In Inference Rule 10.3, if $\xi_1, \xi_2, \eta_1, \eta_2$ are independent uncertain sets with membership functions $\mu_1, \mu_2, \nu_1, \nu_2$, respectively, then

$$\eta^* = \frac{\mu_1(a)}{\mu_1(a) + \mu_2(a)} \eta_1^* + \frac{\mu_2(a)}{\mu_1(a) + \mu_2(a)} \eta_2^*$$

(10.10)

where $\eta_1^*$ and $\eta_2^*$ are uncertain sets whose membership functions are respectively given by

$$\nu_1^*(y) = \begin{cases} \frac{\nu_1(y)}{\mu_1(a)}, & \text{if } \nu_1(y) < \frac{\mu_1(a)}{2} \\ \frac{\nu_1(y) + \mu_1(a)}{\mu_1(a)} - 1, & \text{if } \nu_1(y) > 1 - \frac{\mu_1(a)}{2} \\ 0.5, & \text{otherwise} \end{cases}$$

(10.11)

$$\nu_2^*(y) = \begin{cases} \frac{\nu_2(y)}{\mu_2(a)}, & \text{if } \nu_2(y) < \frac{\mu_2(a)}{2} \\ \frac{\nu_2(y) + \mu_2(a)}{\mu_2(a)} - 1, & \text{if } \nu_2(y) > 1 - \frac{\mu_2(a)}{2} \\ 0.5, & \text{otherwise} \end{cases}$$

(10.12)

Proof: It follows from Inference Rule 10.3 that the uncertain set $\eta^*$ is just

$$\eta^* = \frac{\mathcal{M}\{a \in \xi_1\} \cdot \eta_1|_{a \in \xi_1}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}} + \frac{\mathcal{M}\{a \in \xi_2\} \cdot \eta_2|_{a \in \xi_2}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}}.$$

The theorem follows from $\mathcal{M}\{a \in \xi_1\} = \mu_1(a)$ and $\mathcal{M}\{a \in \xi_2\} = \mu_2(a)$ immediately.

Inference Rule 10.4 (Gao-Gao-Ralescu [47]) Let $X_1, X_2, \ldots, X_m$ be concepts. Assume rules “if $X_i$ is $\xi_{i1}$ and $\cdots$ and $X_m$ is $\xi_{im}$ then $Y$ is $\eta_i$” for $i = 1, 2, \ldots, k$. From $X_1$ is $a_1$ and $\cdots$ and $X_m$ is $a_m$ we infer that $Y$ is an uncertain set

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i|_{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})}}{c_1 + c_2 + \cdots + c_k}$$

(10.13)
where the coefficients are determined by
\[ c_i = \mathcal{M}\{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})\} \quad (10.14) \]
for \( i = 1, 2, \cdots, k \). The inference rule is represented by

- Rule 1: If \( X_1 \) is \( \xi_{11} \) and \( \cdots \) and \( X_m \) is \( \xi_{1m} \) then \( Y \) is \( \eta_1 \)
- Rule 2: If \( X_1 \) is \( \xi_{21} \) and \( \cdots \) and \( X_m \) is \( \xi_{2m} \) then \( Y \) is \( \eta_2 \)
- \( \cdots \)
- Rule \( k \): If \( X_1 \) is \( \xi_{k1} \) and \( \cdots \) and \( X_m \) is \( \xi_{km} \) then \( Y \) is \( \eta_k \)

From: \( X_1 \) is \( a_1 \) and \( \cdots \) and \( X_m \) is \( a_m \)

Infer: \( Y \) is \( \eta^* \) determined by (10.13)

**Theorem 10.4** (Gao-Gao-Ralescu [47]) In Inference Rule 10.4, if \( \xi_{i1}, \xi_{i2}, \cdots, \xi_{im}, \eta_i \) are independent uncertain sets with membership functions \( \mu_{i1}, \mu_{i2}, \cdots, \mu_{im}, \nu_i, \ i = 1, 2, \cdots, k \), respectively, then
\[ \eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta^*_i}{c_1 + c_2 + \cdots + c_k} \quad (10.16) \]
where \( \eta^*_i \) are uncertain sets whose membership functions are given by
\[ \nu^*_i(y) = \begin{cases} \frac{\nu_i(y)}{c_i}, & \text{if } \nu_i(y) < c_i/2 \\ \frac{\nu_i(y) + c_i - 1}{c_i}, & \text{if } \nu_i(y) > 1 - c_i/2 \\ 0.5, & \text{otherwise} \end{cases} \quad (10.17) \]
and \( c_i \) are constants determined by
\[ c_i = \min_{1 \leq i \leq m} \mu_{ii}(a_l) \quad (10.18) \]
for \( i = 1, 2, \cdots, k \), respectively.

**Proof:** For each \( i \), since \( \{a_1 \in \xi_{i1}\}, \{a_2 \in \xi_{i2}\}, \cdots, \{a_m \in \xi_{im}\} \) are independent events, we immediately have
\[ \mathcal{M}\left\{ \bigcap_{j=1}^{m} (a_j \in \xi_{ij}) \right\} = \min_{1 \leq j \leq m} \mathcal{M}\{a_j \in \xi_{ij}\} = \min_{1 \leq l \leq m} \mu_{ii}(a_l) \]
for \( i = 1, 2, \cdots, k \). From those equations, we may prove the theorem by Inference Rule 10.4 immediately.
10.2 Uncertain System

Uncertain system, proposed by Liu [93], is a function from its inputs to outputs based on the uncertain inference rule. Usually, an uncertain system consists of 5 parts:

1. inputs that are crisp data to be fed into the uncertain system;
2. a rule-base that contains a set of if-then rules provided by the experts;
3. an uncertain inference rule that infers uncertain consequents from the uncertain antecedents;
4. an expected value operator that converts the uncertain consequents to crisp values;
5. outputs that are crisp data yielded from the expected value operator.

Now let us consider an uncertain system in which there are $m$ crisp inputs $\alpha_1, \alpha_2, \ldots, \alpha_m$, and $n$ crisp outputs $\beta_1, \beta_2, \ldots, \beta_n$. At first, we infer $n$ uncertain sets $\eta^*_1, \eta^*_2, \ldots, \eta^*_n$ from the $m$ crisp inputs by the rule-base (i.e., a set of if-then rules),

$$
\text{If } \xi_{i1} \text{ and } \xi_{i2} \text{ and } \ldots \text{ and } \xi_{im} \text{ then } \eta_{i1} \text{ and } \eta_{i2} \text{ and } \ldots \text{ and } \eta_{in}
$$

$$
\text{If } \xi_{k1} \text{ and } \xi_{k2} \text{ and } \ldots \text{ and } \xi_{km} \text{ then } \eta_{k1} \text{ and } \eta_{k2} \text{ and } \ldots \text{ and } \eta_{kn}
$$

and the uncertain inference rule

$$
\eta^*_j = \sum_{i=1}^{k} \frac{c_i \cdot \eta_{ij} | (\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \ldots \cap (\alpha_m \in \xi_{im})}{c_1 + c_2 + \ldots + c_k}
$$

for $j = 1, 2, \ldots, n$, where the coefficients are determined by

$$
c_i = M \{ (\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \ldots \cap (\alpha_m \in \xi_{im}) \}
$$

for $i = 1, 2, \ldots, k$. Thus by using the expected value operator, we obtain

$$
\beta_j = E[\eta^*_j]
$$

for $j = 1, 2, \ldots, n$. Until now we have constructed a function from inputs $\alpha_1, \alpha_2, \ldots, \alpha_m$ to outputs $\beta_1, \beta_2, \ldots, \beta_n$. Write this function by $f$, i.e.,

$$
(\beta_1, \beta_2, \ldots, \beta_n) = f(\alpha_1, \alpha_2, \ldots, \alpha_m).
$$

Then we get an uncertain system $f$. 
Figure 10.1: An Uncertain System

**Theorem 10.5** Assume $\xi_1, \xi_2, \cdots, \xi_{im}, \eta_1, \eta_2, \cdots, \eta_{in}$ are independent uncertain sets with membership functions $\mu_{i1}, \mu_{i2}, \cdots, \mu_{im}, \nu_{i1}, \nu_{i2}, \cdots, \nu_{im}, i = 1, 2, \cdots, k$, respectively. Then the uncertain system from $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ to $(\beta_1, \beta_2, \cdots, \beta_n)$ is

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot E[\eta^*_ij]}{c_1 + c_2 + \cdots + c_k} \quad (10.24)$$

for $j = 1, 2, \cdots, n$, where $\eta^*_ij$ are uncertain sets whose membership functions are given by

$$\nu^*_ij(y) = \begin{cases} \frac{\nu_{ij}(y)}{c_i}, & \text{if } \nu_{ij}(y) < c_i/2 \\ \frac{\nu_{ij}(y) + c_i - 1}{c_i}, & \text{if } \nu_{ij}(y) > 1 - c_i/2 \\ 0.5, & \text{otherwise} \end{cases} \quad (10.25)$$

and $c_i$ are constants determined by

$$c_i = \min_{1 \leq l \leq m} \mu_{il}(\alpha_l) \quad (10.26)$$

for $i = 1, 2, \cdots, k, j = 1, 2, \cdots, n$, respectively.

**Proof:** It follows from Inference Rule 10.4 that the uncertain sets $\eta^*_ij$ are

$$\eta^*_ij = \sum_{i=1}^{k} \frac{c_i \cdot \eta^*_ij}{c_1 + c_2 + \cdots + c_k}$$

for $j = 1, 2, \cdots, n$. Since $\eta^*_ij, i = 1, 2, \cdots, k, j = 1, 2, \cdots, n$ are independent uncertain sets, we get the theorem immediately by the linearity of expected value operator.

**Remark 10.1:** The uncertain system allows the uncertain sets $\eta_{ij}$ in the rule-base (10.19) become constants $b_{ij}$, i.e.,

$$\eta_{ij} = b_{ij} \quad (10.27)$$
for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. In this case, the uncertain system (10.24) becomes
\[
\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot b_{ij}}{c_1 + c_2 + \cdots + c_k}
\]
for $j = 1, 2, \cdots, n$.

**Remark 10.2:** The uncertain system allows the uncertain sets $\eta_{ij}$ in the rule-base (10.19) become functions $h_{ij}$ of inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$, i.e.,
\[
\eta_{ij} = h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m)
\]
for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. In this case, the uncertain system (10.24) becomes
\[
\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m)}{c_1 + c_2 + \cdots + c_k}
\]
for $j = 1, 2, \cdots, n$.

**Uncertain Systems are Universal Approximator**

Uncertain systems are capable of approximating any continuous function on a compact set (i.e., bounded and closed set) to arbitrary accuracy. This is the reason why uncertain systems may play a controller. The following theorem shows this fact.

**Theorem 10.6** (Peng-Chen [139]) For any given continuous function $g$ on a compact set $D \subset \mathbb{R}^m$ and any given $\varepsilon > 0$, there exists an uncertain system $f$ such that
\[
\|f(\alpha_1, \alpha_2, \cdots, \alpha_m) - g(\alpha_1, \alpha_2, \cdots, \alpha_m)\| < \varepsilon
\]
for any $(\alpha_1, \alpha_2, \cdots, \alpha_m) \in D$.

**Proof:** Without loss of generality, we assume that the function $g$ is a real-valued function with only two variables $\alpha_1$ and $\alpha_2$, and the compact set is a unit rectangle $D = [0, 1] \times [0, 1]$. Since $g$ is continuous on $D$ and then is uniformly continuous, for any given number $\varepsilon > 0$, there is a number $\delta > 0$ such that
\[
|g(\alpha_1, \alpha_2) - g(\alpha'_1, \alpha'_2)| < \varepsilon
\]
whenever $\| (\alpha_1, \alpha_2) - (\alpha'_1, \alpha'_2) \| < \delta$. Let $k$ be an integer larger than $\sqrt{2}/\delta$, and write
\[
D_{ij} = \left\{ (\alpha_1, \alpha_2) \mid \frac{i-1}{k} < \alpha_1 \leq \frac{i}{k}, \frac{j-1}{k} < \alpha_2 \leq \frac{j}{k} \right\}
\]
(10.33)
for \( i, j = 1, 2, \cdots, k \). Note that \( \{D_{ij}\} \) is a sequence of disjoint rectangles whose “diameter” is less than \( \delta \). Define uncertain sets
\[
\xi_i = \left( \frac{i - 1}{k}, \frac{i}{k} \right), \quad i = 1, 2, \cdots, k, \quad (10.34)
\]
\[
\eta_j = \left( \frac{j - 1}{k}, \frac{j}{k} \right), \quad j = 1, 2, \cdots, k. \quad (10.35)
\]
Then we assume a rule-base with \( k \times k \) if-then rules,
\[
\text{Rule } ij: \text{ If } \xi_i \text{ and } \eta_j \text{ then } g(i/k, j/k), \quad i, j = 1, 2, \cdots, k. \quad (10.36)
\]
According to the uncertain inference rule, the corresponding uncertain system from \( D \) to \( \mathbb{R} \) is
\[
f(\alpha_1, \alpha_2) = g(i/k, j/k), \quad \text{if } (\alpha_1, \alpha_2) \in D_{ij}, \quad i, j = 1, 2, \cdots, k. \quad (10.37)
\]
It follows from (10.32) that for any \( (\alpha_1, \alpha_2) \in D_{ij} \subset D \), we have
\[
|f(\alpha_1, \alpha_2) - g(\alpha_1, \alpha_2)| = |g(i/k, j/k) - g(\alpha_1, \alpha_2)| < \varepsilon. \quad (10.38)
\]
The theorem is thus verified. Hence uncertain systems are universal approximators.

### 10.3 Uncertain Control

Uncertain controller, designed by Liu [93], is a special uncertain system that maps the state variables of a process under control to the action variables. Thus an uncertain controller consists of the same 5 parts of uncertain system: inputs, a rule-base, an uncertain inference rule, an expected value operator, and outputs. The distinguished point is that the inputs of uncertain controller are the state variables of the process under control, and the outputs are the action variables.

Figure 10.2 shows an uncertain control system consisting of an uncertain controller and a process. Note that \( t \) represents time, \( \alpha_1(t), \alpha_2(t), \cdots, \alpha_m(t) \) are not only the inputs of uncertain controller but also the outputs of process, and \( \beta_1(t), \beta_2(t), \cdots, \beta_n(t) \) are not only the outputs of uncertain controller but also the inputs of process.

### 10.4 Inverted Pendulum

Inverted pendulum system is a nonlinear unstable system that is widely used as a benchmark for testing control algorithms. Many good techniques already exist for balancing inverted pendulum. Among others, Gao [52] successfully balanced an inverted pendulum by the uncertain controller with \( 5 \times 5 \) if-then rules.
Thus we have an if-then rule, a large clockwise angular velocity, we should give it a large force to the right.

Rule Base

\begin{align*}
\alpha_1(t) & \quad \eta_1^*(t) & \quad \beta_1(t) = E[\eta_1^*(t)] \\
\alpha_2(t) & \quad \eta_2^*(t) & \quad \beta_2(t) = E[\eta_2^*(t)] \\
\vdots & \quad \vdots & \quad \vdots \\
\alpha_m(t) & \quad \eta_m^*(t) & \quad \beta_m(t) = E[\eta_m^*(t)]
\end{align*}

Figure 10.2: An Uncertain Control System

\begin{align*}
\alpha_1(t) & \quad \eta_1^*(t) & \quad \beta_1(t) \\
\alpha_2(t) & \quad \eta_2^*(t) & \quad \beta_2(t) \\
\vdots & \quad \vdots & \quad \vdots \\
\alpha_m(t) & \quad \eta_m^*(t) & \quad \beta_m(t)
\end{align*}

Figure 10.3: An Inverted Pendulum in which \( A(t) \) represents the angular position and \( F(t) \) represents the force that moves the cart at time \( t \).

The uncertain controller has two inputs ("angle" and "angular velocity") and one output ("force"). Three of them will be represented by uncertain sets labeled by

- "negative large" \( \text{NL} \)
- "negative small" \( \text{NS} \)
- "zero" \( \text{Z} \)
- "positive small" \( \text{PS} \)
- "positive large" \( \text{PL} \)

The membership functions of those uncertain sets are shown in Figures 10.4, 10.5 and 10.6.

Intuitively, when the inverted pendulum has a large clockwise angle and a large clockwise angular velocity, we should give it a large force to the right. Thus we have an if-then rule,

**If** the angle is negative large

and the angular velocity is negative large,

**then** the force is positive large.

Similarly, when the inverted pendulum has a large counterclockwise angle...
and a large counterclockwise angular velocity, we should give it a large force to the left. Thus we have an if-then rule,

\begin{align*}
\text{If} & \text{ the angle is positive large} \\
& \text{ and } \text{ the angular velocity is positive large}, \\
& \text{ then } \text{ the force is negative large.}
\end{align*}

Note that each input or output has 5 states and each state is represented by an uncertain set. This implies that the rule-base contains $5 \times 5$ if-then rules. In order to balance the inverted pendulum, the 25 if-then rules in Table 10.1 are accepted.

A lot of simulation results show that the uncertain controller may balance the inverted pendulum successfully.
### 10.5 Bibliographic Notes

The basic uncertain inference rule was initialized by Liu [93] in 2010 by the tool of conditional uncertain set. After that, Gao-Gao-Ralescu [47] extended the uncertain inference rule to the case with multiple antecedents and multiple if-then rules.

Based on the uncertain inference rules, Liu [93] suggested the concept of uncertain system, and then presented the tool of uncertain controller. As an important contribution, Peng-Chen [139] proved that uncertain systems are universal approximator and then demonstrated that the uncertain controller is a reasonable tool. As a successful application, Gao [52] balanced an inverted pendulum by using the uncertain controller.

<table>
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<th>Z</th>
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Chapter 11

Uncertain Process

The study of uncertain process was started by Liu [89] in 2008 for modelling the evolution of uncertain phenomena. This chapter will give the concept of uncertain process, and introduce sample path, uncertainty distribution, independent increment process, extreme value, first hitting time, time integral, and stationary increment process.

11.1 Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time. A formal definition is given below.

**Definition 11.1** (Liu [89]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let $T$ be a totally ordered set (e.g. time). An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set $B$ of real numbers at each time $t$.

**Remark 11.1:** The above definition says $X_t$ is an uncertain process if and only if it is an uncertain variable at each time $t$.

**Example 11.1:** Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.4$. Then

$$X_t(\gamma) = \begin{cases} 
  t, & \text{if } \gamma = \gamma_1 \\
  t + 1, & \text{if } \gamma = \gamma_2 
\end{cases} \quad (11.1)$$

is an uncertain process.

**Example 11.2:** Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$X_t(\gamma) = t - \gamma, \quad \forall \gamma \in \Gamma \quad (11.2)$$
is an uncertain process.

**Example 11.3:** A real-valued function $f(t)$ with respect to time $t$ may be regarded as a special uncertain process on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, i.e.,

$$X_t(\gamma) = f(t), \quad \forall \gamma \in \Gamma.$$ (11.3)

**Sample Path**

**Definition 11.2** (Liu [89]) Let $X_t$ be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a sample path of $X_t$.

Note that each sample path is a real-valued function of time $t$. In addition, an uncertain process may also be regarded as a function from an uncertainty space to a collection of sample paths.

![Figure 11.1: A Sample Path of Uncertain Process](image)

**Definition 11.3** An uncertain process $X_t$ is said to be sample-continuous if almost all sample paths are continuous functions with respect to time $t$.

**11.2 Uncertainty Distribution**

An uncertainty distribution of uncertain process is a sequence of uncertainty distributions of uncertain variables indexed by time. Thus an uncertainty distribution of uncertain process is a surface rather than a curve. A formal definition is given below.

**Definition 11.4** (Liu [105]) The uncertainty distribution $\Phi_t(x)$ of an uncertain process $X_t$ is defined by

$$\Phi_t(x) = \mathcal{M}\{X_t \leq x\}$$ (11.4)

for any time $t$ and any number $x$. 
That is, the uncertain process $X_t$ has an uncertainty distribution $\Phi_t(x)$ if at each time $t$, the uncertain variable $X_t$ has the uncertainty distribution $\Phi_t(x)$.

**Example 11.4:** The linear uncertain process $X_t \sim \mathcal{L}(at, bt)$ has an uncertainty distribution,

$$\Phi_t(x) = \begin{cases} 
0, & \text{if } x \leq at \\
\frac{x - at}{(b - a)t}, & \text{if } at < x \leq bt \\
1, & \text{if } x > bt.
\end{cases} \quad (11.5)$$

**Example 11.5:** The zigzag uncertain process $X_t \sim \mathcal{Z}(at, bt, ct)$ has an uncertainty distribution,

$$\Phi_t(x) = \begin{cases} 
0, & \text{if } x \leq at \\
\frac{x - at}{2(b - a)t}, & \text{if } at < x \leq bt \\
x + ct - 2bt \quad & \text{if } bt < x \leq ct \\
\frac{2(c - b)t}{2(c - b)t}, & \text{if } x > ct.
\end{cases} \quad (11.6)$$

**Example 11.6:** The normal uncertain process $X_t \sim \mathcal{N}(et, \sigma t)$ has an uncertainty distribution,

$$\Phi_t(x) = \left(1 + \exp \left(\frac{\pi(et - x)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \quad (11.7)$$

**Example 11.7:** The lognormal uncertain process $X_t \sim \mathcal{LOGN}(et, \sigma t)$ has an uncertainty distribution,

$$\Phi_t(x) = \left(1 + \exp \left(\frac{\pi(et - \ln x)}{\sqrt{3}t}\right)\right)^{-1}. \quad (11.8)$$

**Exercise 11.1:** Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.4$. Derive the uncertainty distribution of the uncertain process

$$X_t(\gamma) = \begin{cases} 
t, & \text{if } \gamma = \gamma_1 \\
t + 1, & \text{if } \gamma = \gamma_2.
\end{cases} \quad (11.9)$$
Exercise 11.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Derive the uncertainty distribution of the uncertain process

$$X_t(\gamma) = t - \gamma, \quad \forall \gamma \in \Gamma.$$  

(11.10)

Exercise 11.3: A real-valued function $f(t)$ with respect to time $t$ is a special uncertain process. What is the uncertainty distribution of $f(t)$?

Exercise 11.4: Let $X_t$ be an uncertain process with uncertainty distribution $\Phi_t(x)$, and let $a$ and $b$ be real numbers with $a > 0$. Show that $aX_t + b$ has an uncertainty distribution,

$$\Psi_t(x) = \Phi_t((x - b)/a).$$  

(11.11)

Exercise 11.5: Let $X_t$ be an uncertain process with uncertainty distribution $\Phi_t(x)$, and let $a$ and $b$ be real numbers with $a < 0$. Show that $aX_t + b$ has an uncertainty distribution,

$$\Psi_t(x) = 1 - \Phi_t((x - b)/a).$$  

(11.12)

Exercise 11.6: Let $X_t$ be an uncertain process with uncertainty distribution $\Phi_t(x)$, and let $f(x)$ be a continuous and strictly increasing function. Show that $f(X_t)$ has an uncertainty distribution

$$\Psi_t(x) = \Phi_t(f^{-1}(x)).$$  

(11.13)

Exercise 11.7: Let $X_t$ be an uncertain process with continuous uncertainty distribution $\Phi_t(x)$, and let $f(x)$ be a continuous and strictly decreasing function. Show that $f(X_t)$ has an uncertainty distribution

$$\Psi_t(x) = 1 - \Phi_t(f^{-1}(x)).$$  

(11.14)

Theorem 11.1 (Liu [105], Sufficient and Necessary Condition) A function $\Phi_t(x) : T \times \mathbb{R} \to [0, 1]$ is an uncertainty distribution of uncertain process if and only if at each time $t$, it is a monotone increasing function with respect to $x$ except $\Phi_t(x) \equiv 0$ and $\Phi_t(x) \equiv 1$.

Proof: If $\Phi_t(x)$ is an uncertainty distribution of some uncertain process $X_t$, then at each time $t$, $\Phi_t(x)$ is the uncertainty distribution of uncertain variable $X_t$. It follows from Peng-Iwamura theorem that $\Phi_t(x)$ is a monotone increasing function with respect to $x$ and $\Phi_t(x) \neq 0$, $\Phi_t(x) \neq 1$. Conversely, if at each time $t$, $\Phi_t(x)$ is a monotone increasing function except $\Phi_t(x) \equiv 0$ and $\Phi_t(x) \equiv 1$, it follows from Peng-Iwamura theorem that there exists an uncertain variable $\xi_t$ whose uncertainty distribution is just $\Phi_t(x)$. Define

$$X_t = \xi_t, \quad \forall t \in T.$$  

Then $X_t$ is an uncertain process and has the uncertainty distribution $\Phi_t(x)$. The theorem is verified.
Regular Uncertainty Distribution

Definition 11.5 (Liu [105]) An uncertainty distribution $\Phi_t(x)$ is said to be regular if at each time $t$, it is a continuous and strictly increasing function with respect to $x$ at which $0 < \Phi_t(x) < 1$, and

$$\lim_{x \to -\infty} \Phi_t(x) = 0, \quad \lim_{x \to +\infty} \Phi_t(x) = 1. \quad (11.15)$$

It is clear that linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution of uncertain process are all regular.

Inverse Uncertainty Distribution

Definition 11.6 (Liu [105]) Let $X_t$ be an uncertain process with regular uncertainty distribution $\Phi_t(x)$. Then the inverse function $\Phi_t^{-1}(\alpha)$ is called the inverse uncertainty distribution of $X_t$.

Note that at each time $t$, the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$. If needed, we may extend the domain to $[0, 1]$ via

$$\Phi_t^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi_t^{-1}(\alpha), \quad \Phi_t^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi_t^{-1}(\alpha). \quad (11.16)$$

Figure 11.2: Inverse Uncertainty Distribution of Uncertain Process

Example 11.8: The linear uncertain process $X_t \sim \mathcal{L}(at, bt)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = (1 - \alpha)at + \alpha bt. \quad (11.17)$$
Example 11.9: The zigzag uncertain process $X_t \sim Z(at, bt, ct)$ has an inverse uncertainty distribution,

$$
\Phi_t^{-1}(\alpha) = \begin{cases} 
(1 - 2\alpha)at + 2\alpha bt, & \text{if } \alpha < 0.5 \\
(2 - 2\alpha)bt + (2\alpha - 1)ct, & \text{if } \alpha \geq 0.5.
\end{cases} 
$$  \hspace{1cm} (11.18)

Example 11.10: The normal uncertain process $X_t \sim N(et, \sigma t)$ has an inverse uncertainty distribution,

$$
\Phi_t^{-1}(\alpha) = et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
$$  \hspace{1cm} (11.19)

Example 11.11: The lognormal uncertain process $X_t \sim LOGN(et, \sigma t)$ has an inverse uncertainty distribution,

$$
\Phi_t^{-1}(\alpha) = \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).
$$  \hspace{1cm} (11.20)

Exercise 11.8: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Derive the inverse uncertainty distribution of the uncertain process

$$
X_t(\gamma) = t - \gamma, \quad \forall \gamma \in \Gamma.
$$  \hspace{1cm} (11.21)

Exercise 11.9: Let $X_t$ be an uncertain process with regular uncertainty distribution $\Phi_t(x)$, and let $a$ and $b$ be real numbers. Show that (i) if $a > 0$, then $aX_t + b$ has an inverse uncertainty distribution,

$$
\Psi_t^{-1}(\alpha) = a\Phi_t^{-1}(\alpha) + b; 
$$  \hspace{1cm} (11.22)

and (ii) if $a < 0$, then $aX_t + b$ has an inverse uncertainty distribution,

$$
\Psi_t^{-1}(\alpha) = a\Phi_t^{-1}(1 - \alpha) + b.
$$  \hspace{1cm} (11.23)

Exercise 11.10: Let $X_t$ be an uncertain process with regular uncertainty distribution $\Phi_t(x)$, and let $f(x)$ be a continuous and strictly increasing function. Show that $f(X_t)$ has an inverse uncertainty distribution

$$
\Psi_t^{-1}(\alpha) = f(\Phi_t^{-1}(\alpha)).
$$  \hspace{1cm} (11.24)

Exercise 11.11: Let $X_t$ be an uncertain process with regular uncertainty distribution $\Phi_t(x)$, and let $f(x)$ be a continuous and strictly decreasing function. Show that $f(X_t)$ has an inverse uncertainty distribution

$$
\Psi_t^{-1}(\alpha) = f(\Phi_t^{-1}(1 - \alpha)).
$$  \hspace{1cm} (11.25)
Theorem 11.2 (Liu [105]) A function $\Phi_t^{-1}(\alpha): T \times (0, 1) \to \mathbb{R}$ is an inverse uncertainty distribution of uncertain process if at each time $t$, it is a continuous and strictly increasing function with respect to $\alpha$.

Proof: At each time $t$, since $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha$, it follows from Theorem 2.7 that there exists an uncertain variable $\xi_t$ whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$. Define

$$X_t = \xi_t, \quad \forall t \in T.$$ 

Then $X_t$ is an uncertain process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is proved.

11.3 Independence and Operational Law

Definition 11.7 (Liu [105]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are said to be independent if for any positive integer $k$ and any times $t_1, t_2, \cdots, t_k$, the uncertain vectors

$$\xi_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k}), \quad i = 1, 2, \cdots, n \quad (11.26)$$

are independent, i.e., for any Borel sets $B_1, B_2, \cdots, B_n$ of $k$-dimensional real vectors, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}. \quad (11.27)$$

Exercise 11.12: Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes, and let $t_1, t_2, \cdots, t_n$ be any times. Show that

$$X_{1t_1}, X_{2t_2}, \cdots, X_{nt_n} \quad (11.28)$$

are independent uncertain variables.

Exercise 11.13: Let $X_t$ and $Y_t$ be independent uncertain processes. For any times $t_1, t_2, \cdots, t_k$ and $s_1, s_2, \cdots, s_m$, show that

$$(X_{t_1}, X_{t_2}, \cdots, X_{t_k}) \text{ and } (Y_{s_1}, Y_{s_2}, \cdots, Y_{s_m}) \quad (11.29)$$

are independent uncertain vectors.

Theorem 11.3 (Liu [105]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent if and only if for any positive integer $k$, any times $t_1, t_2, \cdots, t_k$, and any Borel sets $B_1, B_2, \cdots, B_n$ of $k$-dimensional real vectors, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \quad (11.30)$$

where $\xi_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k})$ for $i = 1, 2, \cdots, n$. 

Proof: It follows from Theorem 2.62 that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain vectors if and only if (11.30) holds. The theorem is thus verified.

**Theorem 11.4 (Liu [105], Operational Law)** Let $X_{1t}, X_{2t}, \ldots, X_{nt}$ be independent uncertain processes with regular uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \ldots, \Phi_{nt}$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then

$$X_t = f(X_{1t}, X_{2t}, \ldots, X_{nt})$$

(11.31)

has an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = f(\Phi_{1t}^{-1}(\alpha), \ldots, \Phi_{nt}^{-1}(1-\alpha)).$$

(11.32)

Proof: At any time $t$, it is clear that $X_{1t}, X_{2t}, \ldots, X_{nt}$ are independent uncertain variables with inverse uncertainty distributions $\Phi_{1t}^{-1}(\alpha), \Phi_{2t}^{-1}(\alpha), \ldots, \Phi_{nt}^{-1}(\alpha)$, respectively. The theorem follows from the operational law of uncertain variables immediately.

**Theorem 11.5 (Operational Law)** Let $X_{1t}, X_{2t}, \ldots, X_{nt}$ be independent uncertain processes with continuous uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \ldots, \Phi_{nt}$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then

$$X_t = f(X_{1t}, X_{2t}, \ldots, X_{nt})$$

(11.33)

has an uncertainty distribution

$$\Phi_t(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \left( \min_{1 \leq i \leq m} \Phi_{it}(x_i) \wedge \min_{m+1 \leq i \leq n} (1 - \Phi_{it}(x_i)) \right).$$

(11.34)

Proof: At any time $t$, it is clear that $X_{1t}, X_{2t}, \ldots, X_{nt}$ are independent uncertain variables. The theorem follows from the operational law of uncertain variables immediately.

### 11.4 Independent Increment Process

An independent increment process is an uncertain process that has independent increments. A formal definition is given below.

**Definition 11.8 (Liu [89])** An uncertain process $X_t$ is said to have independent increments if

$$X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}}$$

(11.35)

are independent uncertain variables where $t_1, t_2, \ldots, t_k$ are any times with $t_1 < t_2 < \cdots < t_k$. 
That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. Please note that the increments are also independent of the initial state.

**Theorem 11.6** (Liu [105]) Let \( X_t \) be an independent increment process with regular uncertainty distribution \( \Phi_t(x) \). Then for any times \( s \) and \( t \) with \( s < t \), the increment \( X_t - X_s \) has an inverse uncertainty distribution

\[
\Phi^{-1}_{s,t}(\alpha) = \Phi^{-1}_t(\alpha) - \Phi^{-1}_s(\alpha).
\]

**(11.36)**

**Proof:** Since \( X_t \) is an independent increment process, \( X_s \) and \( X_t - X_s \) are independent uncertain variables. It follows from

\[
X_t = X_s + (X_t - X_s)
\]

that

\[
\Phi^{-1}_t(\alpha) = \Phi^{-1}_s(\alpha) + \Psi^{-1}_{s,t}(\alpha).
\]

The theorem is thus proved.

**Remark 11.2:** It follows from (11.36) that \( \Phi^{-1}_t(\alpha) - \Phi^{-1}_s(\alpha) \) is a monotone increasing function with respect to \( \alpha \) for any times \( s \) and \( t \) with \( s < t \). Thus for any \( \alpha < \beta \), we immediately have

\[
\Phi^{-1}_t(\beta) - \Phi^{-1}_s(\beta) \geq \Phi^{-1}_t(\alpha) - \Phi^{-1}_s(\alpha).
\]

That is,

\[
\Phi^{-1}_t(\beta) - \Phi^{-1}_t(\alpha) \geq \Phi^{-1}_s(\beta) - \Phi^{-1}_s(\alpha).
\]

Therefore, the uncertainty distribution of independent increment process has a horn-like shape. See Figure 11.3.

**Theorem 11.7** (Liu [105]) Let \( \Phi^{-1}_t(\alpha) : T \times (0, 1) \to \mathbb{R} \) be a function. If (i) \( \Phi^{-1}_t(\alpha) \) is a continuous and strictly increasing function with respect to \( \alpha \) at each time \( t \), and (ii) \( \Phi^{-1}_t(\alpha) - \Phi^{-1}_s(\alpha) \) is a monotone increasing function with respect to \( \alpha \) for any times \( s \) and \( t \) with \( s < t \), then there exists an independent increment process whose inverse uncertainty distribution is just \( \Phi^{-1}_t(\alpha) \).

**Proof:** Without loss of generality, we only consider the range of \( t \in [0, 1] \). Let \( n \) be a positive integer. Since \( \Phi^{-1}_t(\alpha) \) is a continuous and strictly increasing function and \( \Phi^{-1}_t(\alpha) - \Phi^{-1}_s(\alpha) \) is a monotone increasing function with respect to \( \alpha \), there exist independent uncertain variables \( \xi_0, \xi_1, \ldots, \xi_n \) such that \( \xi_0 \) has an inverse uncertainty distribution

\[
\Phi_0^{-1}(\alpha) = \Phi^{-1}_0(\alpha)
\]

and \( \xi_i \) have inverse uncertainty distributions

\[
\Phi_i^{-1}(\alpha) = \Phi^{-1}_i(\alpha) - \Phi^{-1}_{(i-1)/n}(\alpha),
\]
\( i = 1, 2, \ldots, n, \) respectively. Define an uncertain process

\[
X^n_t = \begin{cases} 
\sum_{i=0}^{k} \xi_{in}, & \text{if } t = \frac{k}{n} \quad (k = 0, 1, \ldots, n) \\
\text{linear}, & \text{otherwise}.
\end{cases}
\]

It may prove that \( X^n_t \) converges in distribution as \( n \to \infty \). Furthermore, we may verify that the limit is indeed an independent increment process and has the inverse uncertainty distribution \( \Phi_t^{-1}(\alpha) \). The theorem is verified.

**Theorem 11.8** Let \( X_t \) be a sample-continuous independent increment process with regular uncertainty distribution \( \Phi_t(x) \). Then for any \( \alpha \in (0, 1) \), we have

\[
\mathcal{M}\{X_t \leq \Phi_t^{-1}(\alpha), \forall t\} = \alpha, \quad (11.37) \\
\mathcal{M}\{X_t > \Phi_t^{-1}(\alpha), \forall t\} = 1 - \alpha. \quad (11.38)
\]

**Proof:** It is still a conjecture.

**Remark 11.3:** It is also showed that for any \( \alpha \in (0, 1) \), the following two equations are true,

\[
\mathcal{M}\{X_t < \Phi_t^{-1}(\alpha), \forall t\} = \alpha, \quad (11.39) \\
\mathcal{M}\{X_t \geq \Phi_t^{-1}(\alpha), \forall t\} = 1 - \alpha. \quad (11.40)
\]

Please mention that \( \{X_t < \Phi_t^{-1}(\alpha), \forall t\} \) and \( \{X_t \geq \Phi_t^{-1}(\alpha), \forall t\} \) are disjoint events but not opposite. Although it is always true that

\[
\mathcal{M}\{X_t < \Phi_t^{-1}(\alpha), \forall t\} + \mathcal{M}\{X_t \geq \Phi_t^{-1}(\alpha), \forall t\} = 1, \quad (11.41)
\]
the union of \( \{ X_t < \Phi_t^{-1}(\alpha), \forall t \} \) and \( \{ X_t \geq \Phi_t^{-1}(\alpha), \forall t \} \) does not make the universal set, and it is possible that
\[
M\{(X_t < \Phi_t^{-1}(\alpha), \forall t) \cup (X_t \geq \Phi_t^{-1}(\alpha), \forall t)\} < 1. \tag{11.42}
\]

### 11.5 Extreme Value Theorem

This section will present a series of extreme value theorems for sample-continuous independent increment processes.

**Theorem 11.9** *(Liu [101], Extreme Value Theorem)* Let \( X_t \) be a sample-continuous independent increment process with uncertainty distribution \( \Phi_t(x) \). Then the supremum
\[
\sup_{0 \leq t \leq s} X_t \tag{11.43}
\]
has an uncertainty distribution
\[
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(x); \tag{11.44}
\]
and the infimum
\[
\inf_{0 \leq t \leq s} X_t \tag{11.45}
\]
has an uncertainty distribution
\[
\Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(x). \tag{11.46}
\]

**Proof:** Let \( 0 = t_1 < t_2 < \cdots < t_n = s \) be a partition of the closed interval \([0, s]\). It is clear that
\[
X_{t_i} = X_{t_1} + (X_{t_2} - X_{t_1}) + \cdots + (X_{t_i} - X_{t_{i-1}})
\]
for \( i = 1, 2, \cdots, n \). Since the increments
\[
X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}
\]
are independent uncertain variables, it follows from Theorem 2.19 that the maximum
\[
\max_{1 \leq i \leq n} X_{t_i}
\]
has an uncertainty distribution
\[
\min_{1 \leq i \leq n} \Phi_{t_i}(x).
\]

Since \( X_t \) is sample-continuous, we have
\[
\max_{1 \leq i \leq n} X_{t_i} \to \sup_{0 \leq t \leq s} X_t
\]
and
\[
\min_{1 \leq i \leq n} \Phi_{t_i}(x) \to \inf_{0 \leq t \leq s} \Phi_t(x)
\]
as \(n \to \infty\). Thus (11.44) is proved. Similarly, it follows from Theorem 2.19 that the minimum
\[
\min_{1 \leq i \leq n} X_{t_i}
\]
has an uncertainty distribution
\[
\max_{1 \leq i \leq n} \Phi_{t_i}(x).
\]
Since \(X_t\) is sample-continuous, we have
\[
\min_{1 \leq i \leq n} X_{t_i} \to \inf_{0 \leq t \leq s} X_t
\]
and
\[
\max_{1 \leq i \leq n} \Phi_{t_i}(x) \to \sup_{0 \leq t \leq s} \Phi_t(x)
\]
as \(n \to \infty\). Thus (11.46) is verified.

**Example 11.12:** The sample-continuity condition in Theorem 11.9 cannot be removed. For example, take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([0, 1]\) with Borel algebra and Lebesgue measure. Define a sample-discontinuous uncertain process
\[
X_t(\gamma) = \begin{cases} 
0, & \text{if } \gamma \neq t \\
1, & \text{if } \gamma = t.
\end{cases}
\]
(11.47)

Since all increments are 0 almost surely, \(X_t\) is an independent increment process. On the one hand, \(X_t\) has an uncertainty distribution
\[
\Phi_t(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1, & \text{if } x \geq 0.
\end{cases}
\]
(11.48)

On the other hand, the supremum
\[
\sup_{0 \leq t \leq 1} X_t(\gamma) \equiv 1
\]
(11.49)
has an uncertainty distribution
\[
\Psi(x) = \begin{cases} 
0, & \text{if } x < 1 \\
1, & \text{if } x \geq 1.
\end{cases}
\]
(11.50)
Thus
\[
\Psi(x) \neq \inf_{0 \leq t \leq 1} \Phi_t(x).
\]
(11.51)
Therefore, the sample-continuity condition cannot be removed.

**Exercise 11.14:** Let $X_t$ be a sample-continuous independent increment process with uncertainty distribution $\Phi_t(x)$. Assume $f$ is a continuous and strictly increasing function. Show that the supremum

$$
\sup_{0 \leq t \leq s} f(X_t)
$$

has an uncertainty distribution

$$
\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(x));
$$

and the infimum

$$
\inf_{0 \leq t \leq s} f(X_t)
$$

has an uncertainty distribution

$$
\Psi(x) = \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).
$$

**Exercise 11.15:** Let $X_t$ be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Assume $f$ is a continuous and strictly decreasing function. Show that the supremum

$$
\sup_{0 \leq t \leq s} f(X_t)
$$

has an uncertainty distribution

$$
\Psi(x) = 1 - \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x));
$$

and the infimum

$$
\inf_{0 \leq t \leq s} f(X_t)
$$

has an uncertainty distribution

$$
\Psi(x) = 1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).
$$

### 11.6 First Hitting Time

**Definition 11.9** (Liu [101]) Let $X_t$ be an uncertain process and let $z$ be a given level. Then the uncertain variable

$$
\tau_z = \inf \{ t \geq 0 \mid X_t = z \}
$$

is called the first hitting time that $X_t$ reaches the level $z$. 
Theorem 11.10 (Liu [101]) Let $X_t$ be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Then the first hitting time $\tau_z$ that $X_t$ reaches the level $z$ has an uncertainty distribution,

$$
\Upsilon(s) = \begin{cases} 
1 - \inf_{0 \leq t \leq s} \Phi_t(z), & \text{if } z > X_0 \\
\sup_{0 \leq t \leq s} \Phi_t(z), & \text{if } z < X_0.
\end{cases} 
\tag{11.61}
$$

**Proof:** When $X_0 < z$, it follows from the definition of first hitting time that $\tau_z \leq s$ if and only if $\sup_{0 \leq t \leq s} X_t \geq z$.

Thus the uncertainty distribution of $\tau_z$ is

$$
\Upsilon(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{\sup_{0 \leq t \leq s} X_t \geq z\right\}.
$$

By using the extreme value theorem, we obtain

$$
\Upsilon(s) = 1 - \inf_{0 \leq t \leq s} \Phi_t(z).
$$

When $X_0 > z$, it follows from the definition of first hitting time that $\tau_z \leq s$ if and only if $\inf_{0 \leq t \leq s} X_t \leq z$.

Thus the uncertainty distribution of $\tau_z$ is

$$
\Upsilon(s) = \mathcal{M}\{\tau_z \leq s\} = \mathcal{M}\left\{\inf_{0 \leq t \leq s} X_t \leq z\right\} = \sup_{0 \leq t \leq s} \Phi_t(z).
$$

The theorem is verified.
Exercise 11.16: Let $X_t$ be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Assume $f$ is a continuous and strictly increasing function. Show that the first hitting time $\tau_z$ that $f(X_t)$ reaches the level $z$ has an uncertainty distribution,

$$
\Upsilon(s) = \begin{cases} 
1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\
\sup_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0).
\end{cases}
$$

(11.62)

Exercise 11.17: Let $X_t$ be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Assume $f$ is a continuous and strictly decreasing function. Show that the first hitting time $\tau_z$ that $f(X_t)$ reaches the level $z$ has an uncertainty distribution,

$$
\Upsilon(s) = \begin{cases} 
\sup_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\
1 - \inf_{0 \leq t \leq s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0).
\end{cases}
$$

(11.63)

Exercise 11.18: Show that the sample-continuity condition in Theorem 11.10 cannot be removed.

11.7 Time Integral

This section will give a definition of time integral that is an integral of uncertain process with respect to time.

Definition 11.10 (Liu [89]) Let $X_t$ be an uncertain process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
$$

(11.64)

Then the time integral of $X_t$ with respect to $t$ is

$$
\int_a^b X_t \, dt = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (t_{i+1} - t_i)
$$

(11.65)

provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be time integrable.

Since $X_t$ is an uncertain variable at each time $t$, the limit in (11.65) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process $X_t$ is time integrable if and only if the limit in (11.65) is an uncertain variable.
Theorem 11.11 If $X_t$ is a sample-continuous uncertain process on $[a, b]$, then it is time integrable on $[a, b]$.

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a, b]$. Since the uncertain process $X_t$ is sample-continuous, almost all sample paths are continuous functions with respect to $t$. Hence the limit

$$
\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i)
$$

exists almost surely and is finite. On the other hand, since $X_t$ is an uncertain variable at each time $t$, the above limit is also a measurable function. Hence the limit is an uncertain variable and then $X_t$ is time integrable.

Theorem 11.12 If $X_t$ is a time integrable uncertain process on $[a, b]$, then it is time integrable on each subinterval of $[a, b]$. Moreover, if $c \in [a, b]$, then

$$
\int_a^b X_t dt = \int_a^c X_t dt + \int_c^b X_t dt.
$$

(11.66)

Proof: Let $[a', b']$ be a subinterval of $[a, b]$. Since $X_t$ is a time integrable uncertain process on $[a, b]$, for any partition

$$
a = t_1 < \cdots < t_m = a' < t_{m+1} < \cdots < t_n = b' < t_{n+1} < \cdots < t_{k+1} = b,
$$

the limit

$$
\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i)
$$

exists almost surely and is finite. Thus the limit

$$
\lim_{\Delta \to 0} \sum_{i=m}^{n-1} X_{t_i}(t_{i+1} - t_i)
$$

exists almost surely and is finite. Hence $X_t$ is time integrable on the subinterval $[a', b']$. Next, for the partition

$$
a = t_1 < \cdots < t_m = c < t_{m+1} < \cdots < t_{k+1} = b,
$$

we have

$$
\sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i) = \sum_{i=1}^{m-1} X_{t_i}(t_{i+1} - t_i) + \sum_{i=m}^{k} X_{t_i}(t_{i+1} - t_i).
$$

Note that

$$
\int_a^b X_t dt = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i),
$$
\[
\int_a^c X_t dt = \lim_{\Delta \to 0} \sum_{i=1}^{m-1} X_{t_i}(t_{i+1} - t_i),
\]
\[
\int_c^b X_t dt = \lim_{\Delta \to 0} \sum_{i=m}^{k} X_{t_i}(t_{i+1} - t_i).
\]
Hence the equation (11.66) is proved.

**Theorem 11.13** (Linearity of Time Integral) Let \(X_t\) and \(Y_t\) be time integrable uncertain processes on \([a, b]\), and let \(\alpha\) and \(\beta\) be real numbers. Then
\[
\int_a^b (\alpha X_t + \beta Y_t) dt = \alpha \int_a^b X_t dt + \beta \int_a^b Y_t dt. \tag{11.67}
\]

**Proof:** Let \(a = t_1 < t_2 < \cdots < t_{k+1} = b\) be a partition of the closed interval \([a, b]\). It follows from the definition of time integral that
\[
\int_a^b (\alpha X_t + \beta Y_t) dt = \lim_{\Delta \to 0} \sum_{i=1}^{k} (\alpha X_{t_i} + \beta Y_{t_i})(t_{i+1} - t_i)
\]
\[
= \lim_{\Delta \to 0} \alpha \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i) + \lim_{\Delta \to 0} \beta \sum_{i=1}^{k} Y_{t_i}(t_{i+1} - t_i)
\]
\[
= \alpha \int_a^b X_t dt + \beta \int_a^b Y_t dt.
\]
Hence the equation (11.67) is proved.

**Theorem 11.14** (Yao [206]) Let \(X_t\) be a sample-continuous independent increment process with regular uncertainty distribution \(\Phi_t(x)\). Then the time integral
\[
Y_s = \int_0^s X_t dt \tag{11.68}
\]
has an inverse uncertainty distribution
\[
\Psi_s^{-1}(\alpha) = \int_0^s \Phi_t^{-1}(\alpha) dt. \tag{11.69}
\]

**Proof:** For any given time \(s > 0\), it follows from the basic property of time integral that
\[
\left\{ \int_0^s X_t dt \leq \int_0^s \Phi_t^{-1}(\alpha) dt \right\} \supset \{X_t \leq \Phi_t^{-1}(\alpha), \forall t\}.
\]
By using Theorem 11.8, we obtain
\[
\mathcal{M} \left\{ \int_0^s X_t dt \leq \int_0^s \Phi_t^{-1}(\alpha) dt \right\} \geq \mathcal{M}\{X_t \leq \Phi_t^{-1}(\alpha), \forall t\} = \alpha.
\]
Similarly, since
\[
\left\{ \int_0^s X_t \, dt > \int_0^s \Phi_t^{-1}(\alpha) \, dt \right\} \supset \{ X_t > \Phi_t^{-1}(\alpha), \forall t \},
\]
we have
\[
M \left\{ \int_0^s X_t \, dt > \int_0^s \Phi_t^{-1}(\alpha) \, dt \right\} \geq M \{ X_t > \Phi_t^{-1}(\alpha), \forall t \} = 1 - \alpha.
\]
It follows from the above two inequalities and the duality axiom that
\[
M \left\{ \int_0^s X_t \, dt \leq \int_0^s \Phi_t^{-1}(\alpha) \, dt \right\} = \alpha.
\]
Thus the time integral \( Y_s \) has the inverse uncertainty distribution \( \Psi_s^{-1}(\alpha) \).

**Exercise 11.19:** Let \( X_t \) be a sample-continuous independent increment process with regular uncertainty distribution \( \Phi_t(x) \), and let \( J(x) \) be a continuous and strictly increasing function. Show that the time integral
\[
Y_s = \int_0^s J(X_t) \, dt \quad (11.70)
\]
has an inverse uncertainty distribution
\[
\Psi_s^{-1}(\alpha) = \int_0^s J(\Phi_t^{-1}(\alpha)) \, dt. \quad (11.71)
\]

**Exercise 11.20:** Let \( X_t \) be a sample-continuous independent increment process with regular uncertainty distribution \( \Phi_t(x) \), and let \( J(x) \) be a continuous and strictly decreasing function. Show that the time integral
\[
Y_s = \int_0^s J(X_t) \, dt \quad (11.72)
\]
has an inverse uncertainty distribution
\[
\Psi_s^{-1}(\alpha) = \int_0^s J(\Phi_t^{-1}(1 - \alpha)) \, dt. \quad (11.73)
\]

### 11.8 Stationary Increment Process

An uncertain process \( X_t \) is said to have **stationary increments** if its increments are identically distributed uncertain variables whenever the time intervals have the same length, i.e., for any given \( t > 0 \), the increments \( X_{s+t} - X_s \) are identically distributed uncertain variables for all \( s > 0 \).
**Definition 11.11** *(Liu [89])* An uncertain process is said to be a stationary independent increment process if it has not only stationary increments but also independent increments.

It is clear that a stationary independent increment process is a special independent increment process.

**Theorem 11.15** Let $X_t$ be a stationary independent increment process. Then for any real numbers $a$ and $b$, the uncertain process

\[ Y_t = aX_t + b \]  

(11.74)

is also a stationary independent increment process.

**Proof:** Since $X_t$ is an independent increment process, the uncertain variables

\[ X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}} \]

are independent. It follows from $Y_t = aX_t + b$ and Theorem 2.9 that

\[ Y_{t_1}, Y_{t_2} - Y_{t_1}, Y_{t_3} - Y_{t_2}, \ldots, Y_{t_k} - Y_{t_{k-1}} \]

are also independent. That is, $Y_t$ is an independent increment process. On the other hand, since $X_t$ is a stationary increment process, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$. Thus

\[ Y_{s+t} - Y_s = a(X_{s+t} - X_s) \]

are also identically distributed uncertain variables for all $s > 0$, and $Y_t$ is a stationary increment process. Hence $Y_t$ is a stationary independent increment process.

**Remark 11.4:** Generally speaking, a nonlinear function of stationary independent increment process is not necessarily a stationary independent increment process. A typical example is the square of a stationary independent increment process.

**Theorem 11.16** *(Chen [11])* Suppose $X_t$ is a stationary independent increment process. Then $X_t$ and $(1 - t)X_0 + tX_1$ are identically distributed uncertain variables for any time $t \geq 0$.

**Proof:** We first prove the theorem when $t$ is a rational number. Assume $t = q/p$ where $p$ and $q$ are irreducible integers. Let $\Phi$ be the common uncertainty distribution of increments

\[ X_{1/p} - X_{0/p}, X_{2/p} - X_{1/p}, X_{3/p} - X_{2/p}, \ldots \]

Then

\[ X_t - X_0 = (X_{1/p} - X_{0/p}) + (X_{2/p} - X_{1/p}) + \cdots + (X_{q/p} - X_{(q-1)/p}) \]
has an uncertainty distribution
\[ \Psi(x) = \Phi(x/q). \] (11.75)

In addition,
\[ t(X_1 - X_0) = t\left((X_{1/p} - X_{0/p}) + (X_{2/p} - X_{1/p}) + \cdots + (X_{p/p} - X_{(p-1)/p})\right) \]
has an uncertainty distribution
\[ \Upsilon(x) = \Phi(x/p/t) = \Phi(x/p/(q/p)) = \Phi(x/q). \] (11.76)

It follows from (11.75) and (11.76) that \( X_t - X_0 \) and \( t(X_1 - X_0) \) are identically distributed, and so are \( X_t \) and \( (1-t)X_0 + tX_1 \).

**Remark 11.5:** If \( X_t \) is a stationary independent increment process with \( X_0 = 0 \), then \( X_t/t \) and \( X_1 \) are identically distributed uncertain variables. In other words, there is an uncertainty distribution \( \Phi \) such that
\[ \frac{X_t}{t} \sim \Phi(x) \] (11.77)
or equivalently,
\[ X_t \sim \Phi\left(\frac{x}{t}\right) \] (11.78)
for any time \( t > 0 \). Note that \( \Phi \) is just the uncertainty distribution of \( X_1 \).

**Theorem 11.17** (Liu [105]) Let \( X_t \) be a stationary independent increment process whose initial value and increments have inverse uncertainty distributions. Then there exist two continuous and strictly increasing functions \( \mu \) and \( \nu \) such that \( X_t \) has an inverse uncertainty distribution
\[ \Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t. \] (11.79)

**Proof:** Note that \( X_0 \) and \( X_1 - X_0 \) are independent uncertain variables whose inverse uncertainty distributions exist and are denoted by \( \mu(\alpha) \) and \( \nu(\alpha) \), respectively. It is clear that \( \mu(\alpha) \) and \( \nu(\alpha) \) are continuous and strictly increasing functions. Furthermore, it follows from Theorem 11.16 that \( X_t \) and \( X_0 + (X_1 - X_0)t \) are identically distributed uncertain variables. Hence \( X_t \) has the inverse uncertainty distribution \( \Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t. \) The theorem is verified.

**Remark 11.6:** The inverse uncertainty distribution of stationary independent increment process is a family of linear functions of \( t \) indexed by \( \alpha \). See Figure 11.5.

**Theorem 11.18** (Liu [105]) Let \( \mu \) and \( \nu \) be continuous and strictly increasing functions on \((0,1)\). Then there exists a stationary independent increment process \( X_t \) whose inverse uncertainty distribution is
\[ \Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t. \] (11.80)

Furthermore, \( X_t \) has a Lipschitz continuous version.
Section 11.8 - Stationary Increment Process

Theorem 11.19 (Liu [95]) Let $X_t$ be a stationary independent increment process. Then there exist two real numbers $a$ and $b$ such that

$$E[X_t] = a + bt$$

(11.81)

for any time $t \geq 0$.

Proof: It follows from Theorem 11.16 that $X_t$ and $X_0 + (X_1 - X_0)t$ are identically distributed uncertain variables. Thus we have

$$E[X_t] = E[X_0 + (X_1 - X_0)t].$$

Since $X_0$ and $X_1 - X_0$ are independent uncertain variables, we obtain

$$E[X_t] = E[X_0] + E[X_1 - X_0]t.$$

Hence (11.81) holds for $a = E[X_0]$ and $b = E[X_1 - X_0]$.

Proof: Without loss of generality, we only consider the range of $t \in [0, 1]$. Let

$$\{\xi(r) \mid r \text{ represents rational numbers in } [0, 1]\}$$

be a countable sequence of independent uncertain variables, where $\xi(0)$ has an inverse uncertainty distribution $\mu(\alpha)$ and $\xi(r)$ have a common inverse uncertainty distribution $\nu(\alpha)$ for all rational numbers $r$ in $(0, 1]$. For each positive integer $n$, we define an uncertain process

$$X^n_t = \begin{cases} 
\xi(0) + \frac{1}{n} \sum_{i=1}^{k} \xi\left(\frac{i}{n}\right), & \text{if } t = \frac{k}{n} (k = 1, 2, \ldots, n) \\
\text{linear,} & \text{otherwise.}
\end{cases}$$

It may prove that $X^n_t$ converges in distribution as $n \to \infty$. Furthermore, we may verify that the limit is a stationary independent increment process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is verified.
**Theorem 11.20** (Liu [95]) Let \( X_t \) be a stationary independent increment process with an initial value 0. Then for any times \( s \) and \( t \), we have

\[
E[X_{s+t}] = E[X_s] + E[X_t].
\]

**(Proof):** It follows from Theorem 11.19 that there exists a real number \( b \) such that \( E[X_t] = bt \) for any time \( t \geq 0 \). Hence

\[
E[X_{s+t}] = b(s + t) = bs + bt = E[X_s] + E[X_t].
\]

**Theorem 11.21** (Chen [11]) Let \( X_t \) be a stationary independent increment process with a crisp initial value \( X_0 \). Then there exists a real number \( b \) such that

\[
V[X_t] = bt^2
\]

for any time \( t \geq 0 \).

**(Proof):** It follows from Theorem 11.16 that \( X_t \) and \( (1 - t)X_0 + tX_1 \) are identically distributed uncertain variables. Since \( X_0 \) is a constant, we have

\[
V[X_t] = V[(1 - t)X_0 + tX_1] = t^2V[X_1].
\]

Hence (11.83) holds for \( b = V[X_1] \).

**Theorem 11.22** (Chen [11]) Let \( X_t \) be a stationary independent increment process with a crisp initial value \( X_0 \). Then for any times \( s \) and \( t \), we have

\[
\sqrt{V[X_{s+t}]} = \sqrt{V[X_s]} + \sqrt{V[X_t]}.
\]

**(Proof):** It follows from Theorem 11.21 that there exists a real number \( b \) such that \( V[X_t] = bt^2 \) for any time \( t \geq 0 \). Hence

\[
\sqrt{V[X_{s+t}]} = \sqrt{b(s + t)} = \sqrt{bs} + \sqrt{bt} = \sqrt{V[X_s]} + \sqrt{V[X_t]}.
\]

**Exercise 11.21:** (Gao-Ahmadzade [41]) Let \( X_t \) be a stationary independent increment process with a crisp initial value \( X_0 \). Show that there exists a real number \( b \) such that the \( k \)-th central moment

\[
E[(X_t - E[X_t])^k] = bt^k
\]

for any time \( t \geq 0 \).

**Exercise 11.22:** (Gao-Ahmadzade [41]) Let \( X_t \) be a stationary independent increment process with a crisp initial value \( X_0 \). Show that for any times \( s \) and \( t \), we have

\[
\sqrt[k]{E[(X_{s+t} - E[X_{s+t}])^k]} = \sqrt[k]{E[(X_s - E[X_s])^k]} + \sqrt[k]{E[(X_t - E[X_t])^k]}.
\]
11.9 Bibliographic Notes

The study of uncertain process was started by Liu [89] in 2008 for modelling the evolution of uncertain phenomena. In order to describe uncertain process, Liu [105] proposed the uncertainty distribution and inverse uncertainty distribution. In addition, the independence concept of uncertain processes was introduced by Liu [105].

Independent increment process was initialized by Liu [89], and a sufficient and necessary condition was proved by Liu [105] for its inverse uncertainty distribution. In addition, Liu [101] presented an extreme value theorem and obtained the uncertainty distribution of first hitting time, and Yao [206] provided a formula for calculating the inverse uncertainty distribution of time integral of independent increment process.

Stationary independent increment process was initialized by Liu [89], and its inverse uncertainty distribution was investigated by Liu [105]. Furthermore, Liu [95] showed that the expected value is a linear function of time, and Chen [11] verified that the variance is proportional to the square of time.
Chapter 12
Uncertain Renewal Process

Uncertain renewal process is an uncertain process in which events occur continuously and independently of one another in uncertain times. This chapter will introduce uncertain renewal process, renewal reward process, and alternating renewal process. This chapter will also provide block replacement policy, age replacement policy, and uncertain insurance model.

12.1 Uncertain Renewal Process

Definition 12.1 (Liu [89]) Let $\xi_1, \xi_2, \ldots$ be iid uncertain interarrival times. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process

$$N_t = \max_{n \geq 0} \{n \mid S_n \leq t\}$$  \hspace{1cm} (12.1)

is called an uncertain renewal process.

It is clear that $S_n$ is a stationary independent increment process with respect to $n$. Since $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events, $S_n$ can be regarded as the waiting time until the occurrence of the $n$th event. In this case, the renewal process $N_t$ is the number of renewals in $(0, t]$. Note that $N_t$ is not sample-continuous, but each sample path of $N_t$ is a right-continuous and increasing step function taking only nonnegative integer values. Furthermore, since the interarrival times are always assumed to be positive uncertain variables, the size of each jump of $N_t$ is always 1. In other words, $N_t$ has at most one renewal at each time. In particular, $N_t$ does not jump at time 0.

Theorem 12.1 (Fundamental Relationship) Let $N_t$ be a renewal process with positive uncertain interarrival times $\xi_1, \xi_2, \cdots$, and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$. 
Then we have
\[ N_t \geq n \iff S_n \leq t \quad (12.2) \]
for any time \( t \) and integer \( n \). Furthermore, we also have
\[ N_t \leq n \iff S_{n+1} > t. \quad (12.3) \]

**Proof:** Since \( N_t \) is the largest \( n \) such that \( S_n \leq t \), we have \( S_{N_t} \leq t < S_{N_t+1} \). If \( N_t \geq n \), then \( S_n \leq S_{N_t} \leq t \). Conversely, if \( S_n \leq t \), then \( S_n < S_{N_t+1} \) that implies \( N_t \geq n \). Thus (12.2) is verified. Similarly, if \( N_t \leq n \), then \( N_t + 1 \leq n + 1 \) and \( S_{n+1} \geq S_{N_t+1} > t \). Conversely, if \( S_{n+1} > t \), then \( S_{n+1} > S_{N_t} \) that implies \( N_t \leq n \). Thus (12.3) is verified.

**Exercise 12.1:** Let \( N_t \) be a renewal process with positive uncertain interarrival times \( \xi_1, \xi_2, \cdots \), and \( S_n = \xi_1 + \xi_2 + \cdots + \xi_n \). Show that
\[ M\{N_t \geq n\} = M\{S_n \leq t\}, \quad (12.4) \]
\[ M\{N_t \leq n\} = 1 - M\{S_{n+1} \leq t\}. \quad (12.5) \]

**Theorem 12.2** (Liu [95]) Let \( N_t \) be a renewal process with iid positive uncertain interarrival times \( \xi_1, \xi_2, \cdots \). If \( \Phi \) is the common uncertainty distribution of those interarrival times, then \( N_t \) has an uncertainty distribution
\[ \Upsilon_t(x) = 1 - \Phi \left( \frac{t}{\lfloor x \rfloor + 1} \right), \quad \forall x \geq 0 \quad (12.6) \]
where \( \lfloor x \rfloor \) represents the maximal integer less than or equal to \( x \).

**Proof:** Note that \( S_{n+1} \) has an uncertainty distribution \( \Phi(x/(n+1)) \). It follows from (12.5) that
\[ M\{N_t \leq n\} = 1 - M\{S_{n+1} \leq t\} = 1 - \Phi \left( \frac{t}{n+1} \right). \]
Since $N_t$ takes integer values, for any $x \geq 0$, we have
\[
\Upsilon_t(x) = M\{N_t \leq x\} = M\{N_t \leq \lfloor x \rfloor\} = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right).
\]

The theorem is verified.

Figure 12.2: Uncertainty Distribution $\Upsilon_t(x)$ of Renewal Process $N_t$

**Theorem 12.3** (Liu [95], Elementary Renewal Theorem) Let $N_t$ be a renewal process with iid positive uncertain interarrival times $\xi_1, \xi_2, \ldots$. Assume those interarrival times have a continuous uncertainty distribution $\Phi$. Then the average renewal number $N_t/t \to \frac{1}{\xi_1}$ in the sense of convergence in distribution as $t \to \infty$.

**Proof:** The uncertainty distribution $\Upsilon_t$ of $N_t$ has been given by Theorem 12.2 as follows,
\[
\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right).
\]
It follows from the operational law that the uncertainty distribution of $N_t/t$ is
\[
\Psi_t(x) = 1 - \Phi\left(\frac{t}{\lfloor tx \rfloor + 1}\right)
\]
where $\lfloor tx \rfloor$ represents the maximal integer less than or equal to $tx$. It is clear that at each point $x$, we have
\[
\lim_{t \to \infty} \Psi_t(x) = 1 - \Phi\left(\frac{1}{x}\right)
\]
which is just the uncertainty distribution of $1/\xi_1$. Hence $N_t/t$ converges in distribution to $1/\xi_1$ as $t \to \infty$. 


Theorem 12.4 (Liu [95], Elementary Renewal Theorem) Let $N_t$ be a renewal process with iid positive uncertain interarrival times $\xi_1, \xi_2, \cdots$. Then

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = E \left[ \frac{1}{\xi_1} \right].$$  \hfill (12.8)

If $\Phi$ is the common uncertainty distribution of those interarrival times, then

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^{+\infty} \Phi \left( \frac{1}{x} \right) dx.$$  \hfill (12.9)

If the uncertainty distribution $\Phi$ is regular, then

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)} d\alpha.$$  \hfill (12.10)

Proof: Without loss of generality, suppose $\Phi$ is continuous. Otherwise, we may give it a small perturbation such that it becomes continuous. Write the uncertainty distributions of $N_t/t$ and $1/\xi_1$ by $\Psi_t(x)$ and $G(x)$, respectively. Theorem 12.3 says that $\Psi_t(x) \to G(x)$ as $t \to \infty$ at each point $x$. Note that $\Psi_t(x) \geq G(x)$. It follows from the Lebesgue dominated convergence theorem and the existence of $E[1/\xi_1]$ that

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \lim_{t \to \infty} \int_0^{+\infty} (1 - \Psi_t(x)) dx = \int_0^{+\infty} (1 - G(x)) dx = E \left[ \frac{1}{\xi_1} \right].$$

Since $1/\xi_1$ has an uncertainty distribution $1 - \Phi(1/x)$, we have

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = E \left[ \frac{1}{\xi_1} \right] = \int_0^{+\infty} \Phi \left( \frac{1}{x} \right) dx.$$  \hfill (12.11)

Furthermore, since $1/\xi_1$ has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1 - \alpha)},$$

we get

$$E \left[ \frac{1}{\xi_1} \right] = \int_0^1 \frac{1}{\Phi^{-1}(1 - \alpha)} d\alpha = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)} d\alpha.$$  \hfill (12.12)

The theorem is proved.

Exercise 12.2: A renewal process $N_t$ is called linear if $\xi_1, \xi_2, \cdots$ are iid linear uncertain variables $L(a, b)$ with $a > 0$. Show that

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \frac{\ln b - \ln a}{b - a}.$$  \hfill (12.11)
Exercise 12.3: A renewal process $N_t$ is called zigzag if $\xi_1, \xi_2, \cdots$ are iid zigzag uncertain variables $Z(a, b, c)$ with $a > 0$. Show that
\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = \frac{1}{2} \left( \frac{\ln b - \ln a}{b - a} + \frac{\ln c - \ln b}{c - b} \right). \tag{12.12}
\]

Exercise 12.4: A renewal process $N_t$ is called lognormal if $\xi_1, \xi_2, \cdots$ are iid lognormal uncertain variables $LOGN(e, \sigma)$. Show that
\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = \begin{cases} 
\sqrt{3} \sigma \exp(-e) \csc(\sqrt{3} \sigma), & \text{if } \sigma < \pi/\sqrt{3} \\
+\infty, & \text{if } \sigma \geq \pi/\sqrt{3}.
\end{cases} \tag{12.13}
\]

12.2 Block Replacement Policy

Block replacement policy means that an element is always replaced at failure or periodically with time $s$. Assume that the lifetimes of elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution $\Phi$. Then the replacement times form an uncertain renewal process $N_t$. Let $a$ denote the “failure replacement” cost of replacing an element when it fails earlier than $s$, and $b$ the “planned replacement” cost of replacing an element at planned time $s$. Note that $a > b > 0$ is always assumed. It is clear that the cost of one period is $aN_s + b$ and the average cost is
\[
\frac{aN_s + b}{s}. \tag{12.14}
\]

Theorem 12.5 (Ke-Yao [77]) Assume the lifetimes of elements are iid positive uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution $\Phi$, and $N_t$ is the uncertain renewal process representing the replacement times. Then the average cost has an expected value
\[
E\left[\frac{aN_s + b}{s}\right] = \frac{1}{s} \left( a \sum_{n=1}^{\infty} \Phi\left(\frac{s}{n}\right) + b \right). \tag{12.15}
\]

Proof: Note that the uncertainty distribution of $N_t$ is a step function. It follows from Theorem 12.2 that
\[
E[N_s] = \int_{0}^{+\infty} \Phi\left(\frac{s}{x} + 1\right) \, dx = \sum_{n=1}^{\infty} \Phi\left(\frac{s}{n}\right).
\]
Thus (12.15) is verified by
\[
E\left[\frac{aN_s + b}{s}\right] = \frac{aE[N_s] + b}{s}. \tag{12.16}
\]
Chapter 12 - Uncertain Renewal Process

What is the optimal time $s$?

When the block replacement policy is accepted, one problem is concerned with finding an optimal time $s$ in order to minimize the average cost, i.e.,

$$\min_s \frac{1}{s} \left( a \sum_{n=1}^{\infty} \Phi \left( \frac{s}{n} \right) + b \right).$$

(12.17)

12.3 Renewal Reward Process

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \cdots$ be a sequence of pairs of uncertain variables. We shall interpret $\eta_i$ as the rewards (or costs) associated with the $i$-th interarrival times $\xi_i$ for $i = 1, 2, \cdots$, respectively.

**Definition 12.2** (Liu [95]) Let $\xi_1, \xi_2, \cdots$ be iid uncertain interarrival times, and let $\eta_1, \eta_2, \cdots$ be iid uncertain rewards. Then

$$R_t = \sum_{i=1}^{N_t} \eta_i$$

(12.18)

is called a renewal reward process, where $N_t$ is the renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$

A renewal reward process $R_t$ denotes the total reward earned by time $t$. In addition, if $\eta_i \equiv 1$, then $R_t$ degenerates to a renewal process $N_t$.

**Theorem 12.6** (Liu [95]) Let $R_t$ be a renewal reward process with iid positive uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid positive uncertain rewards $\eta_1, \eta_2, \cdots$. Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors, and those interarrival times and rewards have uncertainty distributions $\Phi$ and $\Psi$, respectively. Then $R_t$ has an uncertainty distribution

$$\Upsilon_t(x) = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \wedge \Psi \left( \frac{x}{k} \right).$$

(12.19)

Here we set $\Psi(x/0) = 1$ for any $x \geq 0$.

**Proof:** It follows from the definition of renewal reward process that the renewal process $N_t$ is independent of uncertain rewards $\eta_1, \eta_2, \cdots$, and $R_t$
has an uncertainty distribution

\[ \Upsilon_t(x) = \mathcal{M} \left\{ \sum_{i=1}^{N_t} \eta_i \leq x \right\} = \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} (N_t = k) \cap \sum_{i=1}^{k} \eta_i \leq x \right\} \]

\[ = \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} (N_t \leq k) \cap \sum_{i=1}^{k} \eta_i \leq x \right\} \quad \text{(this is a polyrectangle)} \]

\[ = \max_{k \geq 0} \mathcal{M} \left\{ N_t \leq k \right\} \wedge \mathcal{M} \left\{ \sum_{i=1}^{k} \eta_i \leq x \right\} \quad \text{(polyrectangular theorem)} \]

\[ = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \wedge \Psi \left( \frac{x}{k} \right) . \]

The theorem is proved.

![Figure 12.3: Uncertainty Distribution \( \Upsilon_t(x) \) of Renewal Reward Process \( R_t \) in which the dashed horizontal lines are \( 1 - \Phi(t/(k+1)) \) and the dashed curves are \( \Psi(x/k) \) for \( k = 0, 1, 2, \cdots \).](image)

**Theorem 12.7 (Liu [95], Renewal Reward Theorem)** Let \( R_t \) be a renewal reward process with iid positive uncertain interarrival times \( \xi_1, \xi_2, \cdots \) and iid positive uncertain rewards \( \eta_1, \eta_2, \cdots \). Assume \( (\xi_1, \xi_2, \cdots) \) and \( (\eta_1, \eta_2, \cdots) \) are independent uncertain vectors, and those interarrival times and rewards have continuous uncertainty distributions \( \Phi \) and \( \Psi \), respectively. Then the reward rate

\[ \frac{R_t}{t} \xrightarrow{\mathcal{D}} \frac{\eta_1}{\xi_1} \quad (12.20) \]

in the sense of convergence in distribution as \( t \to \infty \).
Chapter 12 - Uncertain Renewal Process

**Proof:** Assume those interarrival times and rewards have uncertainty distributions $\Phi$ and $\Psi$, respectively. It follows from Theorem 12.6 that the uncertainty distribution of $R_t$ is

$$\Upsilon_t(x) = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \wedge \Psi \left( \frac{x}{k} \right).$$

Then $R_t/t$ has an uncertainty distribution

$$\Psi_t(x) = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \wedge \Psi \left( \frac{tx}{k} \right).$$

When $t \to \infty$, we have

$$\Psi_t(x) \to \sup_{y \geq 0} (1 - \Phi(y)) \wedge \Psi(xy)$$

which is just the uncertainty distribution of $\eta_1/\xi_1$. Hence $R_t/t$ converges in distribution to $\eta_1/\xi_1$ as $t \to \infty$.

**Theorem 12.8** (Liu [95], Renewal Reward Theorem) Let $R_t$ be a renewal reward process with iid positive uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid positive uncertain rewards $\eta_1, \eta_2, \cdots$. Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors. Then

$$\lim_{t \to \infty} E \left[ \frac{R_t}{t} \right] = E \left[ \frac{\eta_1}{\xi_1} \right]. \quad (12.21)$$

If those interarrival times and rewards have regular uncertainty distributions $\Phi$ and $\Psi$, respectively, then

$$\lim_{t \to \infty} E \left[ \frac{R_t}{t} \right] = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1 - \alpha)} \, d\alpha. \quad (12.22)$$

**Proof:** Without loss of generality, suppose $\Phi$ and $\Psi$ are continuous. Otherwise, we may give them a small perturbation such that they become continuous. It follows from Theorem 12.6 that $R_t/t$ has an uncertainty distribution

$$F_t(x) = \max_{k \geq 0} \left( 1 - \Phi \left( \frac{t}{k+1} \right) \right) \wedge \Psi \left( \frac{tx}{k} \right)$$

and $\eta_1/\xi_1$ has an uncertainty distribution

$$G(x) = \sup_{y \geq 0} (1 - \Phi(y)) \wedge \Psi(xy).$$

Note that $F_t(x) \to G(x)$ and $F_t(x) \geq G(x)$. It follows from Lebesgue dominated convergence theorem and the existence of $E[\eta_1/\xi_1]$ that

$$\lim_{t \to \infty} E \left[ \frac{R_t}{t} \right] = \lim_{t \to \infty} \int_0^+ (1 - F_t(x)) \, dx = \int_0^+ (1 - G(x)) \, dx = E \left[ \frac{\eta_1}{\xi_1} \right].$$
Finally, since $\eta_1/\xi_1$ has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)},$$

we get

$$E\left[\frac{\eta_1}{\xi_1}\right] = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)} \, d\alpha.$$

The theorem is proved.

### 12.4 Uncertain Insurance Model

Liu [101] assumed that $a$ is the initial capital of an insurance company, $b$ is the premium rate, $bt$ is the total income up to time $t$, and the uncertain claim process is a renewal reward process

$$R_t = \sum_{i=1}^{N_t} \eta_i$$

(12.23)

with iid uncertain interarrival times $\xi_1, \xi_2, \ldots$ and iid uncertain claim amounts $\eta_1, \eta_2, \ldots$ Then the capital of the insurance company at time $t$ is

$$Z_t = a + bt - R_t$$

(12.24)

and $Z_t$ is called an insurance risk process.

![Figure 12.4: An Insurance Risk Process](image-url)
Ruin Index

Definition 12.3 (Liu [101]) Let $Z_t$ be an insurance risk process. Then the ruin index is defined as the uncertain measure that the capital $Z_t$ eventually becomes negative, i.e.,

$$Ruin = M\left\{\inf_{t \geq 0} Z_t < 0\right\}. \quad (12.25)$$

Theorem 12.9 (Liu [101], Ruin Index Theorem) Let $Z_t = a + bt - R_t$ be an insurance risk process where $a$ and $b$ are positive numbers, and $R_t$ is a renewal reward process with iid positive uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid positive uncertain claim amounts $\eta_1, \eta_2, \cdots$. Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors, and those interarrival times and claim amounts have continuous uncertainty distributions $\Phi$ and $\Psi$, respectively. Then the ruin index is

$$Ruin = \max_{k \geq 1} \sup_{x \geq 0} \Phi\left(\frac{x}{k}\right) \wedge \left(1 - \Psi\left(\frac{a + bx}{k}\right)\right). \quad (12.26)$$

Proof: At first, we define an uncertain process indexed by positive integer $k$ as follows,

$$Y_k = a + b \sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} \eta_i.$$ 

It is easy to verify that $Y_k$ is an independent increment process, and has an uncertainty distribution

$$F_k(z) = \sup_{x \geq 0} \Phi\left(\frac{x}{k}\right) \wedge \left(1 - \Psi\left(\frac{a + bx + z}{k}\right)\right).$$

It follows from the extreme value theorem that

$$\min_{k \geq 1} Y_k$$

has an uncertainty distribution

$$G(z) = \max_{k \geq 1} F_k(z).$$

Since $Y_k$ is just the capital $Z_t$ at the arrival time of the $k$th claim for each $k$, and a ruin occurs only at the arrival times, we have

$$Ruin = M\left\{\inf_{t \geq 0} Z_t < 0\right\} = M\left\{\min_{k \geq 1} Y_k < 0\right\} = G(0) = \max_{k \geq 1} F_k(0).$$

The theorem is proved.
**Ruin Time**

**Definition 12.4** (Liu [101]) Let $Z_t$ be an insurance risk process. Then the ruin time is defined as the first hitting time that the capital $Z_t$ becomes negative, i.e.,

$$
\tau = \inf \left\{ t \geq 0 \mid Z_t < 0 \right\} .
$$

(12.27)

**Theorem 12.10** (Yao-Zhou [201]) Let $Z_t = a + bt - R_t$ be an insurance risk process where $a$ and $b$ are positive numbers, and $R_t$ is a renewal reward process with iid positive uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid positive uncertain claim amounts $\eta_1, \eta_2, \cdots$. Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors, and those interarrival times and claim amounts have regular uncertainty distributions $\Phi$ and $\Psi$, respectively. Then the ruin time has an uncertainty distribution

$$
\Upsilon(t) = \max_{k \geq 1} \sup_{x \leq t} \Phi \left( \frac{x}{k} \right) \wedge \left( 1 - \Psi \left( \frac{a + bx}{k} \right) \right) .
$$

(12.28)

**Proof:** At first, for fixed positive integer $k$ and time $t$, we define a number,

$$
\alpha_k = \sup_{x \leq t} \Phi \left( \frac{x}{k} \right) \wedge \left( 1 - \Psi \left( \frac{a + bx}{k} \right) \right) .
$$

(12.29)

Keep in mind that $\Phi(x/k)$ is an increasing function with respect to $x$, and

$$
1 - \Psi \left( \frac{a + bx}{k} \right)
$$

is a decreasing function with respect to $x$. Let $x^*$ be the supremum solution of (12.29). If $x^* = t$, then

$$
\alpha_k = \Phi \left( \frac{t}{k} \right) , \quad \alpha_k \leq 1 - \Psi \left( \frac{a + bt}{k} \right) .
$$

That is,

$$
\Phi^{-1}(\alpha_k) = \frac{t}{k} , \quad \frac{a + bt}{k} \leq \Psi^{-1}(1 - \alpha_k) .
$$

Thus,

$$
k\Phi^{-1}(\alpha_k) = t , \quad a + bk\Phi^{-1}(\alpha_k) - k\Psi^{-1}(1 - \alpha_k) \leq 0 .
$$

(12.30)

If $x^* < t$, then

$$
\alpha_k = \Phi \left( \frac{x^*}{k} \right) = 1 - \Psi \left( \frac{a + bx^*}{k} \right) .
$$

That is,

$$
\Phi^{-1}(\alpha_k) = \frac{x^*}{k} , \quad \frac{a + bx^*}{k} = \Psi^{-1}(1 - \alpha_k) .
$$
Thus,

\[ k\Phi^{-1}(\alpha_k) < t, \quad a + bk\Phi^{-1}(\alpha_k) - k\Psi^{-1}(1 - \alpha_k) = 0. \quad (12.31) \]

Therefore, one of the alternatives (12.30) and (12.31) holds. Let us turn our attention to the uncertainty distribution of ruin time. It follows from the definition of ruin time that for each \( t \), we have

\[ \tau \leq t \text{ if and only if } \inf_{0 \leq s \leq t} Z_s < 0. \]

On the one hand, by using (12.30) or (12.31), we obtain

\[
\Upsilon(t) = \mathcal{M}\{\tau \leq t\} = \mathcal{M}\left\{ \inf_{0 \leq s \leq t} Z_s < 0 \right\} \\
= \mathcal{M}\left\{ \bigcup_{k=1}^{\infty} \left( \sum_{i=1}^{k} \xi_i \leq t, \ a + b \sum_{i=1}^{k} \xi_i - k \sum_{i=1}^{k} \eta_i < 0 \right) \right\} \\
\geq \bigvee_{k=1}^{\infty} \mathcal{M}\left\{ \sum_{i=1}^{k} \xi_i \leq t, \ a + b \sum_{i=1}^{k} \xi_i - k \sum_{i=1}^{k} \eta_i < 0 \right\} \\
\geq \bigvee_{k=1}^{\infty} \mathcal{M}\left\{ \bigcap_{i=1}^{k} (\xi_i \leq \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1 - \alpha_k)) \right\} \\
= \bigvee_{k=1}^{\infty} \bigwedge_{i=1}^{k} \mathcal{M}\{\xi_i \leq \Phi^{-1}(\alpha_k)\} \cap \mathcal{M}\{\eta_i > \Psi^{-1}(1 - \alpha_k)\} \\
= \bigvee_{k=1}^{\infty} \bigwedge_{i=1}^{k} \alpha_k \wedge \alpha_k \wedge \bigvee_{k=1}^{\infty} \alpha_k = \bigvee_{k=1}^{\infty} \alpha_k. 
\]
On the other hand, by using (12.30) or (12.31) and Fubini theorem, we obtain

\[ \Upsilon(t) = \mathcal{M} \left\{ \bigcup_{k=1}^{\infty} \left( \sum_{i=1}^{k} \xi_i \leq t, a + b \sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} \eta_i < 0 \right) \right\} \]

\[ \leq \mathcal{M} \left\{ \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \left( \xi_i \leq \Phi^{-1}(\alpha_k) \cup (\eta_i > \Psi^{-1}(1 - \alpha_k)) \right) \right\} \]

\[ = \mathcal{M} \left\{ \bigcup_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \left( \xi_i \leq \Phi^{-1}(\alpha_k) \cup (\eta_i > \Psi^{-1}(1 - \alpha_k)) \right) \right\} \]

\[ \leq \mathcal{M} \left\{ \bigcup_{i=1}^{\infty} \left( \xi_i \leq \sum_{k=i}^{\infty} \Phi^{-1}(\alpha_k) \cup (\eta_i > \sum_{k=i}^{\infty} \Psi^{-1}(1 - \alpha_k)) \right) \right\} \]

\[ = \bigvee_{i=1}^{\infty} \mathcal{M} \left\{ \xi_i \leq \sum_{k=i}^{\infty} \Phi^{-1}(\alpha_k) \cup (\eta_i > \sum_{k=i}^{\infty} \Psi^{-1}(1 - \alpha_k)) \right\} \]

\[ = \bigvee_{i=1}^{\infty} \left( \bigvee_{k=i}^{\infty} \alpha_k \right) \cup \bigvee_{k=i}^{\infty} = \bigvee_{k=1}^{\infty} \alpha_k. \]

It follows that

\[ \Upsilon(t) = \bigvee_{k=1}^{\infty} \alpha_k \]

and the theorem is verified.

### 12.5 Age Replacement Policy

Age replacement means that an element is always replaced at failure or at an age \( s \). Assume that the lifetimes of the elements are iid uncertain variables \( \xi_1, \xi_2, \ldots \) with a common uncertainty distribution \( \Phi \). Then the actual lifetimes of the elements are iid uncertain variables

\[ \xi_1 \land s, \xi_2 \land s, \ldots \]  \hspace{1cm} (12.32)

which may generate an uncertain renewal process

\[ N_t = \max_{n\geq0} \left\{ n \mid \sum_{i=1}^{n} (\xi_i \land s) \leq t \right\}. \]  \hspace{1cm} (12.33)

Let \( a \) denote the “failure replacement” cost of replacing an element when it fails earlier than \( s \), and \( b \) the “planned replacement” cost of replacing an
element at the age \( s \). Note that \( a > b > 0 \) is always assumed. Define

\[
f(x) = \begin{cases} 
a, & \text{if } x < s \\
b, & \text{if } x = s.
\end{cases} \tag{12.34}
\]

Then \( f(\xi_i \wedge s) \) is just the cost of replacing the \( i \)th element, and the average replacement cost before the time \( t \) is

\[
\frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s). \tag{12.35}
\]

**Theorem 12.11** (Yao-Ralescu [187]) Assume \( \xi_1, \xi_2, \ldots \) are iid positive uncertain lifetimes and \( s \) is a positive number. Then

\[
\frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) \to \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \tag{12.36}
\]

in the sense of convergence in distribution as \( t \to \infty \).

**Proof:** Without of generality, assume \( t > s \). At first, we have \( N_t \geq 1 \). Then for any real number \( x \), on the one hand, we obtain

\[
\mathcal{M} \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge s) / \sum_{i=1}^{N_t} (\xi_i \wedge s) \leq x \right\}
\]

\[
= \mathcal{M} \left\{ \bigcup_{n=1}^{\infty} \left( (N_t = n) \cap \left( \sum_{i=1}^{n} f(\xi_i \wedge s) / \sum_{i=1}^{n} (\xi_i \wedge s) \leq x \right) \right) \right\}
\]

\[
\geq \mathcal{M} \left\{ \bigcup_{n=1}^{\infty} \left( (N_t = n) \cap \left( \bigcap_{i=1}^{n} \left( f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right) \right) \right) \right\}
\]

\[
\geq \mathcal{M} \left\{ \bigcup_{n=1}^{\infty} \left( (N_t = n) \cap \left( \bigcap_{i=1}^{\infty} \left( f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right) \right) \right) \right\}
\]

\[
= \mathcal{M} \left\{ \bigcap_{i=1}^{\infty} \left( f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right) \right\}
\]

\[
= \bigwedge_{i=1}^{\infty} \mathcal{M} \left\{ f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right\}
\]

\[
= \mathcal{M} \left\{ f(\xi_1 \wedge s) / (\xi_1 \wedge s) \leq x \right\}.
\]
On the other hand, we obtain

\[
M \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge s) / \sum_{i=1}^{N_t} (\xi_i \wedge s) \leq x \right\}
= M \left\{ \bigcup_{n=1}^{\infty} \left( (N_t = n) \cap \left( \sum_{i=1}^{n} f(\xi_i \wedge s) / \sum_{i=1}^{n} (\xi_i \wedge s) \leq x \right) \right) \right\}
\leq M \left\{ \bigcup_{n=1}^{\infty} \left( (N_t = n) \cap \left( \bigcup_{i=1}^{n} \left( f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right) \right) \right\}
\leq M \left\{ \bigcup_{i=1}^{\infty} \left( f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right) \right\}
= \bigvee_{i=1}^{\infty} M \left\{ f(\xi_i \wedge s) / (\xi_i \wedge s) \leq x \right\}
= M \left\{ f(\xi_1 \wedge s) / (\xi_1 \wedge s) \leq x \right\}.
\]

Thus for any real number \( x \) and \( t > s \), we have

\[
M \left\{ \frac{\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\sum_{i=1}^{N_t} (\xi_i \wedge s)} \leq x \right\} = M \left\{ \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \leq x \right\}. \tag{12.37}
\]

Note that the average replacement cost before time \( t \) may be rewritten as

\[
\frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) = \sum_{i=1}^{N_t} f(\xi_i \wedge s) \times \frac{\sum_{i=1}^{N_t} (\xi_i \wedge s)}{t} \tag{12.38}
\]

and

\[
\frac{t-s}{t} \leq \frac{\sum_{i=1}^{N_t} (\xi_i \wedge s)}{t} \leq 1. \tag{12.39}
\]

Let \( x_0 \) be any point at which

\[
M \left\{ \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \leq x \right\}
\]
is continuous. It follows from (12.37), (12.38) and (12.39) that
\[
M\left\{ \frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) \leq x_0 \right\} = M \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge s) \leq x_0 \right\} = M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) \leq x_0 \right\}
\]
and
\[
M\left\{ \frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) \leq x_0 \right\} \leq M \left[ \frac{\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\sum_{i=1}^{N_t} (\xi_i \wedge s)} \times \frac{t-s}{t} \leq x_0 \right]
\]

Thus
\[
\lim_{t \to \infty} M\left\{ \frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) \leq x_0 \right\} = M \left\{ \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \leq x_0 \right\}
\]
and (12.36) is verified.

**Theorem 12.12** (Yao-Ralescu [187]) Assume \(\xi_1, \xi_2, \cdots\) are iid positive uncertain lifetimes with a common continuous uncertainty distribution \(\Phi\), and \(s\) is a positive number. Then the long-run average replacement cost is
\[
\lim_{t \to \infty} E \left[ \frac{1}{t} \sum_{i=1}^{N_t} f(\xi_i \wedge s) \right] = \frac{b}{s} + \frac{a-b}{s} \Phi(s) + a \int_0^s \frac{\Phi(x)}{x^2} \, dx. \tag{12.40}
\]

**Proof:** Let \(\Psi(x)\) be the uncertainty distribution of \(f(\xi_1 \wedge s)/(\xi_1 \wedge s)\). It follows from (12.34) that \(f(\xi_1 \wedge s) \geq b\) and \(\xi_1 \wedge s \leq s\). Thus
\[
\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \geq \frac{b}{s}
\]
almost surely. If $x < b/s$, then
\[
\Psi(x) = \mathcal{M}\left\{ f\left(\frac{x_1 \wedge s}{x_1\wedge s}\right) < x \right\} = 0.
\]
If $b/s \leq x < a/s$, then
\[
\Psi(x) = \mathcal{M}\left\{ f\left(\frac{x_1 \wedge s}{x_1\wedge s}\right) < x \right\} = \mathcal{M}\{x_1 \geq s\} = 1 - \Phi(s).
\]
If $x \geq a/s$, then
\[
\Psi(x) = \mathcal{M}\left\{ f\left(\frac{x_1 \wedge s}{x_1\wedge s}\right) < x \right\} = \mathcal{M}\{x_1 \geq a\} = 1 - \Phi\left(\frac{a}{x}\right).
\]
Hence
\[
\Psi(x) = \begin{cases} 
0, & \text{if } x < b/s \\
1 - \Phi(s), & \text{if } b/s \leq x < a/s \\
1 - \Phi\left(\frac{a}{x}\right), & \text{if } x \geq a/s 
\end{cases}
\]
and
\[
E\left[ f\left(\frac{x_1 \wedge s}{x_1\wedge s}\right) \right] = \int_{0}^{+\infty} (1 - \Psi(x))dx = \frac{b}{s} - \frac{a - b}{s}\Phi(s) + \frac{a}{s} \int_{0}^{s} \frac{\Phi(x)}{x^2}dx.
\]
Since
\[
\mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t} f(x_i \wedge s) \leq x \right\} \geq \mathcal{M}\left\{ \frac{f(x_1 \wedge s)}{x_1 \wedge s} \leq x \right\}
\]
for any real number $x$ and $t > s$, by using the Lebesgue dominated convergence theorem, we get
\[
\lim_{t \to \infty} E\left[ \frac{1}{t} \sum_{i=1}^{N_t} f(x_i \wedge s) \right] = \lim_{t \to \infty} \int_{0}^{+\infty} \left(1 - \mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t} f(x_i \wedge s) \leq x \right\} \right) dx
\]
\[
= \int_{0}^{+\infty} \left(1 - \mathcal{M}\left\{ \frac{f(x_1 \wedge s)}{x_1 \wedge s} \leq x \right\} \right) dx
\]
\[
= E\left[ f\left(\frac{x_1 \wedge s}{x_1\wedge s}\right) \right].
\]
Hence the theorem is proved.

**What is the optimal age $s$?**

When the age replacement policy is accepted, one problem is to find the optimal age $s$ such that the average replacement cost is minimized. That is, the optimal age $s$ should solve
\[
\min_{s \geq 0} \left( \frac{b}{s} + \frac{a - b}{s}\Phi(s) + \frac{a}{s} \int_{0}^{s} \frac{\Phi(x)}{x^2}dx \right).
\] (12.41)
12.6 Alternating Renewal Process

Let \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots\) be a sequence of pairs of uncertain variables. We shall interpret \(\xi_i\) as the “on-times” and \(\eta_i\) as the “off-times” in the \(i\)-th cycles, \(i = 1, 2, \ldots\), respectively.

**Definition 12.5** (Yao-Li [184]) Let \(\xi_1, \xi_2, \ldots\) be iid uncertain on-times, and let \(\eta_1, \eta_2, \ldots\) be iid uncertain off-times. Then

\[
A_t = \begin{cases}
    t - \sum_{i=1}^{N_t} \eta_i, & \text{if } \sum_{i=1}^{N_t} (\xi_i + \eta_i) \leq t < \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} \\
    \sum_{i=1}^{N_t+1} \xi_i, & \text{if } \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} \leq t < \sum_{i=1}^{N_t+1} (\xi_i + \eta_i)
\end{cases}
\]  

(12.42)

is called an alternating renewal process, where \(N_t\) is the renewal process with uncertain interarrival times \(\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots\)

Note that the alternating renewal process \(A_t\) is just the total time at which the system is on up to time \(t\). It is clear that

\[
\sum_{i=1}^{N_t} \xi_i \leq A_t \leq \sum_{i=1}^{N_t+1} \xi_i
\]

(12.43)

for each time \(t\). We are interested in the limit property of the rate at which the system is on.

**Theorem 12.13** (Yao-Li [184], Alternating Renewal Theorem) Let \(A_t\) be an alternating renewal process with iid positive uncertain on-times \(\xi_1, \xi_2, \ldots\) and iid positive uncertain off-times \(\eta_1, \eta_2, \ldots\). Assume \((\xi_1, \xi_2, \ldots)\) and \((\eta_1, \eta_2, \ldots)\) are independent uncertain vectors, and those on-times and off-times have continuous uncertainty distributions \(\Phi\) and \(\Psi\), respectively. Then the availability rate

\[
\frac{A_t}{t} \to \frac{\xi_1}{\xi_1 + \eta_1}
\]

(12.44)

in the sense of convergence in distribution as \(t \to \infty\).

**Proof:** It follows from the operational law that \(\xi_1/(\xi_1 + \eta_1)\) has an uncertainty distribution

\[
\Upsilon(x) = \sup_{y > 0} \Phi(xy) \wedge (1 - \Psi(y - xy)).
\]

(12.45)
On the one hand, we have

\[
M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \xi_i \leq x \right\}
\]

\[
= M \left\{ \bigcup_{k=0}^{\infty} (N_t = k) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \xi_i \leq x \right) \right\}
\]

\[
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k+1} (\xi_i + \eta_i) > t \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \xi_i \leq x \right) \right\}
\]

\[
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( tx + \xi_{k+1} + \sum_{i=1}^{k+1} \eta_i > t \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \xi_i \leq x \right) \right\}
\]

\[
= M \left\{ \bigcup_{k=0}^{\infty} \left( \frac{\xi_{k+1}}{t} + \frac{1}{t} \sum_{i=1}^{k+1} \eta_i > 1 - x \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \xi_i \leq x \right) \right\}.
\]

Since

\[
\frac{\xi_{k+1}}{t} \to 0, \quad \text{as} \quad t \to \infty
\]

and

\[
\sum_{i=1}^{k+1} \eta_i \sim (k+1)\eta_1, \quad \sum_{i=1}^{k} \xi_i \sim k\xi_1,
\]

we have

\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \xi_i \leq x \right\}
\]

\[
\leq \lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( \eta_1 > \frac{t(1-x)}{k+1} \right) \cap \left( \xi_1 \leq \frac{tx}{k} \right) \right\}
\]

\[
= \lim_{t \to \infty} \sup_{k \geq 0} \left\{ \eta_1 > \frac{t(1-x)}{k+1} \right\} \land \left\{ \xi_1 \leq \frac{tx}{k} \right\}
\]

\[
= \lim_{t \to \infty} \sup_{k \geq 0} \left( 1 - \Psi \left( \frac{t(1-x)}{k+1} \right) \right) \land \Phi \left( \frac{tx}{k} \right)
\]

\[
= \sup_{y > 0} \Phi(xy) \land (1 - \Psi(y - xy)) = \Upsilon(x).
\]

That is,

\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \xi_i \leq x \right\} \leq \Upsilon(x). \quad (12.46)
\]
On the other hand, we have
\[
M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \xi_i > x \right\}
\]
\[
= M \left\{ \bigcup_{k=0}^{\infty} (N_t = k) \cap \left( \frac{1}{t} \sum_{i=1}^{k+1} \xi_i > x \right) \right\}
\]
\[
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k} (\xi_i + \eta_i) \leq t \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k+1} \xi_i > x \right) \right\}
\]
\[
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( tx - \xi_{k+1} + \sum_{i=1}^{k} \eta_i \leq t \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k+1} \xi_i > x \right) \right\}
\]
\[
= M \left\{ \bigcup_{k=0}^{\infty} \left( \frac{1}{t} \sum_{i=1}^{k} \eta_i - \frac{\xi_{k+1}}{t} \leq 1 - x \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k+1} \xi_i > x \right) \right\}
\].

Since
\[
\frac{\xi_{k+1}}{t} \to 0, \quad \text{as } t \to \infty
\]
and
\[
\sum_{i=1}^{k} \eta_i \sim k \eta_1, \quad \sum_{i=1}^{k+1} \xi_i \sim (k+1) \xi_1,
\]
we have
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \xi_i > x \right\}
\]
\[
\leq \lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( \eta_1 \leq \frac{t(1-x)}{k} \right) \cap \left( \xi_1 > \frac{tx}{k+1} \right) \right\}
\]
\[
= \lim_{t \to \infty} \sup_{k \geq 0} M \left\{ \eta_1 \leq \frac{t(1-x)}{k} \right\} \wedge M \left\{ \xi_1 > \frac{tx}{k+1} \right\}
\]
\[
= \lim_{t \to \infty} \sup_{y > 0} \Psi \left( \frac{t(1-x)}{k+1} \right) \wedge \left( 1 - \Phi \left( \frac{tx}{k+1} \right) \right)
\]
\[
= \sup_{y > 0} \left( 1 - \Phi(xy) \right) \wedge \Psi(y - xy).
\]

By using the duality axiom, we get
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \xi_i \leq x \right\} \geq 1 - \sup_{y > 0} \left( 1 - \Phi(xy) \right) \wedge \Psi(y - xy)
\]
\[
= \inf_{y > 0} \Phi(xy) \vee \left( 1 - \Psi(y - xy) \right) = \Upsilon(x).
\]
That is,
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \xi_i \leq x \right\} \geq \Upsilon(x). \tag{12.47}
\]

Since
\[
\frac{1}{t} \sum_{i=1}^{N_t} \xi_i \leq \frac{A_t}{t} \leq \frac{1}{t} \sum_{i=1}^{N_t+1} \xi_i,
\]
we obtain
\[
M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \xi_i \leq x \right\} \geq M \left\{ \frac{A_t}{t} \leq x \right\} \geq M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \xi_i \leq x \right\}.
\]

It follows from (12.46) and (12.47) that for any real number \( x \), we have
\[
\lim_{t \to \infty} \left\{ \frac{A_t}{t} \leq x \right\} = \Upsilon(x).
\]

Hence the availability rate \( A_t/t \) converges in distribution to \( \xi_1/(\xi_1 + \eta_1) \). The theorem is proved.

**Theorem 12.14** (Yao-Li [184], Alternating Renewal Theorem) Let \( A_t \) be an alternating renewal process with iid positive uncertain on-times \( \xi_1, \xi_2, \ldots \) and iid positive uncertain off-times \( \eta_1, \eta_2, \ldots \) Assume \( (\xi_1, \xi_2, \ldots) \) and \( (\eta_1, \eta_2, \ldots) \) are independent uncertain vectors. Then
\[
\lim_{t \to \infty} \frac{E[A_t]}{t} = E \left[ \frac{\xi_1}{\xi_1 + \eta_1} \right]. \tag{12.48}
\]

If those on-times and off-times have regular uncertainty distributions \( \Phi \) and \( \Psi \), respectively, then
\[
\lim_{t \to \infty} \frac{E[A_t]}{t} = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} d\alpha. \tag{12.49}
\]

**Proof:** Without loss of generality, suppose \( \Phi \) and \( \Psi \) are continuous. Otherwise, we may give them a small perturbation such that they become continuous. Write the uncertainty distributions of \( A_t/t \) and \( \xi_1/(\xi_1 + \eta_1) \) by \( F_t(x) \) and \( G(x) \), respectively. Since \( A_t/t \) converges in distribution to \( \xi_1/(\xi_1 + \eta_1) \), we have \( F_t(x) \to G(x) \) as \( t \to \infty \). It follows from the Lebesgue dominated convergence theorem that
\[
\lim_{t \to \infty} \frac{E[A_t]}{t} = \lim_{t \to \infty} \int_0^1 (1-F_t(x))dx = \int_0^1 (1-G(x))dx = E \left[ \frac{\xi_1}{\xi_1 + \eta_1} \right].
\]

Finally, since the uncertain variable \( \xi_1/(\xi_1 + \eta_1) \) is strictly increasing with respect to \( \xi_1 \) and strictly decreasing with respect to \( \eta_1 \), it has an inverse uncertainty distribution
\[
G^{-1}(\alpha) = \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi(1-\alpha)}.
\]
The equation (12.49) is thus obtained.

12.7 Bibliographic Notes

Uncertain renewal process was first proposed by Liu [89] in 2008. Two years later, Liu [95] proved some elementary renewal theorems for determining the average renewal number. Liu [95] also provided an uncertain renewal reward process and verified some renewal reward theorems for determining the long-run reward rate. In addition, Yao-Li [184] presented an uncertain alternating renewal process and proved some alternating renewal theorems for determining the availability rate.

Based on the theory of uncertain renewal process, Liu [101] started studying uncertain insurance models in 2013. After that, uncertain insurance models were further developed by Yao-Qin [197], and Yao-Zhou [201], among others. Besides, Ke-Yao [77] discussed the uncertain block replacement policy, and Yao-Ralescu [187] investigated the uncertain age replacement policy and obtained the long-run average replacement cost.
Chapter 13

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes. This chapter will introduce Liu process, Liu integral, fundamental theorem, chain rule, change of variables, and integration by parts.

13.1 Liu Process

In 2009, Liu [91] investigated a type of stationary independent increment process whose increments are normal uncertain variables. Later, this process was named by the academic community as Liu process due to its importance and usefulness. A formal definition is given below.

**Definition 13.1 (Liu [91])** An uncertain process $C_t$ is said to be a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$.

It is clear that a Liu process $C_t$ is a stationary independent increment process and has a normal uncertainty distribution with expected value 0 and variance $t^2$. The uncertainty distribution of $C_t$ is

$$
\Phi_t(x) = \left(1 + \exp \left(-\frac{\pi x}{\sqrt{3} t}\right)\right)^{-1}
$$

and inverse uncertainty distribution is

$$
\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}
$$
that are homogeneous linear functions of time $t$ for any given $\alpha$. See Figure 13.1.

A Liu process is described by three properties in the above definition. Does such an uncertain process exist? The following theorem will answer this question.

**Theorem 13.1** (Liu [95], Existence Theorem) There exists a Liu process.

**Proof:** It follows from Theorem 11.18 that there exists a stationary independent increment process $C_t$ whose inverse uncertainty distribution is

$$
\Phi_t^{-1}(\alpha) = \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
$$

Furthermore, $C_t$ has a Lipschitz continuous version. It is also easy to verify that every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$. Hence there exists a Liu process.

**Theorem 13.2** Let $C_t$ be a Liu process. Then for each time $t > 0$, the ratio $C_t/t$ is a normal uncertain variable with expected value 0 and variance 1. That is,

$$
\frac{C_t}{t} \sim \mathcal{N}(0, 1)
$$

for any $t > 0$.

**Proof:** Since $C_t$ is a normal uncertain variable $\mathcal{N}(0, t)$, the operational law tells us that $C_t/t$ has an uncertainty distribution

$$
\Psi(x) = \Phi_t(tx) = \left(1 + \exp \left(-\frac{\pi x}{\sqrt{3}} \right) \right)^{-1}.
$$

Hence $C_t/t$ is a normal uncertain variable with expected value 0 and variance 1. The theorem is verified.
**Theorem 13.3** (Liu [95]) Let $C_t$ be a Liu process. Then for each time $t$, we have
\[ \frac{t^2}{2} \leq E[C_t^2] \leq t^2. \] (13.4)

**Proof:** Note that $C_t$ is a normal uncertain variable and has an uncertainty distribution $\Phi_t(x)$ in (13.1). It follows from the definition of expected value that
\[ E[C_t^2] = \int_0^{+\infty} M\{C_t^2 \geq x\}dx = \int_0^{+\infty} M\{(C_t \geq \sqrt{x}) \cup (C_t \leq -\sqrt{x})\}dx. \]

On the one hand, we have
\[ E[C_t^2] \leq \int_0^{+\infty} (M\{C_t \geq \sqrt{x}\} + M\{C_t \leq -\sqrt{x}\})dx \]
\[ = \int_0^{+\infty} (1 - \Phi_t(\sqrt{x}) + \Phi_t(-\sqrt{x}))dx = t^2. \]

On the other hand, we have
\[ E[C_t^2] \geq \int_0^{+\infty} M\{C_t \geq \sqrt{x}\}dx = \int_0^{+\infty} (1 - \Phi_t(\sqrt{x}))dx = \frac{t^2}{2}. \]

Hence (13.4) is proved.

**Theorem 13.4** (Iwamura-Xu [66]) Let $C_t$ be a Liu process. Then for each time $t$, we have
\[ 1.24t^4 < V[C_t^2] < 4.31t^4. \] (13.5)

**Proof:** Let $q$ be the expected value of $C_t^2$. On the one hand, it follows from the definition of variance that
\[ V[C_t^2] = \int_0^{+\infty} M\{(C_t^2 - q)^2 \geq x\}dx \]
\[ \leq \int_0^{+\infty} M\left\{ C_t \geq \sqrt{q + \sqrt{x}} \right\} dx \]
\[ + \int_0^{+\infty} M\left\{ C_t \leq -\sqrt{q + \sqrt{x}} \right\} dx \]
\[ + \int_0^{+\infty} M\left\{ -\sqrt{q - \sqrt{x}} \leq C_t \leq \sqrt{q - \sqrt{x}} \right\} dx. \]
Since $t^2/2 \leq q \leq t^2$, we have

First Term = \( \int_0^{+\infty} M \left\{ C_t \geq \sqrt{q + \sqrt{x}} \right\} dx \)
\[ \leq \int_0^{+\infty} M \left\{ C_t \geq \sqrt{t^2/2 + \sqrt{x}} \right\} dx \]
\[ = \int_0^{+\infty} \left( 1 - \left( 1 + \exp \left( - \frac{\pi \sqrt{t^2/2 + \sqrt{x}}}{\sqrt{3t}} \right) \right)^{-1} \right) dx \]
\[ \leq 1.725t^4, \]

Second Term = \( \int_0^{+\infty} M \left\{ C_t \leq -\sqrt{q + \sqrt{x}} \right\} dx \)
\[ \leq \int_0^{+\infty} M \left\{ C_t \leq -\sqrt{t^2/2 + \sqrt{x}} \right\} dx \]
\[ = \int_0^{+\infty} \left( 1 + \exp \left( \frac{\pi \sqrt{t^2/2 + \sqrt{x}}}{\sqrt{3t}} \right) \right)^{-1} dx \]
\[ \leq 1.725t^4, \]

Third Term = \( \int_0^{+\infty} M \left\{ -\sqrt{q - \sqrt{x}} \leq C_t \leq \sqrt{q - \sqrt{x}} \right\} dx \)
\[ \leq \int_0^{+\infty} M \left\{ C_t \leq \sqrt{q - \sqrt{x}} \right\} dx \]
\[ \leq \int_0^{+\infty} M \left\{ C_t \leq \sqrt{t^2 - \sqrt{x}} \right\} dx \]
\[ = \int_0^{+\infty} \left( 1 + \exp \left( - \frac{\pi \sqrt{t^2 - \sqrt{x}}}{\sqrt{3t}} \right) \right)^{-1} dx \]
\[ < 0.86t^4. \]

It follows from the above three upper bounds that

\[ V[C_t^2] < 1.725t^4 + 1.725t^4 + 0.86t^4 = 4.31t^4. \]
On the other hand, we have

\[ V[C_t^2] = \int_{0}^{+\infty} M\{(C_t^2 - q)^2 \geq x\} \, dx \]

\[ \geq \int_{0}^{+\infty} M\left\{ C_t \geq \sqrt{q + \sqrt{x}} \right\} \, dx \]

\[ \geq \int_{0}^{+\infty} M\left\{ C_t \geq \sqrt{t^2 + \sqrt{x}} \right\} \, dx \]

\[ = \int_{0}^{+\infty} \left( 1 - \left( 1 + \exp\left( -\frac{\pi \sqrt{t^2 + \sqrt{x}}}{\sqrt{3} \pi} \right) \right)^{-1} \right) \, dx \]

\[ > 1.24t^4. \]

The theorem is thus verified. An open problem is to improve the bounds of the variance of the square of Liu process.

**Definition 13.2** Let \( C_t \) be a Liu process. Then for any real numbers \( e \) and \( \sigma > 0 \), the uncertain process

\[ A_t = et + \sigma C_t \] (13.6)

is called an arithmetic Liu process, where \( e \) is called the drift and \( \sigma \) is called the diffusion.

It is clear that the arithmetic Liu process \( A_t \) is a type of stationary independent increment process. In addition, the arithmetic Liu process \( A_t \) has a normal uncertainty distribution with expected value \( et \) and variance \( \sigma^2t^2 \), i.e.,

\[ A_t \sim \mathcal{N}(et, \sigma t) \] (13.7)

whose uncertainty distribution is

\[ \Phi_t(x) = \left( 1 + \exp\left( \frac{\pi (et - x)}{\sqrt{3} \sigma t} \right) \right)^{-1} \] (13.8)

and inverse uncertainty distribution is

\[ \Phi_t^{-1}(\alpha) = et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \] (13.9)

**Definition 13.3** Let \( C_t \) be a Liu process. Then for any real numbers \( e \) and \( \sigma > 0 \), the uncertain process

\[ G_t = \exp(et + \sigma C_t) \] (13.10)

is called a geometric Liu process, where \( e \) is called the log-drift and \( \sigma \) is called the log-diffusion.
Note that the geometric Liu process $G_t$ has a lognormal uncertainty distribution, i.e.,

$$G_t \sim \mathcal{LOGN}(et, \sigma t) \quad (13.11)$$

whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi (et - \ln x)}{\sqrt{3} \sigma t}\right)\right)^{-1} \quad (13.12)$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \exp\left(et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right). \quad (13.13)$$

Furthermore, the geometric Liu process $G_t$ has an expected value,

$$E[G_t] = \begin{cases} 
\sigma t \sqrt{3} \exp(et) \csc(\sigma t \sqrt{3}), & \text{if } t < \frac{\pi}{(\sigma \sqrt{3})} \\
+\infty, & \text{if } t \geq \frac{\pi}{(\sigma \sqrt{3})}.
\end{cases} \quad (13.14)$$

### 13.2 Liu Integral

As the most popular topic of uncertain integral, Liu integral allows us to integrate an uncertain process (the integrand) with respect to Liu process (the integrator). The result of Liu integral is another uncertain process.

**Definition 13.4 (Liu [91])** Let $X_t$ be an uncertain process and let $C_t$ be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|. \quad (13.15)$$

Then Liu integral of $X_t$ with respect to $C_t$ is defined as

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \quad (13.16)$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be integrable.

Since $X_t$ and $C_t$ are uncertain variables at each time $t$, the limit in (13.16) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process $X_t$ is integrable with respect to $C_t$ if and only if the limit in (13.16) is an uncertain variable.
Example 13.1: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (13.16) that

$$\int_0^s dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k (C_{t_{i+1}} - C_{t_i}) \equiv C_s - C_0 = C_s.$$  

That is,

$$\int_0^s dC_t = C_s. \quad (13.17)$$

Example 13.2: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (13.16) that

$$C_s^2 = \sum_{i=1}^k (C_{t_{i+1}}^2 - C_{t_i}^2)$$

$$= \sum_{i=1}^k (C_{t_{i+1}} - C_{t_i})^2 + 2 \sum_{i=1}^k C_{t_i} (C_{t_{i+1}} - C_{t_i})$$

$$\to 0 + 2 \int_0^s C_t dC_t$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t dC_t = \frac{1}{2} C_s^2. \quad (13.18)$$

Example 13.3: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (13.16) that

$$sC_s = \sum_{i=1}^k (t_{i+1}C_{t_{i+1}} - t_iC_{t_i})$$

$$= \sum_{i=1}^k C_{t_{i+1}} (t_{i+1} - t_i) + \sum_{i=1}^k t_i (C_{t_{i+1}} - C_{t_i})$$

$$\to \int_0^s C_t dt + \int_0^s t dC_t$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t dt + \int_0^s t dC_t = sC_s. \quad (13.19)$$

Theorem 13.5 If $X_t$ is a sample-continuous uncertain process on $[a, b]$, then it is integrable with respect to $C_t$ on $[a, b]$. 

**Proof:** Let \( a = t_1 < t_2 < \cdots < t_{k+1} = b \) be a partition of the closed interval \([a, b]\). Since the uncertain process \( X_t \) is sample-continuous, almost all sample paths are continuous functions with respect to \( t \). Hence the limit

\[
\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})
\]

exists almost surely and is finite. On the other hand, since \( X_t \) and \( C_t \) are uncertain variables at each time \( t \), the above limit is also a measurable function. Hence the limit is an uncertain variable and then \( X_t \) is integrable with respect to \( C_t \).

**Theorem 13.6** If \( X_t \) is an integrable uncertain process on \([a, b]\), then it is integrable on each subinterval of \([a, b]\). Moreover, if \( c \in [a, b] \), then

\[
\int_{a}^{b} X_t dC_t = \int_{a}^{c} X_t dC_t + \int_{c}^{b} X_t dC_t. \tag{13.20}
\]

**Proof:** Let \([a', b']\) be a subinterval of \([a, b]\). Since \( X_t \) is an integrable uncertain process on \([a, b]\), for any partition

\( a = t_1 < \cdots < t_m = a' < t_{m+1} < \cdots < t_n = b' < t_{n+1} < \cdots < t_{k+1} = b \),

the limit

\[
\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})
\]

exists almost surely and is finite. Thus the limit

\[
\lim_{\Delta \to 0} \sum_{i=m}^{n-1} X_{t_i}(C_{t_{i+1}} - C_{t_i})
\]

exists almost surely and is finite. Hence \( X_t \) is integrable on the subinterval \([a', b']\). Next, for the partition

\( a = t_1 < \cdots < t_m = c < t_{m+1} < \cdots < t_{k+1} = b \),

we have

\[
\sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}) = \sum_{i=1}^{m-1} X_{t_i}(C_{t_{i+1}} - C_{t_i}) + \sum_{i=m}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}).
\]

Note that

\[
\int_{a}^{b} X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}),
\]
\[
\int_a^c X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{m-1} X_{t_i}(C_{t_{i+1}} - C_{t_i}),
\]
\[
\int_c^b X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=m}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}).
\]

Hence the equation (13.20) is proved.

**Theorem 13.7 (Linearity of Liu Integral)** Let \(X_t\) and \(Y_t\) be integrable uncertain processes on \([a, b]\), and let \(\alpha\) and \(\beta\) be real numbers. Then
\[
\int_a^b (\alpha X_t + \beta Y_t) \, dC_t = \alpha \int_a^b X_t \, dC_t + \beta \int_a^b Y_t \, dC_t.
\]

**Proof:** Let \(a = t_1 < t_2 < \cdots < t_{k+1} = b\) be a partition of the closed interval \([a, b]\). It follows from the definition of Liu integral that
\[
\int_a^b (\alpha X_t + \beta Y_t) \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k (\alpha X_{t_i} + \beta Y_{t_i})(C_{t_{i+1}} - C_{t_i})
\]
\[
= \lim_{\Delta \to 0} \alpha \sum_{i=1}^k X_{t_i}(C_{t_{i+1}} - C_{t_i}) + \lim_{\Delta \to 0} \beta \sum_{i=1}^k Y_{t_i}(C_{t_{i+1}} - C_{t_i})
\]
\[
= \alpha \int_a^b X_t \, dC_t + \beta \int_a^b Y_t \, dC_t.
\]

Hence the equation (13.21) is proved.

**Theorem 13.8** Let \(f(t)\) be an integrable function with respect to \(t\). Then the Liu integral
\[
\int_0^s f(t) \, dC_t
\]
is a normal uncertain variable at each time \(s\), and
\[
\int_0^s f(t) \, dC_t \sim \mathcal{N}\left(0, \int_0^s |f(t)| \, dt\right).
\]

**Proof:** Since the increments of \(C_t\) are stationary and independent normal uncertain variables, for any partition of closed interval \([0, s]\) with \(0 = t_1 < t_2 < \cdots < t_{k+1} = s\), it follows from Theorem 2.13 that
\[
\sum_{i=1}^k f(t_i)(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \sum_{i=1}^k |f(t_i)|(t_{i+1} - t_i)\right).
\]
That is, the sum is also a normal uncertain variable. Since \(f\) is an integrable function, we have
\[
\sum_{i=1}^k |f(t_i)|(t_{i+1} - t_i) \to \int_0^s |f(t)| \, dt
\]
as the mesh $\Delta \to 0$. Hence we obtain

$$\int_0^s f(t) dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k f(t_i)(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \int_0^s |f(t)| dt\right).$$

The theorem is proved.

**Exercise 13.1:** Let $s$ be a given time with $s > 0$. Show that the Liu integral

$$\int_0^s t dC_t$$

is a normal uncertain variable $\mathcal{N}(0, s^2/2)$ and has an uncertainty distribution

$$\Phi_s(x) = \left(1 + \exp\left(-\frac{2\pi x}{\sqrt{3}s^2}\right)\right)^{-1}.$$  \hspace{1cm} (13.24)

**Exercise 13.2:** For any real number $\alpha$ with $0 < \alpha < 1$, the uncertain process

$$F_s = \int_0^s (s - t)^{-\alpha} dC_t$$

is called a *fractional Liu process* with index $\alpha$. Show that $F_s$ is a normal uncertain variable and

$$F_s \sim \mathcal{N}\left(0, \frac{s^{1-\alpha}}{1-\alpha}\right)$$

whose uncertainty distribution is

$$\Phi_s(x) = \left(1 + \exp\left(-\frac{\pi(1-\alpha)x}{\sqrt{3}s^{1-\alpha}}\right)\right)^{-1}.$$ \hspace{1cm} (13.27)

**Definition 13.5** (Chen-Ralescu [14]) Let $C_t$ be a Liu process and let $Z_t$ be an uncertain process. If there exist uncertain processes $\mu_t$ and $\sigma_t$ such that

$$Z_t = Z_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dC_s$$

for any $t \geq 0$, then $Z_t$ is called a general Liu process with drift $\mu_t$ and diffusion $\sigma_t$. Furthermore, $Z_t$ has an uncertain differential

$$dZ_t = \mu_t dt + \sigma_t dC_t.$$ \hspace{1cm} (13.29)

**Example 13.4:** It follows from the equation (13.17) that Liu process $C_t$ can be written as

$$C_t = \int_0^t dC_s.$$
Thus $C_t$ is a general Liu process with drift 0 and diffusion 1, and has an uncertain differential $dC_t$.

**Example 13.5:** It follows from the equation (13.18) that $C_t^2$ can be written as

$$C_t^2 = 2 \int_0^t C_s dC_s.$$ 

Thus $C_t^2$ is a general Liu process with drift 0 and diffusion $2C_t$, and has an uncertain differential

$$d(C_t^2) = 2C_t dC_t.$$

**Example 13.6:** It follows from the equation (13.19) that $tC_t$ can be written as

$$tC_t = \int_0^t C_s ds + \int_0^t s dC_s.$$ 

Thus $tC_t$ is a general Liu process with drift $C_t$ and diffusion $t$, and has an uncertain differential

$$d(tC_t) = C_t dt + t dC_t.$$

**Theorem 13.9** *(Chen-Ralescu [14])* Any general Liu process is a sample-continuous uncertain process.

**Proof:** Let $Z_t$ be a general Liu process with drift $\mu_t$ and diffusion $\sigma_t$. Then we immediately have

$$Z_t = Z_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dC_s.$$ 

For each $\gamma \in \Gamma$, it is obvious that

$$|Z_t(\gamma) - Z_r(\gamma)| = \left| \int_r^t \mu_s(\gamma) ds + \int_r^t \sigma_s(\gamma) dC_s(\gamma) \right| \to 0$$

as $r \to t$. Hence $Z_t$ is sample-continuous and the theorem is proved.

### 13.3 Fundamental Theorem

**Theorem 13.10** *(Liu [91], Fundamental Theorem of Uncertain Calculus)* Let $h(t,c)$ be a continuously differentiable function. Then $Z_t = h(t,C_t)$ is a general Liu process and has an uncertain differential

$$dZ_t = \frac{\partial h}{\partial t}(t,C_t) dt + \frac{\partial h}{\partial c}(t,C_t) dC_t.$$ 

(13.31)
Proof: Write $\Delta C_t = C_{t+\Delta t} - C_t = C_{\Delta t}$. It follows from Theorems 13.3 and 13.4 that $\Delta t$ and $\Delta C_t$ are infinitesimals with the same order. Since the function $h$ is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of $Z_t$ has a first-order approximation,

$$\Delta Z_t = \frac{\partial h}{\partial t}(t,C_t)\Delta t + \frac{\partial h}{\partial c}(t,C_t)\Delta C_t.$$ 

Hence we obtain the uncertain differential (13.31) because it makes

$$Z_s = Z_0 + \int_0^s \frac{\partial h}{\partial t}(t,C_t)dt + \int_0^s \frac{\partial h}{\partial c}(t,C_t)dC_t. \quad (13.32)$$

This formula is an integral form of the fundamental theorem.

**Example 13.7:** Let us calculate the uncertain differential of $tC_t$. In this case, we have $h(t,c) = tc$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t,c) = c, \quad \frac{\partial h}{\partial c}(t,c) = t.$$ 

It follows from the fundamental theorem of uncertain calculus that

$$d(tC_t) = C_t dt + tdC_t. \quad (13.33)$$

Thus $tC_t$ is a general Liu process with drift $C_t$ and diffusion $t$.

**Example 13.8:** Let us calculate the uncertain differential of the arithmetic Liu process $A_t = et + \sigma C_t$. In this case, we have $h(t,c) = et + \sigma c$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t,c) = e, \quad \frac{\partial h}{\partial c}(t,c) = \sigma.$$ 

It follows from the fundamental theorem of uncertain calculus that

$$dA_t = edt + \sigma dC_t. \quad (13.34)$$

Thus $A_t$ is a general Liu process with drift $e$ and diffusion $\sigma$.

**Example 13.9:** Let us calculate the uncertain differential of the geometric Liu process $G_t = \exp(et + \sigma C_t)$. In this case, we have $h(t,c) = \exp(et + \sigma c)$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t,c) = eh(t,c), \quad \frac{\partial h}{\partial c}(t,c) = \sigma h(t,c).$$ 

It follows from the fundamental theorem of uncertain calculus that

$$dG_t = eG_t dt + \sigma G_t dC_t. \quad (13.35)$$

Thus $G_t$ is a general Liu process with drift $eG_t$ and diffusion $\sigma G_t$. 
13.4 Chain Rule

Chain rule is a special case of the fundamental theorem of uncertain calculus.

**Theorem 13.11** (Liu [91], Chain Rule) Let \( f(c) \) be a continuously differentiable function. Then \( f(C_t) \) has an uncertain differential

\[
df(C_t) = f'(C_t) dC_t. \tag{13.36}
\]

**Proof:** Since \( f(c) \) is a continuously differentiable function, we immediately have

\[
\frac{\partial}{\partial t} f(c) = 0, \quad \frac{\partial}{\partial c} f(c) = f'(c).
\]

It follows from the fundamental theorem of uncertain calculus that the equation (13.36) holds.

**Example 13.10:** Let us calculate the uncertain differential of \( C_t^2 \). In this case, we have \( f(c) = c^2 \) and \( f'(c) = 2c \). It follows from the chain rule that

\[
dC_t^2 = 2C_t dC_t. \tag{13.37}
\]

**Example 13.11:** Let us calculate the uncertain differential of \( \sin(C_t) \). In this case, we have \( f(c) = \sin(c) \) and \( f'(c) = \cos(c) \). It follows from the chain rule that

\[
d \sin(C_t) = \cos(C_t) dC_t. \tag{13.38}
\]

**Example 13.12:** Let us calculate the uncertain differential of \( \exp(C_t) \). In this case, we have \( f(c) = \exp(c) \) and \( f'(c) = \exp(c) \). It follows from the chain rule that

\[
d \exp(C_t) = \exp(C_t) dC_t. \tag{13.39}
\]

13.5 Change of Variables

**Theorem 13.12** (Liu [91], Change of Variables) Let \( f \) be a continuously differentiable function. Then for any \( s > 0 \), we have

\[
\int_0^s f'(C_t) dC_t = \int_{C_0}^{C_s} f'(c) dc. \tag{13.40}
\]

That is,

\[
\int_0^s f'(C_t) dC_t = f(C_s) - f(C_0). \tag{13.41}
\]

**Proof:** Since \( f \) is a continuously differentiable function, it follows from the chain rule that

\[
df(C_t) = f'(C_t) dC_t.
\]
This formula implies that
\[ f(C_s) = f(C_0) + \int_0^s f'(C_t) dC_t. \]

Hence the theorem is verified.

**Example 13.13:** Since the function \( f'(c) = c \) has an antiderivative \( f(c) = c^2/2 \), it follows from the change of variables of integral that
\[ \int_0^s C_t dC_t = \frac{1}{2} C_s^2 - \frac{1}{2} C_0^2 = \frac{1}{2} C_s^2. \]

**Example 13.14:** Since the function \( f'(c) = c^2 \) has an antiderivative \( f(c) = c^3/3 \), it follows from the change of variables of integral that
\[ \int_0^s C_t^2 dC_t = \frac{1}{3} C_s^3 - \frac{1}{3} C_0^3 = \frac{1}{3} C_s^3. \]

**Example 13.15:** Since the function \( f'(c) = \exp(c) \) has an antiderivative \( f(c) = \exp(c) \), it follows from the change of variables of integral that
\[ \int_0^s \exp(C_t) dC_t = \exp(C_s) - \exp(C_0) = \exp(C_s) - 1. \]

### 13.6 Integration by Parts

**Theorem 13.13** (Liu [91], Integration by Parts) Suppose \( X_t \) and \( Y_t \) are general Liu processes. Then
\[ d(X_t Y_t) = Y_t dX_t + X_t dY_t. \] (13.42)

**Proof:** Note that \( \Delta X_t \) and \( \Delta Y_t \) are infinitesimals with the same order. Since the function \( xy \) is a continuously differentiable function with respect to \( x \) and \( y \), by using Taylor series expansion, the infinitesimal increment of \( X_t Y_t \) has a first-order approximation,
\[ \Delta(X_t Y_t) = Y_t \Delta X_t + X_t \Delta Y_t. \]

Hence we obtain the uncertain differential (13.42) because it makes
\[ X_s Y_s = X_0 Y_0 + \int_0^s Y_t dX_t + \int_0^s X_t dY_t. \] (13.43)

The theorem is thus proved.
**Example 13.16:** In order to illustrate the integration by parts, let us calculate the uncertain differential of

\[ Z_t = \exp(t)C_t^2. \]

In this case, we define

\[ X_t = \exp(t), \quad Y_t = C_t^2. \]

Then

\[ dX_t = \exp(t)dt, \quad dY_t = 2C_t dC_t. \]

It follows from the integration by parts that

\[ dZ_t = \exp(t)C_t^2 dt + 2\exp(t)C_t dC_t. \]

**Example 13.17:** The integration by parts may also calculate the uncertain differential of

\[ Z_t = \sin(t + 1) \int_0^t sdC_s. \]

In this case, we define

\[ X_t = \sin(t + 1), \quad Y_t = \int_0^t sdC_s. \]

Then

\[ dX_t = \cos(t + 1)dt, \quad dY_t = tdC_t. \]

It follows from the integration by parts that

\[ dZ_t = \left(\int_0^t sdC_s\right) \cos(t + 1)dt + \sin(t + 1)tdC_t. \]

**Example 13.18:** Let \( f \) and \( g \) be continuously differentiable functions. It is clear that

\[ Z_t = f(t)g(C_t) \]

is an uncertain process. In order to calculate the uncertain differential of \( Z_t \), we define

\[ X_t = f(t), \quad Y_t = g(C_t). \]

Then

\[ dX_t = f'(t)dt, \quad dY_t = g'(C_t)dC_t. \]

It follows from the integration by parts that

\[ dZ_t = f'(t)g(C_t)dt + f(t)g'(C_t)dC_t. \]
13.7 Bibliographic Notes

Uncertain integral was proposed by Liu [89] in 2008 in order to integrate uncertain processes with respect to Liu process. One year later, Liu [91] presented the fundamental theorem of uncertain calculus from which the techniques of chain rule, change of variables, and integration by parts were derived.

Note that uncertain integral may also be defined with respect to other integrators. For example, Yao [183] defined an uncertain integral with respect to uncertain renewal process, and Chen [17] investigated an uncertain integral with respect to finite variation processes. Since then, the theory of uncertain calculus was well developed.
Chapter 14

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation involving uncertain processes. This chapter will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and introduce Yao-Chen formula that represents the solution of an uncertain differential equation by a family of solutions of ordinary differential equations. On the basis of this formula, some formulas to calculate extreme value, first hitting time, and time integral of solution are provided. Furthermore, some numerical methods for solving general uncertain differential equations are designed.

14.1 Uncertain Differential Equation

Definition 14.1 (Liu [89]) Suppose $C_t$ is a Liu process, and $f$ and $g$ are measurable functions. Then
\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t \] (14.1)
is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies (14.1) identically in $t$.

Remark 14.1: The uncertain differential equation (14.1) is equivalent to the uncertain integral equation
\[ X_s = X_0 + \int_0^s f(t, X_t)dt + \int_0^s g(t, X_t)dC_t. \] (14.2)

Theorem 14.1 Let $u_t$ and $v_t$ be two integrable uncertain processes. Then the uncertain differential equation
\[ dX_t = u_t dt + v_t dC_t \] (14.3)
has a solution
\[ X_t = X_0 + \int_0^t u_s ds + \int_0^t v_s dC_s. \] (14.4)

**Proof:** This theorem is essentially the definition of uncertain differential or a direct deduction of the fundamental theorem of uncertain calculus.

**Example 14.1:** Let \( a \) and \( b \) be real numbers. Consider the uncertain differential equation
\[ dX_t = adt + b dC_t. \] (14.5)
It follows from Theorem 14.1 that the solution is
\[ X_t = X_0 + \int_0^t a ds + \int_0^t b dC_s. \]
That is,
\[ X_t = X_0 + at + b C_t. \] (14.6)

**Theorem 14.2** Let \( u_t \) and \( v_t \) be two integrable uncertain processes. Then the uncertain differential equation
\[ dX_t = u_t X_t dt + v_t X_t dC_t \] (14.7)
has a solution
\[ X_t = X_0 \exp \left( \int_0^t u_s ds + \int_0^t v_s dC_s \right). \] (14.8)

**Proof:** At first, the original uncertain differential equation is equivalent to
\[ \frac{dX_t}{X_t} = u_t dt + v_t dC_t. \]
It follows from the fundamental theorem of uncertain calculus that
\[ d\ln X_t = \frac{dX_t}{X_t} = u_t dt + v_t dC_t \]
and then
\[ \ln X_t = \ln X_0 + \int_0^t u_s ds + \int_0^t v_s dC_s. \]
Therefore the uncertain differential equation has a solution (14.8).

**Example 14.2:** Let \( a \) and \( b \) be real numbers. Consider the uncertain differential equation
\[ dX_t = a X_t dt + b X_t dC_t. \] (14.9)
It follows from Theorem 14.2 that the solution is
\[ X_t = X_0 \exp \left( \int_0^t a ds + \int_0^t b dC_s \right). \]
That is,
\[ X_t = X_0 \exp \left( at + b C_t \right). \] (14.10)
Linear Uncertain Differential Equation

**Theorem 14.3 (Chen-Liu [6])** Let $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ be integrable uncertain processes. Then the linear uncertain differential equation

$$dX_t = (u_{1t} X_t + u_{2t}) dt + (v_{1t} X_t + v_{2t}) dC_t$$  

(14.11)

has a solution

$$X_t = U_t \left( X_0 + \int_0^t \frac{u_{2s}}{U_s} ds + \int_0^t \frac{v_{2s}}{U_s} dC_s \right)$$  

(14.12)

where

$$U_t = \exp \left( \int_0^t u_{1s} ds + \int_0^t v_{1s} dC_s \right).$$  

(14.13)

**Proof:** At first, we define two uncertain processes $U_t$ and $V_t$ via uncertain differential equations,

$$dU_t = u_{1t} U_t dt + v_{1t} U_t dC_t, \quad dV_t = \frac{u_{2t}}{U_t} dt + \frac{v_{2t}}{U_t} dC_t.$$

It follows from the integration by parts that

$$d(U_t V_t) = V_t dU_t + U_t dV_t = (u_{1t} U_t V_t + u_{2t}) dt + (v_{1t} U_t V_t + v_{2t}) dC_t.$$

That is, the uncertain process $X_t = U_t V_t$ is a solution of the uncertain differential equation (14.11). Note that

$$U_t = U_0 \exp \left( \int_0^t u_{1s} ds + \int_0^t v_{1s} dC_s \right),$$

$$V_t = V_0 + \int_0^t \frac{u_{2s}}{U_s} ds + \int_0^t \frac{v_{2s}}{U_s} dC_s.$$

Taking $U_0 = 1$ and $V_0 = X_0$, we get the solution (14.12). The theorem is proved.

**Example 14.3:** Let $m, a, \sigma$ be real numbers. Consider a linear uncertain differential equation

$$dX_t = (m - aX_t) dt + \sigma dC_t.$$

(14.14)

At first, we have

$$U_t = \exp \left( \int_0^t (-a) ds + \int_0^t 0 dC_s \right) = \exp(-at).$$

It follows from Theorem 14.3 that the solution is

$$X_t = \exp(-at) \left( X_0 + \int_0^t m \exp(as) ds + \int_0^t \sigma \exp(as) dC_s \right).$$
That is,
\[ X_t = \frac{m}{a} + \exp(-at)\left( X_0 - \frac{m}{a} \right) + \sigma \exp(-at) \int_0^t \exp(as)dC_s \] (14.15)
provided that \( a \neq 0 \). Note that \( X_t \) is a normal uncertain variable, i.e.,
\[ X_t \sim \mathcal{N}\left( \frac{m}{a} + \exp(-at)\left( X_0 - \frac{m}{a} \right), \frac{\sigma}{a} - \exp(-at)\frac{\sigma}{a} \right). \] (14.16)

**Example 14.4:** Let \( m \) and \( \sigma \) be real numbers. Consider a linear uncertain differential equation
\[ dX_t = mdt + \sigma X_t dC_t. \] (14.17)
At first, we have
\[ U_t = \exp\left( \int_0^t 0ds + \int_0^t \sigma dC_s \right) = \exp(\sigma C_t). \]
It follows from Theorem 14.3 that the solution is
\[ X_t = \exp(\sigma C_t)\left( X_0 + \int_0^t m \exp(-\sigma C_s)ds + \int_0^t 0dC_s \right). \]
That is,
\[ X_t = \exp(\sigma C_t)\left( X_0 + m \int_0^t \exp(-\sigma C_s)ds \right). \] (14.18)

### 14.2 Analytic Methods

This section will provide two analytic methods for solving some nonlinear uncertain differential equations.

**First Analytic Method**

This subsection will introduce an analytic method for solving nonlinear uncertain differential equations like
\[ dX_t = f(t, X_t)dt + \sigma_t X_t dC_t \] (14.19)
and
\[ dX_t = \alpha_t X_t dt + g(t, X_t) dC_t. \] (14.20)

**Theorem 14.4** (Liu [116]) Let \( f \) be a measurable function of two variables, and let \( \sigma_t \) be an integrable uncertain process. Then the uncertain differential equation
\[ dX_t = f(t, X_t) dt + \sigma_t X_t dC_t \] (14.21)
has a solution
\[ X_t = Y_t^{-1}Z_t \]  \hspace{1cm} (14.22)

where
\[ Y_t = \exp\left(-\int_0^t \sigma_s dC_s\right) \]  \hspace{1cm} (14.23)

and \( Z_t \) is the solution of the uncertain differential equation
\[ dZ_t = Y_t f(t, Y_t^{-1}Z_t) dt \]  \hspace{1cm} (14.24)

with initial value \( Z_0 = X_0 \).

**Proof:** At first, by using the chain rule, the uncertain process \( Y_t \) has an uncertain differential
\[ dY_t = -\exp\left(-\int_0^t \sigma_s dC_s\right) \sigma_t dC_t = -Y_t \sigma_t dC_t. \]

It follows from the integration by parts that
\[ d(X_tY_t) = X_t dY_t + Y_t dX_t = -X_t Y_t \sigma_t dC_t + Y_t f(t, X_t) dt + Y_t \sigma_t X_t dC_t. \]

That is,
\[ d(X_tY_t) = Y_t f(t, X_t) dt. \]

Defining \( Z_t = X_t Y_t \), we obtain \( X_t = Y_t^{-1}Z_t \) and \( dZ_t = Y_t f(t, Y_t^{-1}Z_t) dt \). Furthermore, since \( Y_0 = 1 \), the initial value \( Z_0 \) is just \( X_0 \). The theorem is thus verified.

**Example 14.5:** Let \( \alpha \) and \( \sigma \) be real numbers with \( \alpha \neq 1 \). Consider the uncertain differential equation
\[ dX_t = X_t^\alpha dt + \sigma X_t dC_t. \]  \hspace{1cm} (14.25)

At first, we have \( Y_t = \exp(-\sigma C_t) \) and \( Z_t \) satisfies the uncertain differential equation,
\[ dZ_t = \exp(-\sigma C_t)(\exp(\sigma C_t)Z_t)^\alpha dt = \exp((\alpha - 1)\sigma C_t)Z_t^\alpha dt. \]

Since \( \alpha \neq 1 \), we have
\[ dZ_t^{1-\alpha} = (1 - \alpha) \exp((\alpha - 1)\sigma C_t) dt. \]

It follows from the fundamental theorem of uncertain calculus that
\[ Z_t^{1-\alpha} = Z_0^{1-\alpha} + (1 - \alpha) \int_0^t \exp((\alpha - 1)\sigma C_s) ds. \]

Since the initial value \( Z_0 \) is just \( X_0 \), we have
\[ Z_t = \left(X_0^{1-\alpha} + (1 - \alpha) \int_0^t \exp((\alpha - 1)\sigma C_s) ds\right)^{1/(1-\alpha)}. \]
Theorem 14.4 says the uncertain differential equation (14.25) has a solution
\[ X_t = Y_t^{-1}Z_t, \]
i.e.,
\[ X_t = \exp(\sigma C_t) \left( X_0^{1-\alpha} + (1 - \alpha) \int_0^t \exp((\alpha - 1)\sigma C_s)ds \right)^{1/(1-\alpha)}. \]

**Theorem 14.5** (Liu [116]) Let \( g \) be a measurable function of two variables, and let \( \alpha_t \) be an integrable uncertain process. Then the uncertain differential equation
\[ dX_t = \alpha_t X_t dt + g(t, X_t) dC_t \]
has a solution
\[ X_t = Y_t^{-1}Z_t \]
where
\[ Y_t = \exp \left( -\int_0^t \alpha_s ds \right) \]
and \( Z_t \) is the solution of the uncertain differential equation
\[ dZ_t = Y_t g(t, Y_t^{-1}Z_t) dC_t \]
with initial value \( Z_0 = X_0 \).

**Proof:** At first, by using the chain rule, the uncertain process \( Y_t \) has an uncertain differential
\[ dY_t = -\exp \left( -\int_0^t \alpha_s ds \right) \alpha_t dt = -Y_t \alpha_t dt. \]
It follows from the integration by parts that
\[ d(X_t Y_t) = X_t dY_t + Y_t dX_t = -X_t Y_t \alpha_t dt + Y_t \alpha_t X_t dt + Y_t g(t, X_t) dC_t. \]
That is,
\[ d(X_t Y_t) = Y_t g(t, X_t) dC_t. \]
Defining \( Z_t = X_t Y_t \), we obtain \( X_t = Y_t^{-1}Z_t \) and \( dZ_t = Y_t g(t, Y_t^{-1}Z_t) dC_t \). Furthermore, since \( Y_0 = 1 \), the initial value \( Z_0 \) is just \( X_0 \). The theorem is thus verified.

**Example 14.6:** Let \( \alpha \) and \( \beta \) be real numbers with \( \beta \neq 1 \). Consider the uncertain differential equation
\[ dX_t = \alpha X_t dt + X_t^\beta dC_t. \]
At first, we have \( Y_t = \exp(-\alpha t) \) and \( Z_t \) satisfies the uncertain differential equation,
\[ dZ_t = \exp(-\alpha t)(\exp(\alpha t)Z_t)^\beta dC_t = \exp((\beta - 1)\alpha t)Z_t^\beta dC_t. \]
Since $\beta \neq 1$, we have
\[ dZ_t^{1-\beta} = (1 - \beta) \exp((\beta - 1)\alpha t) dC_t. \]

It follows from the fundamental theorem of uncertain calculus that
\[ Z_t^{1-\beta} = Z_0^{1-\beta} + (1 - \beta) \int_0^t \exp((\beta - 1)\alpha s) dC_s. \]

Since the initial value $Z_0$ is just $X_0$, we have
\[ Z_t = \left( X_0^{1-\beta} + (1 - \beta) \int_0^t \exp((\beta - 1)\alpha s) dC_s \right)^{1/(1-\beta)}. \]

Theorem 14.5 says the uncertain differential equation (14.30) has a solution $X_t = Y_t^{-1} Z_t$, i.e.,
\[ X_t = \exp(\alpha t) \left( X_0^{1-\beta} + (1 - \beta) \int_0^t \exp((\beta - 1)\alpha s) dC_s \right)^{1/(1-\beta)}. \]

**Second Analytic Method**

This subsection will introduce an analytic method for solving nonlinear uncertain differential equations like
\[ dX_t = f(t, X_t) dt + \sigma_t dC_t \quad (14.31) \]
and
\[ dX_t = \alpha_t dt + g(t, X_t) dC_t. \quad (14.32) \]

**Theorem 14.6** (Yao [189]) Let $f$ be a measurable function of two variables, and let $\sigma_t$ be an integrable uncertain process. Then the uncertain differential equation
\[ dX_t = f(t, X_t) dt + \sigma_t dC_t \quad (14.33) \]
has a solution
\[ X_t = Y_t + Z_t \quad (14.34) \]
where
\[ Y_t = \int_0^t \sigma_s dC_s \quad (14.35) \]
and $Z_t$ is the solution of the uncertain differential equation
\[ dZ_t = f(t, Y_t + Z_t) dt \quad (14.36) \]
with initial value $Z_0 = X_0$. 
**Proof:** At first, $Y_t$ has an uncertain differential $dY_t = \sigma_t dC_t$. It follows that

$$d(X_t - Y_t) = dX_t - dY_t = f(t, X_t)dt + \sigma_t dC_t - \sigma_t dC_t.$$  

That is,

$$d(X_t - Y_t) = f(t, X_t)dt.$$  

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $dZ_t = f(t, Y_t + Z_t)dt$. Furthermore, since $Y_0 = 0$, the initial value $Z_0$ is just $X_0$. The theorem is proved.

**Example 14.7:** Let $\alpha$ and $\sigma$ be real numbers with $\alpha \neq 0$. Consider the uncertain differential equation

$$dX_t = \alpha \exp(X_t)dt + \sigma dC_t.$$  

At first, we have $Y_t = \sigma C_t$ and $Z_t$ satisfies the uncertain differential equation,

$$dZ_t = \alpha \exp(\sigma t + Z_t)dt.$$  

Since $\alpha \neq 0$, we have

$$d \exp(-Z_t) = -\alpha \exp(\sigma t)dt.$$  

It follows from the fundamental theorem of uncertain calculus that

$$\exp(-Z_t) = \exp(-Z_0) - \alpha \int_0^t \exp(\sigma s)ds.$$  

Since the initial value $Z_0$ is just $X_0$, we have

$$Z_t = X_0 - \ln \left(1 - \alpha \int_0^t \exp(X_0 + \sigma s)ds\right).$$  

Hence

$$X_t = X_0 + \sigma C_t - \ln \left(1 - \alpha \int_0^t \exp(X_0 + \sigma s)ds\right).$$  

**Theorem 14.7** (Yao [189]) Let $g$ be a measurable function of two variables, and let $\alpha_t$ be an integrable uncertain process. Then the uncertain differential equation

$$dX_t = \alpha_t dt + g(t, X_t) dC_t$$  

has a solution

$$X_t = Y_t + Z_t$$  

where

$$Y_t = \int_0^t \alpha_s ds$$  

and $Z_t$ is the solution of the uncertain differential equation

$$dZ_t = g(t, Y_t + Z_t) dC_t$$  

with initial value $Z_0 = X_0$. 
Section 14.3 - Existence and Uniqueness

Proof: The uncertain process $Y_t$ has an uncertain differential $dY_t = \alpha_t dt$. It follows that
\[ d(X_t - Y_t) = dX_t - dY_t = \alpha_t dt + g(t, X_t) dC_t - \alpha_t dt. \]
That is,
\[ d(X_t - Y_t) = g(t, X_t) dC_t. \]
Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $dZ_t = g(t, Y_t + Z_t) dC_t$.
Furthermore, since $Y_0 = 0$, the initial value $Z_0$ is just $X_0$. The theorem is proved.

Example 14.8: Let $\alpha$ and $\sigma$ be real numbers with $\sigma \neq 0$. Consider the uncertain differential equation
\[ dX_t = \alpha dt + \sigma \exp(X_t) dC_t. \] (14.42)
At first, we have $Y_t = \alpha t$ and $Z_t$ satisfies the uncertain differential equation,
\[ dZ_t = \sigma \exp(\alpha t + Z_t) dC_t. \]
Since $\sigma \neq 0$, we have
\[ d \exp(-Z_t) = -\sigma \exp(\alpha t) dC_t. \]
It follows from the fundamental theorem of uncertain calculus that
\[ \exp(-Z_t) = \exp(-Z_0) - \sigma \int_0^t \exp(\alpha s) dC_s. \]
Since the initial value $Z_0$ is just $X_0$, we have
\[ Z_t = X_0 - \ln \left(1 - \sigma \int_0^t \exp(X_0 + \alpha s) dC_s\right). \]
Hence
\[ X_t = X_0 + \alpha t - \ln \left(1 - \sigma \int_0^t \exp(X_0 + \alpha s) dC_s\right). \]

14.3 Existence and Uniqueness

Theorem 14.8 (Chen-Liu [6], Existence and Uniqueness Theorem) The uncertain differential equation
\[ dX_t = f(t, X_t) dt + g(t, X_t) dC_t \] (14.43)
has a unique solution if the coefficients $f(t, x)$ and $g(t, x)$ satisfy the linear growth condition
\[ |f(t, x)| + |g(t, x)| \leq L(1 + |x|), \quad \forall x \in \mathbb{R}, t \geq 0 \] (14.44)
and Lipschitz condition

\[ |f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0 \quad (14.45) \]

for some constant \( L \). Moreover, the solution is sample-continuous.

**Proof:** We first prove the existence of solution by a successive approximation method. Define \( X_t^{(0)} = X_0 \), and

\[
X_t^{(n)} = X_0 + \int_0^t f(s, X_s^{(n-1)}) \, ds + \int_0^t g(s, X_s^{(n-1)}) \, dC_s
\]

for \( n = 1, 2, \cdots \) and write

\[
D_t^{(n)}(\gamma) = \max_{0 \leq s \leq t} \left| X_s^{(n+1)}(\gamma) - X_s^{(n)}(\gamma) \right|
\]

for each \( \gamma \in \Gamma \). It follows from the linear growth condition and Lipschitz condition that

\[
D_t^{(0)}(\gamma) = \max_{0 \leq s \leq t} \left| \int_0^s f(v, X_0) \, dv + \int_0^s g(v, X_0) \, dC_v(\gamma) \right|
\]

\[
\leq \int_0^t |f(v, X_0)| \, dv + K_\gamma \int_0^t |g(v, X_0)| \, dv
\]

\[
\leq (1 + |X_0|)L(1 + K_\gamma)t
\]

where \( K_\gamma \) is the Lipschitz constant to the sample path \( C_t(\gamma) \). In fact, by using the induction method, we may verify

\[
D_t^{(n)}(\gamma) \leq (1 + |X_0|) \frac{L^{n+1}(1 + K_\gamma)^{n+1}}{(n + 1)!} t^{n+1}
\]

for each \( n \). This means that, for each \( \gamma \in \Gamma \), the sample paths \( X_t^{(k)}(\gamma) \) converges uniformly on any given time interval. Write the limit by \( X_t(\gamma) \) that is just a solution of the uncertain differential equation because

\[
X_t = X_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dC_s.
\]

Next we prove that the solution is unique. Assume that both \( X_t \) and \( X_t^* \) are solutions of the uncertain differential equation. Then for each \( \gamma \in \Gamma \), it follows from the linear growth condition and Lipschitz condition that

\[
|X_t(\gamma) - X_t^*(\gamma)| \leq L(1 + K_\gamma) \int_0^t |X_v(\gamma) - X_v^*(\gamma)| \, dv.
\]

By using Gronwall inequality, we obtain

\[
|X_t(\gamma) - X_t^*(\gamma)| \leq 0 \cdot \exp(L(1 + K_\gamma)t) = 0.
\]
Hence \(X_t = X^*_t\). The uniqueness is verified. Finally, for each \(\gamma \in \Gamma\), we have
\[
|X_t(\gamma) - X_r(\gamma)| = \left| \int_r^t f(s, X_s(\gamma))ds + \int_r^t g(s, X_s(\gamma))dC_s(\gamma) \right| \to 0
\]
as \(r \to t\). Thus \(X_t\) is sample-continuous and the theorem is proved.

### 14.4 Stability

**Definition 14.2** (Liu [91]) An uncertain differential equation is said to be stable if for any two solutions \(X_t\) and \(Y_t\), we have
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = 1 \quad (14.46)
\]
for any given number \(\varepsilon > 0\).

**Example 14.9:** In order to illustrate the concept of stability, let us consider the uncertain differential equation
\[
dX_t = adt + b dC_t. \quad (14.47)
\]
It is clear that two solutions with initial values \(X_0\) and \(Y_0\) are
\[
X_t = X_0 + at + bC_t,
Y_t = Y_0 + at + bC_t.
\]
Then for any given number \(\varepsilon > 0\), we have
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = \lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_0 - Y_0| < \varepsilon\} = 1.
\]
Hence the uncertain differential equation (14.47) is stable.

**Example 14.10:** Some uncertain differential equations are not stable. For example, consider
\[
dX_t = X_t dt + b dC_t. \quad (14.48)
\]
It is clear that two solutions with different initial values \(X_0\) and \(Y_0\) are
\[
X_t = \exp(t)X_0 + b \exp(t) \int_0^t \exp(-s)dC_s,
Y_t = \exp(t)Y_0 + b \exp(t) \int_0^t \exp(-s)dC_s.
\]
Then for any given number \(\varepsilon > 0\), we have
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = \lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{\exp(t)|X_0 - Y_0| < \varepsilon \text{ for all } t \geq 0\} = 0.
\]
Hence the uncertain differential equation (14.48) is unstable.
Theorem 14.9 (Yao-Gao-Gao [185], Stability Theorem) The uncertain differential equation
\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t \quad (14.49) \]

is stable if the coefficients \( f(t, x) \) and \( g(t, x) \) satisfy the linear growth condition
\[ |f(t, x)| + |g(t, x)| \leq K(1 + |x|), \quad \forall x \in \mathbb{R}, t \geq 0 \quad (14.50) \]

for some constant \( K \) and strong Lipschitz condition
\[ |f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq L(t)|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0 \quad (14.51) \]

for some bounded and integrable function \( L(t) \) on \([0, +\infty)\).

Proof: Since \( L(t) \) is bounded on \([0, +\infty)\), there is a constant \( R \) such that \( L(t) \leq R \) for any \( t \). Then the strong Lipschitz condition (14.51) implies the following Lipschitz condition,
\[ |f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq R|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0. \quad (14.52) \]

It follows from the linear growth condition (14.50), the Lipschitz condition (14.52) and the existence and uniqueness theorem that the uncertain differential equation (14.49) has a unique solution. Let \( X_t \) and \( Y_t \) be two solutions with initial values \( X_0 \) and \( Y_0 \), respectively. Then for each \( \gamma \), we have
\[
\begin{align*}
|X_t(\gamma) - Y_t(\gamma)| &\leq |f(t, X_t(\gamma)) - f(t, Y_t(\gamma))| + |g(t, X_t(\gamma)) - g(t, Y_t(\gamma))|
\leq L(t)|X_t(\gamma) - Y_t(\gamma)|dt + L(t)K(\gamma)|X_t(\gamma) - Y_t(\gamma)|dt
= L(t)(1 + K(\gamma))|X_t(\gamma) - Y_t(\gamma)|dt
\end{align*}
\]

where \( K(\gamma) \) is the Lipschitz constant of the sample path \( C_t(\gamma) \). It follows that
\[ |X_t(\gamma) - Y_t(\gamma)| \leq |X_0 - Y_0|\exp\left((1 + K(\gamma))\int_0^{+\infty} L(s)ds\right). \]

Thus for any given \( \varepsilon > 0 \), we always have
\[
\mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\}
\geq \mathcal{M}\left\{|X_0 - Y_0|\exp\left((1 + K(\gamma))\int_0^{+\infty} L(s)ds\right) < \varepsilon\right\}.
\]
Since
\[
\mathcal{M}\left\{|X_0 - Y_0|\exp\left((1 + K(\gamma))\int_0^{+\infty} L(s)ds\right) < \varepsilon\right\} \to 1
\]
as \( |X_0 - Y_0| \to 0 \), we obtain
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = 1.
\]
Hence the uncertain differential equation is stable.

**Exercise 14.1:** Suppose $u_1, u_2, v_1, v_2$ are bounded functions with respect to $t$ such that
\[
\int_0^{+\infty} |u_1| \, dt < +\infty, \quad \int_0^{+\infty} |v_1| \, dt < +\infty. \tag{14.53}
\]

Show that the linear uncertain differential equation
\[
dX_t = (u_1 X_t + u_2) \, dt + (v_1 X_t + v_2) \, dC_t \tag{14.54}
\]
is stable.

### 14.5 $\alpha$-Path

**Definition 14.3** (Yao-Chen [188]) Let $\alpha$ be a number with $0 < \alpha < 1$. An uncertain differential equation
\[
dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t \tag{14.55}
\]
is said to have an $\alpha$-path $X^\alpha_t$ if it solves the corresponding ordinary differential equation
\[
dX^\alpha_t = f(t, X^\alpha_t) \, dt + |g(t, X^\alpha_t)| \Phi^{-1}(\alpha) \, dt \tag{14.56}
\]
where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,
\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{14.57}
\]

**Remark 14.2:** Note that each $\alpha$-path $X^\alpha_t$ is a real-valued function of time $t$, but is not necessarily one of sample paths. Furthermore, almost all $\alpha$-paths are continuous functions with respect to time $t$.

**Example 14.11:** The uncertain differential equation $dX_t = adt + bdC_t$ with $X_0 = 0$ has an $\alpha$-path
\[
X^\alpha_t = at + |b| \Phi^{-1}(\alpha) t \tag{14.58}
\]
where $\Phi^{-1}$ is the inverse standard normal uncertainty distribution.

**Example 14.12:** The uncertain differential equation $dX_t = aX_t dt + bX_t dC_t$ with $X_0 = 1$ has an $\alpha$-path
\[
X^\alpha_t = \exp \left( at + |b| \Phi^{-1}(\alpha) t \right) \tag{14.59}
\]
where $\Phi^{-1}$ is the inverse standard normal uncertainty distribution.
14.6 Yao-Chen Formula

Yao-Chen formula relates uncertain differential equations and ordinary differential equations, just like that Feynman-Kac formula relates stochastic differential equations and partial differential equations.

**Theorem 14.10 (Yao-Chen Formula [188])** Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

(14.60)

respectively. Then

$$M\{X_t \leq X_t^\alpha, \forall t\} = \alpha,$$

(14.61)

$$M\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$  

(14.62)

**Proof:** At first, for each $\alpha$-path $X_t^\alpha$, we divide the time interval into two parts,

$$T^+ = \{t \mid g(t, X_t^\alpha) \geq 0\},$$

$$T^- = \{t \mid g(t, X_t^\alpha) < 0\}.$$  

It is obvious that $T^+ \cap T^- = \emptyset$ and $T^+ \cup T^- = [0, +\infty)$. Write

$$\Lambda_1^+ = \left\{ \gamma \mid \frac{dC_t(\gamma)}{dt} \leq \Phi^{-1}(\alpha) \text{ for any } t \in T^+ \right\},$$
\[ \Lambda_1^- = \left\{ \gamma \mid \frac{dC_t(\gamma)}{dt} \geq \Phi^{-1}(1 - \alpha) \text{ for any } t \in T^- \right\} \]

where \( \Phi^{-1} \) is the inverse standard normal uncertainty distribution. Since \( T^+ \) and \( T^- \) are disjoint sets and \( C_t \) has independent increments, we get

\[ M\{\Lambda_1^+\} = \alpha, \quad M\{\Lambda_1^-\} = \alpha, \quad M\{\Lambda_1^+ \cap \Lambda_1^-\} = \alpha. \]

For any \( \gamma \in \Lambda_1^+ \cap \Lambda_1^- \), we always have

\[ g(t, X_t(\gamma)) \frac{dC_t(\gamma)}{dt} \leq |g(t, X^\alpha_t)| \Phi^{-1}(\alpha), \forall t. \]

Hence \( X_t(\gamma) \leq X^\alpha_t \) for all \( t \) and

\[ M\{X_t \leq X^\alpha_t, \forall t\} \geq M\{\Lambda_1^+ \cap \Lambda_1^-\} = \alpha. \quad (14.63) \]

On the other hand, let us define

\[ \Lambda_2^+ = \left\{ \gamma \mid \frac{dC_t(\gamma)}{dt} > \Phi^{-1}(\alpha) \text{ for any } t \in T^+ \right\}, \]

\[ \Lambda_2^- = \left\{ \gamma \mid \frac{dC_t(\gamma)}{dt} < \Phi^{-1}(1 - \alpha) \text{ for any } t \in T^- \right\}. \]

Since \( T^+ \) and \( T^- \) are disjoint sets and \( C_t \) has independent increments, we obtain

\[ M\{\Lambda_2^+\} = 1 - \alpha, \quad M\{\Lambda_2^-\} = 1 - \alpha, \quad M\{\Lambda_2^+ \cap \Lambda_2^-\} = 1 - \alpha. \]

For any \( \gamma \in \Lambda_2^+ \cap \Lambda_2^- \), we always have

\[ g(t, X_t(\gamma)) \frac{dC_t(\gamma)}{dt} > |g(t, X^\alpha_t)| \Phi^{-1}(\alpha), \forall t. \]

Hence \( X_t(\gamma) > X^\alpha_t \) for all \( t \) and

\[ M\{X_t > X^\alpha_t, \forall t\} \geq M\{\Lambda_2^+ \cap \Lambda_2^-\} = 1 - \alpha. \quad (14.64) \]

Note that \( \{X_t \leq X^\alpha_t, \forall t\} \) and \( \{X_t \not\leq X^\alpha_t, \forall t\} \) are opposite events with each other. By using the duality axiom, we obtain

\[ M\{X_t \leq X^\alpha_t, \forall t\} + M\{X_t \not\leq X^\alpha_t, \forall t\} = 1. \]

It follows from \( \{X_t > X^\alpha_t, \forall t\} \subset \{X_t \not\leq X^\alpha_t, \forall t\} \) and monotonicity theorem that

\[ M\{X_t \leq X^\alpha_t, \forall t\} + M\{X_t > X^\alpha_t, \forall t\} \leq 1. \quad (14.65) \]

Thus (14.61) and (14.62) follow from (14.63), (14.64) and (14.65) immediately.
Remark 14.3: Please mention that \( \{X_t \leq X_t^\alpha, \forall t\} \) and \( \{X_t > X_t^\alpha, \forall t\} \) are disjoint events but not opposite. That is, their union does not make the universal set. However, we always have
\[
M\{X_t \leq X_t^\alpha, \forall t\} + M\{X_t > X_t^\alpha, \forall t\} = 1.
\] (14.66)

Remark 14.4: It is also showed that for any \( \alpha \in (0, 1) \), the following two equations are true,
\[
M\{X_t < X_t^\alpha, \forall t\} = \alpha,
\] (14.67)
\[
M\{X_t \geq X_t^\alpha, \forall t\} = 1 - \alpha.
\] (14.68)

Uncertainty Distribution of Solution

Theorem 14.11 (Yao-Chen [188]) Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of the uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t,
\] (14.69)
respectively. Then \( X_t \) has an inverse uncertainty distribution
\[
\Psi_t^{-1}(\alpha) = X_t^\alpha.
\] (14.70)

Proof: Note that \( \{X_t \leq X_t^\alpha\} \supset \{X_s \leq X_s^\alpha, \forall s\} \) holds for each \( t \). By using the monotonicity theorem and Yao-Chen formula, we obtain
\[
M\{X_t \leq X_t^\alpha\} \geq M\{X_s \leq X_s^\alpha, \forall s\} = \alpha.
\] (14.71)

Similarly, we also have
\[
M\{X_t > X_t^\alpha\} \geq M\{X_s > X_s^\alpha, \forall s\} = 1 - \alpha.
\] (14.72)

Since \( \{X_t \leq X_t^\alpha\} \) and \( \{X_t > X_t^\alpha\} \) are opposite events for each \( t \), the duality axiom makes
\[
M\{X_t \leq X_t^\alpha\} + M\{X_t > X_t^\alpha\} = 1.
\] (14.73)

It follows from (14.71), (14.72) and (14.73) that \( M\{X_t \leq X_t^\alpha\} = \alpha \). Thus \( \Psi_t^{-1}(\alpha) = X_t^\alpha \) is the inverse uncertainty distribution of \( X_t \).

Exercise 14.2: Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively, and let \( J \) be a continuous and strictly increasing function. Show that \( J(X_t) \) has an inverse uncertainty distribution
\[
\Psi_t^{-1}(\alpha) = J(X_t^\alpha).
\] (14.74)

Exercise 14.3: Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively, and let \( J \) be a continuous and strictly decreasing function. Show that \( J(X_t) \) has an inverse uncertainty distribution
\[
\Psi_t^{-1}(\alpha) = J(X_t^{1-\alpha}).
\] (14.75)
Expected Value of Solution

**Theorem 14.12** (Yao-Chen [188]) Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (14.76)$$

respectively. Then

$$E[X_t] = \int_0^1 X_t^\alpha d\alpha. \quad (14.77)$$

**Proof:** Yao-Chen formula says that $X_t$ has an inverse uncertainty distribution $\Psi_t^{-1}(\alpha) = X_t^\alpha$. It follows from Theorem 2.26 that (14.77) holds.

**Exercise 14.4:** Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of an uncertain differential equation, respectively, and let $J$ be a continuous and monotone (increasing or decreasing) function. Show that

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha) d\alpha. \quad (14.78)$$

Extreme Value of Solution

**Theorem 14.13** (Yao [186]) Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (14.79)$$

respectively. Then for any time $s > 0$, the supremum

$$\sup_{0 \leq t \leq s} X_t \quad (14.80)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} X_t^\alpha; \quad (14.81)$$

and the infimum

$$\inf_{0 \leq t \leq s} X_t \quad (14.82)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \inf_{0 \leq t \leq s} X_t^\alpha. \quad (14.83)$$

**Proof:** For any given time $s > 0$, it follows from the basic property of extreme value that

$$\left\{ \sup_{0 \leq t \leq s} X_t \leq \sup_{0 \leq t \leq s} X_t^\alpha \right\} \supset \{ X_t \leq X_t^\alpha, \forall t \}. \quad (14.84)$$
By using Yao-Chen formula, we obtain
\[ M \left\{ \sup_{0 \leq t \leq s} X_t \leq \sup_{0 \leq t \leq s} X_t^\alpha \right\} \geq M \{ X_t \leq X_t^\alpha, \forall t \} = \alpha. \tag{14.84} \]

Similarly, we have
\[ M \left\{ \sup_{0 \leq t \leq s} X_t > \sup_{0 \leq t \leq s} X_t^\alpha \right\} \geq M \{ X_t > X_t^\alpha, \forall t \} = 1 - \alpha. \tag{14.85} \]

It follows from (14.84), (14.85) and the duality axiom that
\[ M \left\{ \sup_{0 \leq t \leq s} X_t \leq \sup_{0 \leq t \leq s} X_t^\alpha \right\} = \alpha \tag{14.86} \]

which proves (14.81). Next, it follows from the basic property of extreme value that
\[ \left\{ \inf_{0 \leq t \leq s} X_t \leq \inf_{0 \leq t \leq s} X_t^\alpha \right\} \supset \{ X_t \leq X_t^\alpha, \forall t \}. \]

By using Yao-Chen formula, we obtain
\[ M \left\{ \inf_{0 \leq t \leq s} X_t \leq \inf_{0 \leq t \leq s} X_t^\alpha \right\} \geq M \{ X_t \leq X_t^\alpha, \forall t \} = \alpha. \tag{14.87} \]

Similarly, we have
\[ M \left\{ \inf_{0 \leq t \leq s} X_t > \inf_{0 \leq t \leq s} X_t^\alpha \right\} \geq M \{ X_t > X_t^\alpha, \forall t \} = 1 - \alpha. \tag{14.88} \]

It follows from (14.87), (14.88) and the duality axiom that
\[ M \left\{ \inf_{0 \leq t \leq s} X_t \leq \inf_{0 \leq t \leq s} X_t^\alpha \right\} = \alpha \tag{14.89} \]

which proves (14.83). The theorem is thus verified.

**Exercise 14.5:** Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively. Assume \( J \) is a continuous and strictly increasing function. For any time \( s > 0 \), show that the supremum
\[ \sup_{0 \leq t \leq s} J(X_t) \tag{14.90} \]

has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} J(X_t^\alpha); \tag{14.91} \]

and the infimum
\[ \inf_{0 \leq t \leq s} J(X_t) \tag{14.92} \]
has an inverse uncertainty distribution

\[ \Psi_s^{-1}(\alpha) = \inf_{0 \leq t \leq s} J(X^\alpha_t). \] (14.93)

**Exercise 14.6:** Let \( X_t \) and \( X^\alpha_t \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively. Assume \( J \) is a continuous and strictly decreasing function. For any time \( s > 0 \), show that the supremum

\[ \sup_{0 \leq t \leq s} J(X_t) \] (14.94)

has an inverse uncertainty distribution

\[ \Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} J(X^{1-\alpha}_t); \] (14.95)

and the infimum

\[ \inf_{0 \leq t \leq s} J(X_t) \] (14.96)

has an inverse uncertainty distribution

\[ \Psi_s^{-1}(\alpha) = \inf_{0 \leq t \leq s} J(X^{1-\alpha}_t). \] (14.97)

**First Hitting Time of Solution**

**Theorem 14.14** (Yao [186]) Let \( X_t \) and \( X^\alpha_t \) be the solution and \( \alpha \)-path of the uncertain differential equation

\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \] (14.98)

respectively. Then for any given level \( z \), the first hitting time \( \tau_z \) that \( X_t \) reaches \( z \) has an uncertainty distribution

\[ \Psi(s) = \begin{cases} 1 - \inf \left\{ \alpha \mid \sup_{0 \leq t \leq s} X^\alpha_t \geq z \right\}, & \text{if } z > X_0 \\ \sup \left\{ \alpha \mid \inf_{0 \leq t \leq s} X^\alpha_t \leq z \right\}, & \text{if } z < X_0. \end{cases} \] (14.99)

**Proof:** At first, assume \( z > X_0 \) and write

\[ \alpha_0 = \inf \left\{ \alpha \mid \sup_{0 \leq t \leq s} X^\alpha_t \geq z \right\}. \]

Then

\[ \sup_{0 \leq t \leq s} X^\alpha_{\alpha_0} = z, \]
\{ \tau_z \leq s \} = \left\{ \sup_{0 \leq t \leq s} X_t \geq z \right\} \supset \{ X_t \geq X_t^{\alpha_0}, \forall t \},

\{ \tau_z > s \} = \left\{ \sup_{0 \leq t \leq s} X_t < z \right\} \supset \{ X_t < X_t^{\alpha_0}, \forall t \}.

By using Yao-Chen formula, we obtain

\[ M\{ \tau_z \leq s \} \geq M\{ X_t \geq X_t^{\alpha_0}, \forall t \} = 1 - \alpha_0, \]

\[ M\{ \tau_z > s \} \geq M\{ X_t < X_t^{\alpha_0}, \forall t \} = \alpha_0. \]

It follows from \( M\{ \tau_z \leq s \} + M\{ \tau_z > s \} = 1 \) that \( M\{ \tau_z \leq s \} = 1 - \alpha_0 \). Hence the first hitting time \( \tau_z \) has an uncertainty distribution

\[ \Psi(s) = M\{ \tau_z \leq s \} = 1 - \alpha_0 = 1 - \inf \left\{ \alpha \mid \sup_{0 \leq t \leq s} X_t^{\alpha} \geq z \right\}. \]

Similarly, assume \( z < X_0 \) and write

\[ \alpha_0 = \sup \left\{ \alpha \mid \inf_{0 \leq t \leq s} X_t^{\alpha} \leq z \right\}. \]

Then

\[ \inf_{0 \leq t \leq s} X_t^{\alpha_0} = z, \]

\[ \{ \tau_z \leq s \} = \left\{ \inf_{0 \leq t \leq s} X_t \leq z \right\} \supset \{ X_t \leq X_t^{\alpha_0}, \forall t \}, \]

\[ \{ \tau_z > s \} = \left\{ \inf_{0 \leq t \leq s} X_t > z \right\} \supset \{ X_t > X_t^{\alpha_0}, \forall t \}. \]

By using Yao-Chen formula, we obtain

\[ M\{ \tau_z \leq s \} \geq M\{ X_t \leq X_t^{\alpha_0}, \forall t \} = \alpha_0, \]

\[ M\{ \tau_z > s \} \geq M\{ X_t > X_t^{\alpha_0}, \forall t \} = 1 - \alpha_0. \]

It follows from \( M\{ \tau_z \leq s \} + M\{ \tau_z > s \} = 1 \) that \( M\{ \tau_z \leq s \} = \alpha_0 \). Hence the first hitting time \( \tau_z \) has an uncertainty distribution

\[ \Psi(s) = M\{ \tau_z \leq s \} = \alpha_0 = \sup \left\{ \alpha \mid \inf_{0 \leq t \leq s} X_t^{\alpha} \leq z \right\}. \]

The theorem is verified.

**Exercise 14.7:** Let \( X_t \) and \( X_t^{\alpha} \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively. Assume \( J \) is a continuous and strictly
increasing function. For any given level $z$, show that the first hitting time $\tau_z$ that $J(X_t)$ reaches $z$ has an uncertainty distribution

$$
\Psi(s) = \begin{cases} 
1 - \inf \left\{ \alpha \mid \sup_{0 \leq t \leq s} J(X_t^\alpha) \geq z \right\}, & \text{if } z > J(X_0) \\
\sup \left\{ \alpha \mid \inf_{0 \leq t \leq s} J(X_t^\alpha) \leq z \right\}, & \text{if } z < J(X_0).
\end{cases}
$$

(14.100)

**Exercise 14.8:** Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of an uncertain differential equation, respectively. Assume $J$ is a continuous and strictly decreasing function. For any given level $z$, show that the first hitting time $\tau_z$ that $J(X_t)$ reaches $z$ has an uncertainty distribution

$$
\Psi(s) = \begin{cases} 
\sup \left\{ \alpha \mid \sup_{0 \leq t \leq s} J(X_t^\alpha) \geq z \right\}, & \text{if } z > J(X_0) \\
1 - \inf \left\{ \alpha \mid \inf_{0 \leq t \leq s} J(X_t^\alpha) \leq z \right\}, & \text{if } z < J(X_0).
\end{cases}
$$

(14.101)

**Time Integral of Solution**

**Theorem 14.15** (Yao [186]) Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation

$$
dX_t = f(t, X_t)dt + g(t, X_t)dC_t,
$$

(14.102)

respectively. Then for any time $s > 0$, the time integral

$$
\int_0^s X_t dt
$$

(14.103)

has an inverse uncertainty distribution

$$
\Psi_s^{-1}(\alpha) = \int_0^s X_t^\alpha dt.
$$

(14.104)

**Proof:** For any given time $s > 0$, it follows from the basic property of time integral that

$$
\left\{ \int_0^s X_t dt \leq \int_0^s X_t^\alpha dt \right\} \supset \{ X_t \leq X_t^\alpha, \forall t \}.
$$

By using Yao-Chen formula, we obtain

$$
\mathbb{M}\left\{ \int_0^s X_t dt \leq \int_0^s X_t^\alpha dt \right\} \geq \mathbb{M}\{ X_t \leq X_t^\alpha, \forall t \} = \alpha.
$$

(14.105)
Similarly, we have
\[ M \left\{ \int_0^s X_t \, dt > \int_0^s X_t^\alpha \, dt \right\} \geq M \{ X_t > X_t^\alpha, \forall t \} = 1 - \alpha. \] (14.106)

It follows from (14.105), (14.106) and the duality axiom that
\[ M \left\{ \int_0^s X_t \, dt \leq \int_0^s X_t^\alpha \, dt \right\} = \alpha. \] (14.107)

The theorem is thus verified.

**Exercise 14.9:** Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively. Assume \( J \) is a continuous and strictly increasing function. For any time \( s > 0 \), show that the time integral
\[ \int_0^s J(X_t) \, dt \] (14.108)
has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \int_0^s J(X_t^\alpha) \, dt. \] (14.109)

**Exercise 14.10:** Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of an uncertain differential equation, respectively. Assume \( J \) is a continuous and strictly decreasing function. For any time \( s > 0 \), show that the time integral
\[ \int_0^s J(X_t) \, dt \] (14.110)
has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \int_0^s J(X_t^{1-\alpha}) \, dt. \] (14.111)

**14.7 Numerical Methods**

It is almost impossible to find analytic solutions for general uncertain differential equations. This fact provides a motivation to design some numerical methods to solve the uncertain differential equation
\[ dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t. \] (14.112)

In order to do so, a key point is to obtain a spectrum of \( \alpha \)-paths of the uncertain differential equation. For this purpose, Yao-Chen [188] designed an Euler method:
Step 1. Fix $\alpha$ on $(0, 1)$.

Step 2. Solve $dX^\alpha_t = f(t, X^\alpha_t)dt + |g(t, X^\alpha_t)|\Phi^{-1}(\alpha)dt$ by any method of ordinary differential equation and obtain the $\alpha$-path $X^\alpha_t$, for example, by using the recursion formula

$$X^\alpha_{i+1} = X^\alpha_i + f(t_i, X^\alpha_i)h + |g(t_i, X^\alpha_i)|\Phi^{-1}(\alpha)h$$

(14.113)

where $\Phi^{-1}$ is the inverse standard normal uncertainty distribution and $h$ is the step length.

Step 3. The $\alpha$-path $X^\alpha_t$ is obtained.

Remark 14.5: Yang-Shen [176] designed a Runge-Kutta method that replaces the recursion formula (14.113) with

$$X^\alpha_{i+1} = X^\alpha_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

(14.114)

where

$$k_1 = f(t_i, X^\alpha_i) + |g(t_i, X^\alpha_i)|\Phi^{-1}(\alpha),$$

(14.115)

$$k_2 = f(t_i + h/2, X^\alpha_i + hk_1/2) + |g(t_i + h/2, X^\alpha_i + hk_1/2)|\Phi^{-1}(\alpha),$$

(14.116)

$$k_3 = f(t_i + h/2, X^\alpha_i + hk_2/2) + |g(t_i + h/2, X^\alpha_i + hk_2/2)|\Phi^{-1}(\alpha),$$

(14.117)

$$k_4 = f(t_i + h, X^\alpha_i + hk_3) + |g(t_i + h, X^\alpha_i + hk_3)|\Phi^{-1}(\alpha).$$

(14.118)

Example 14.13: In order to illustrate the numerical method, let us consider an uncertain differential equation

$$dX_t = (t - X_t)dt + \sqrt{1 + X_t}dC_t, \quad X_0 = 1.$$  

(14.119)

The Euler method may solve this equation successfully and obtain all $\alpha$-paths of the uncertain differential equation. Furthermore, we may get

$$E[X_1] \approx 0.870.$$  

(14.120)

Example 14.14: Now we consider a nonlinear uncertain differential equation

$$dX_t = \sqrt{X_t}dt + (1 - t)X_t dC_t, \quad X_0 = 1.$$  

(14.121)

Note that $(1 - t)X_t$ takes not only positive values but also negative values. The Euler method may obtain all $\alpha$-paths of the uncertain differential equation. Furthermore, we may get

$$E[(X_2 - 3)^+] \approx 2.845.$$  

(14.122)
14.8 Bibliographic Notes

The study of uncertain differential equation was pioneered by Liu [89] in 2008. This work was immediately followed upon by many researchers. Nowadays, the uncertain differential equation has achieved fruitful results in both theory and practice.

The existence and uniqueness theorem of solution of uncertain differential equation was first proved by Chen-Liu [6] under linear growth condition and Lipschitz condition. Later, the theorem was verified again by Gao [53] under local linear growth condition and local Lipschitz condition.

The first concept of stability of uncertain differential equation was presented by Liu [91], and some stability theorems were proved by Yao-Gao-Gao [185]. Following that, different types of stability of uncertain differential equations were explored, for example, stability in mean (Yao-Ke-Sheng [192]), stability in moment (Sheng-Wang [152]), stability in distribution (Yang-Ni-Zhang [178]), almost sure stability (Liu-Ke-Fei [112]), and exponential stability (Sheng-Gao [156]).


More importantly, Yao-Chen [188] showed that the solution of an uncertain differential equation can be represented by a family of solutions of ordinary differential equations, thus relating uncertain differential equations and ordinary differential equations. On the basis of Yao-Chen formula, Yao [186] presented some formulas to calculate extreme value, first hitting time, and time integral of solution of uncertain differential equation. Furthermore, some numerical methods for solving general uncertain differential equations were designed among others by Yao-Chen [188], Yang-Shen [176], Yang-Ralescu [175], Gao [37], and Zhang-Gao-Huang [221].

Uncertain differential equation has been successfully extended in many directions, including uncertain delay differential equation (Barbacioru [2], Ge-Zhu [56] and Liu-Fei [111]), higher-order uncertain differential equation (Yao [200]), multifactor uncertain differential equation (Li-Peng-Zhang [81], Hassanzadeh-Mehrdoust [61] and Chen-Gao [18]), uncertain differential equation with jumps (Yao [183]), uncertain fractional differential equation (Zhu [228]), and uncertain partial differential equation (Yang-Yao [179]).

Uncertain differential equation has been widely applied in many fields such as finance (Liu [100]), optimal control (Zhu [227][229]), differential game (Yang-Gao [173]), population growth (Zhang-Yang [225]), heat conduction (Yang-Yao [179]), string vibration (Gao [42]), spring vibration (Jia-Gen-Dai [72]), and epidemic spread (Li-Sheng-Teng-Miao [83]).
Chapter 15

Uncertain Finance

This chapter will introduce uncertain stock model, uncertain interest rate model, and uncertain currency model by using the tool of uncertain differential equation. Based on the fair price principle, this chapter will also price European options, American options, Asian options, zero-coupon bond, interest rate ceiling, and interest rate floor.

15.1 Uncertain Stock Model

In 2009 Liu [91] first supposed that the stock price follows an uncertain differential equation and presented an uncertain stock model in which the bond price $X_t$ and the stock price $Y_t$ are determined by

\[
\begin{align*}
    dX_t &= rX_t \, dt \\
    dY_t &= eY_t \, dt + \sigma Y_t \, dC_t
\end{align*}
\] (15.1)

where $r$ is the riskless interest rate, $e$ is the log-drift, $\sigma$ is the log-diffusion, and $C_t$ is a Liu process. Note that the bond price is

\[X_t = X_0 \exp(rt) \] (15.2)

and the stock price is

\[Y_t = Y_0 \exp(et + \sigma C_t) \] (15.3)

whose inverse uncertainty distribution is

\[
\Phi_t^{-1}(\alpha) = Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right). \] (15.4)

15.2 European Options

This section will price European call and put options for the financial market determined by the uncertain stock model (15.1).
European Call Option

**Definition 15.1** A European call option is a contract that gives the holder the right to buy a stock at an expiration time $s$ for a strike price $K$.

Let $f_c$ represent the price of this contract. Then the investor pays $f_c$ for buying the contract at time 0, and has a payoff $(Y_s - K)^+$ at time $s$ since the option is rationally exercised if and only if $Y_s > K$. See Figure 15.1. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-rs)(Y_s - K)^+$. Thus the net return of the investor at time 0 is

$$- f_c + \exp(-rs)(Y_s - K)^+. \quad (15.5)$$

On the other hand, the bank receives $f_c$ for selling the contract at time 0, and pays $(Y_s - K)^+$ at the expiration time $s$. Thus the net return of the bank at the time 0 is

$$f_c - \exp(-rs)(Y_s - K)^+. \quad (15.6)$$

The fair price of this contract should make the investor and the bank have an identical expected return (we will call it fair price principle\(^1\) hereafter), i.e.,

$$- f_c + \exp(-rs)E[(Y_s - K)^+] = f_c - \exp(-rs)E[(Y_s - K)^+]. \quad (15.7)$$

Thus $f_c = \exp(-rs)E[(Y_s - K)^+]$. That is, the European call option price is just the expected present value of the payoff.

---

\(^1\)Fair price principle does not meet no arbitrage principle (i.e., there are never opportunities to make risk-free profit). In fact, I do not agree with no arbitrage principle since it may lead to unreasonable results.
**Definition 15.2** (Liu [91]) Assume a European call option has a strike price $K$ and an expiration time $s$. Then the European call option price is

$$f_c = \exp(-rs)E[(Y_s - K)^+]$$  \hspace{1cm} (15.8)

where $Y_s$ is the stock price at time $s$, and $r$ is the riskless interest rate.

**Theorem 15.1** (Liu [91]) Assume a European call option for the uncertain stock model (15.1) has a strike price $K$ and an expiration time $s$. Then the European call option price is

$$f_c = \exp(-rs) \int_0^1 \left( Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) - K \right)^+ d\alpha.$$  \hspace{1cm} (15.9)

**Proof:** It follows from the uncertain stock model (15.1) that the stock price $Y_s$ has an inverse uncertainty distribution

$$\Phi^{-1}_s(\alpha) = Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).$$

Thus $(Y_s - K)^+$ has an inverse uncertainty distribution

$$\Psi^{-1}_s(\alpha) = \left( Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) - K \right)^+.$$

By using (15.8) and expected value formula, we get (15.9).

**European Put Option**

**Definition 15.3** A European put option is a contract that gives the holder the right to sell a stock at an expiration time $s$ for a strike price $K$.

Let $f_p$ represent the price of this contract. Then the investor pays $f_p$ for buying the contract at time 0, and has a payoff $(K - Y_s)^+$ at time $s$ since the option is rationally exercised if and only if $Y_s < K$. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-rs)(K - Y_s)^+$. Thus the net return of the investor at time 0 is

$$- f_p + \exp(-rs)(K - Y_s)^+.$$  \hspace{1cm} (15.10)

On the other hand, the bank receives $f_p$ for selling the contract at time 0, and pays $(K - Y_s)^+$ at the expiration time $s$. Thus the net return of the bank at the time 0 is

$$f_p - \exp(-rs)(K - Y_s)^+.$$  \hspace{1cm} (15.11)
The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,
\[-f_p + \exp(-rs)E[(K - Y_s)^+] = f_p - \exp(-rs)E[(K - Y_s)^+].\]  
(15.12)
Thus \(f_p = \exp(-rs)E[(K - Y_s)^+].\) That is, the European put option price is just the expected present value of the payoff.

**Definition 15.4 (Liu [91])** Assume a European put option has a strike price \(K\) and an expiration time \(s\). Then the European put option price is
\[f_p = \exp(-rs)E[(K - Y_s)^+]\]  
(15.13)
where \(Y_s\) is the stock price at time \(s\), and \(r\) is the riskless interest rate.

**Theorem 15.2 (Liu [91])** Assume a European put option for the uncertain stock model (15.1) has a strike price \(K\) and an expiration time \(s\). Then the European put option price is
\[f_p = \exp(-rs) \int_0^1 \left( K - Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \right)^+ d\alpha.\]  
(15.14)

**Proof:** It follows from the uncertain stock model (15.1) that the stock price \(Y_s\) has an inverse uncertainty distribution
\[\Phi_s^{-1}(\alpha) = Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).\]
Thus \((K - Y_s)^+\) has an inverse uncertainty distribution
\[\Psi_s^{-1}(\alpha) = \left( K - Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \right)^+.\]
Therefore, by using the expected value formula and the change of variables of integral, we get
\[f_p = \exp(-rs)E[(K - Y_s)^+]\]
\[= \exp(-rs) \int_0^1 \left( K - Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \right)^+ d\alpha\]
\[= \exp(-rs) \int_0^1 \left( K - Y_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \right)^+ d\alpha.\]
The European put option price formula is verified.
15.3 American Options

This section will price American call and put options for the financial market determined by the uncertain stock model (15.1).

American Call Option

Definition 15.5 An American call option is a contract that gives the holder the right to buy a stock at any time prior to an expiration time \( s \) for a strike price \( K \).

Let \( f_c \) represent the price of this contract. Then the net return of the investor at time 0 is

\[
-f_c + \sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+,
\]

(15.15)

and the net return of the bank at the time 0 is

\[
f_c - \sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+.
\]

(15.16)

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

\[
-f_c + E\left[\sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+\right] = f_c - E\left[\sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+\right].
\]

Thus the American call option price is just the expected present value of the payoff.

Definition 15.6 (Chen [7]) Assume an American call option has a strike price \( K \) and an expiration time \( s \). Then the American call option price is

\[
f_c = E\left[\sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+\right]
\]

(15.17)

where \( Y_t \) is the stock price, and \( r \) is the riskless interest rate.

Theorem 15.3 (Chen [7]) Assume an American call option for the uncertain stock model (15.1) has a strike price \( K \) and an expiration time \( s \). Then the American call option price is

\[
f_c = \int_0^1 \sup_{0 \leq t \leq s} \exp(-rt) \left( Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) - K \right)^+ d\alpha.
\]
**Proof:** Note that the stock price $Y_t$ in the uncertain stock model (15.1) has an inverse uncertainty distribution

$$
\Phi_t^{-1}(\alpha) = Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right).
$$

Since $\exp(-rt)(Y_t - K)^+$ is an increasing function of $Y_t$, it follows from the extreme value of solution of uncertain differential equation that

$$
\sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+
$$

has an inverse uncertainty distribution

$$
\Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} \exp(-rt) \left( Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) - K \right)^+.
$$

By using (15.17) and the expected value formula, we get the result.

**American Put Option**

**Definition 15.7** An American put option is a contract that gives the holder the right to sell a stock at any time prior to an expiration time $s$ for a strike price $K$.

Let $f_p$ represent the price of this contract. Then the net return of the investor at time 0 is

$$
-f_p + \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+,
$$

and the net return of the bank at the time 0 is

$$
f_p - \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+.
$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$
-f_p + E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \right] = f_p - E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \right].
$$

Thus the American put option price is just the expected present value of the payoff.

**Definition 15.8** (Chen [7]) Assume an American put option has a strike price $K$ and an expiration time $s$. Then the American put option price is

$$
f_p = E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \right]
$$

where $Y_t$ is the stock price, and $r$ is the riskless interest rate.
Theorem 15.4 (Chen [7]) Assume an American put option for the uncertain stock model (15.1) has a strike price $K$ and an expiration time $s$. Then the American put option price is

$$f_p = \int_0^1 \sup_{0 \leq t \leq s} \exp(-rt) \left( K - Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \right)^+ \, d\alpha.$$ 

Proof: Note that the stock price $Y_t$ in the uncertain stock model (15.1) has an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).$$

Since $\exp(-rt)(K - Y_t)^+$ is a decreasing function of $Y_t$, it follows from the extreme value of solution of uncertain differential equation that

$$\sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} \exp(-rt) \left( K - Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \right)^+.$$ 

By using (15.20), the expected value formula and the change of variables of integral, we get the result.

15.4 Asian Options

This section will price Asian call and put options for the financial market determined by the uncertain stock model (15.1).

Asian Call Option

Definition 15.9 An Asian call option is a contract whose payoff at the expiration time $s$ is

$$\left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+$$

(15.21)

where $K$ is a strike price.

Let $f_c$ represent the price of this contract. Then the investor pays $f_c$ for buying the contract at time 0, and has a payoff

$$\left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+$$

(15.22)
at time $s$. Considering the time value of money resulted from the bond, the present value of the payoff is

$$\exp(-rs) \left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+. \quad (15.23)$$

Thus the net return of the investor at time 0 is

$$- f_c + \exp(-rs) \left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+. \quad (15.24)$$

On the other hand, the bank receives $f_c$ for selling the contract at time 0, and pays

$$\left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+ \quad (15.25)$$

at the expiration time $s$. Thus the net return of the bank at the time 0 is

$$f_c - \exp(-rs) \left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+. \quad (15.26)$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$- f_c + \exp(-rs) E \left[ \left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+ \right]$$

$$= f_c - \exp(-rs) E \left[ \left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+ \right]. \quad (15.27)$$

Thus the Asian call option price is just the expected present value of the payoff.

**Definition 15.10** (Sun-Chen [159]) Assume an Asian call option has a strike price $K$ and an expiration time $s$. Then the Asian call option price is

$$f_c = \exp(-rs) E \left[ \left( \frac{1}{s} \int_0^s Y_t \, dt - K \right)^+ \right] \quad (15.28)$$

where $Y_t$ is the stock price, and $r$ is the riskless interest rate.

**Theorem 15.5** (Sun-Chen [159]) Assume an Asian call option for the uncertain stock model (15.1) has a strike price $K$ and an expiration time $s$. Then the Asian call option price is

$$f_c = \exp(-rs) \int_0^1 \left( \frac{Y_0}{s} \int_0^s \exp \left( et + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \, dt - K \right)^+ \, d\alpha.$$
**Proof:** Note that the stock price $Y_t$ in the uncertain stock model (15.1) has an inverse uncertainty distribution

$$
\Phi_t^{-1}(\alpha) = Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).
$$

It follows from the time integral of solution of uncertain differential equation that

$$
\int_0^s Y_t dt
$$

has an inverse uncertainty distribution

$$
\Psi_s^{-1}(\alpha) = Y_0 \int_0^s \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) dt.
$$

Thus

$$
\left( \frac{1}{s} \int_0^s Y_t dt - K \right)^+
$$

has an inverse uncertainty distribution

$$
\Upsilon_s^{-1}(\alpha) = \left( \frac{Y_0}{s} \int_0^s \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) dt - K \right)^+.
$$

By using (15.28) and the expected value formula, we get the result.

**Asian Put Option**

**Definition 15.11** An Asian put option is a contract whose payoff at the expiration time $s$ is

$$
\left( K - \frac{1}{s} \int_0^s Y_t dt \right)^+ 
$$

where $K$ is a strike price.

Let $f_p$ represent the price of this contract. Then the investor pays $f_p$ for buying the contract at time 0, and has a payoff

$$
\left( K - \frac{1}{s} \int_0^s Y_t dt \right)^+
$$

at time $s$. Considering the time value of money resulted from the bond, the present value of the payoff is

$$
\exp(-rs) \left( K - \frac{1}{s} \int_0^s Y_t dt \right)^+.
$$
Thus the net return of the investor at time 0 is

\[-f_p + \exp(-rs) \left( K - \frac{1}{s} \int_0^s Y_t \, dt \right)^+ . \tag{15.32} \]

On the other hand, the bank receives \( f_p \) for selling the contract at time 0, and pays

\[ \left( K - \frac{1}{s} \int_0^s Y_t \, dt \right)^+ \] \tag{15.33}

at the expiration time \( s \). Thus the net return of the bank at the time 0 is

\[ f_p - \exp(-rs) \left( K - \frac{1}{s} \int_0^s Y_t \, dt \right)^+ . \tag{15.34} \]

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

\[-f_p + \exp(-rs) \mathbb{E} \left[ \left( K - \frac{1}{s} \int_0^s Y_t \, dt \right)^+ \right] = f_p - \exp(-rs) \mathbb{E} \left[ \left( K - \frac{1}{s} \int_0^s Y_t \, dt \right)^+ \right]. \tag{15.35} \]

Thus the Asian put option price should be the expected present value of the payoff.

**Definition 15.12** (Sun-Chen [159]) Assume an Asian put option has a strike price \( K \) and an expiration time \( s \). Then the Asian put option price is

\[ f_p = \exp(-rs) \mathbb{E} \left[ \left( K - \frac{1}{s} \int_0^s Y_t \, dt \right)^+ \right] \tag{15.36} \]

where \( Y_t \) is the stock price, and \( r \) is the riskless interest rate.

**Theorem 15.6** (Sun-Chen [159]) Assume an Asian put option for the uncertain stock model (15.1) has a strike price \( K \) and an expiration time \( s \). Then the Asian put option price is

\[ f_p = \exp(-rs) \int_0^1 \left( K - \frac{Y_0}{s} \int_0^s \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \, dt \right)^+ \, d\alpha. \]

**Proof:** Note that the stock price \( Y_t \) in the uncertain stock model (15.1) has an inverse uncertainty distribution

\[ \Phi_t^{-1}(\alpha) = Y_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right). \]
It follows from the time integral of solution of uncertain differential equation that
\[
\int_0^s Y_t dt
\]
has an inverse uncertainty distribution
\[
\Psi_s^{-1}(\alpha) = Y_0 \int_0^s \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) dt.
\]
Thus
\[
\left( K - \frac{1}{s} \int_0^s Y_t dt \right)^+
\]
has an inverse uncertainty distribution
\[
\Upsilon_s^{-1}(\alpha) = \left( K - \frac{Y_0}{s} \int_0^s \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \right) dt \right)^+.
\]
By using (15.36), the expected value formula and the change of variables of integral, we get the result.

### 15.5 General Stock Model

Generally, we may assume the stock price follows a general uncertain differential equation and obtain a general stock model in which the bond price \(X_t\) and the stock price \(Y_t\) are determined by

\[
\begin{align*}
\text{d}X_t &= rX_t \text{d}t \\
\text{d}Y_t &= F(t,Y_t) \text{d}t + G(t,Y_t) \text{d}C_t
\end{align*}
\]

(15.37)

where \(r\) is the riskless interest rate, \(F\) and \(G\) are measurable functions, and \(C_t\) is a Liu process.

**Theorem 15.7** (Liu [106]) Assume a European option for the uncertain stock model (15.37) has a strike price \(K\) and an expiration time \(s\). Then the European call option price is

\[
f_c = \exp(-rs) \int_0^1 (Y_s^\alpha - K)^+ d\alpha
\]

(15.38)

and the European put option price is

\[
f_p = \exp(-rs) \int_0^1 (K - Y_s^\alpha)^+ d\alpha
\]

(15.39)

where \(Y_s^\alpha\) is the \(\alpha\)-path of the corresponding uncertain differential equation.
Proof: On the one hand, it follows from the fair price principle that the European call option price is
\[ f_c = \exp(-rs)E[(Y_s - K)^+]. \]  \hfill (15.40)

On the other hand, it follows from Yao-Chen formula that \((Y_s - K)^+\) has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = (Y_s^\alpha - K)^+. \]  \hfill (15.41)

By using the expected value formula, we get (15.38). Similarly, the European put option price is
\[ f_p = \exp(-rs)E[(K - Y_s)^+], \]  \hfill (15.42)
and \((K - Y_s)^+\) has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = (K - Y_s^{1-\alpha})^+. \]  \hfill (15.43)

By using the expected value formula and the change of variables of integral, we get (15.39).

**Theorem 15.8** (Liu [106]) Assume an American option for the uncertain stock model (15.37) has a strike price \(K\) and an expiration time \(s\). Then the American call option price is
\[ f_c = \int_0^1 \sup_{0 \leq t \leq s} \exp(-rt)(Y_t^\alpha - K)^+ d\alpha \]  \hfill (15.44)
and the American put option price is
\[ f_p = \int_0^1 \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t^\alpha)^+ d\alpha \]  \hfill (15.45)
where \(Y_t^\alpha\) is the \(\alpha\)-path of the corresponding uncertain differential equation.

Proof: On the one hand, it follows from the fair price principle that the American call option price is
\[ f_c = E\left[ \sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+ \right]. \]  \hfill (15.46)

On the other hand, it follows from the extreme value of solution of uncertain differential equation that
\[ \sup_{0 \leq t \leq s} \exp(-rt)(Y_t - K)^+ \]
has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} \exp(-rt)(Y_t^\alpha - K)^+. \]  \hfill (15.47)
By using the expected value formula, we get (15.44). Similarly, the American put option price is

\[ f_p = E \left[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \right], \]  

(15.48)

and

\[ \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t)^+ \]

has an inverse uncertainty distribution

\[ \Psi^{-1}_s(\alpha) = \sup_{0 \leq t \leq s} \exp(-rt)(K - Y_t^{1-\alpha})^+. \]  

(15.49)

By using the expected value formula and the change of variables of integral, we get (15.45).

**Theorem 15.9** (Liu [106]) Assume an Asian option for the uncertain stock model (15.37) has a strike price \( K \) and an expiration time \( s \). Then the Asian call option price is

\[ f_c = \exp(-rs) \int_0^1 \left( \frac{1}{s} \int_0^s Y_t^\alpha dt - K \right)^+ d\alpha \]  

(15.50)

and the Asian put option price is

\[ f_p = \exp(-rs) \int_0^1 \left( K - \frac{1}{s} \int_0^s Y_t^\alpha dt \right)^+ d\alpha \]  

(15.51)

where \( Y_t^\alpha \) is the \( \alpha \)-path of the corresponding uncertain differential equation.

**Proof:** On the one hand, it follows from the fair price principle that the Asian call option price is

\[ f_c = \exp(-rs)E \left[ \left( \frac{1}{s} \int_0^s Y_t dt - K \right)^+ \right]. \]  

(15.52)

On the other hand, it follows from the time integral of solution of uncertain differential equation that

\[ \left( \frac{1}{s} \int_0^s Y_t dt - K \right)^+ \]

has an inverse uncertainty distribution

\[ \Psi^{-1}_s(\alpha) = \left( \frac{1}{s} \int_0^s Y_t^\alpha dt - K \right)^+. \]  

(15.53)
By using the expected value formula, we get (15.50). Similarly, the Asian put option price is

$$f_p = \exp(-rs)E\left[\left( K - \frac{1}{s} \int_0^s Y_t dt \right)^+ \right],$$  \hspace{1cm} (15.54)

and

$$\left( K - \frac{1}{s} \int_0^s Y_t dt \right)^+$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \left( K - \frac{1}{s} \int_0^s Y_t^{1-\alpha} dt \right)^+. \hspace{1cm} (15.55)$$

By using the expected value formula and the change of variables of integral, we get (15.51).

### 15.6 Multifactor Stock Model

Now we assume that there are multiple stocks whose prices are determined by multiple Liu processes. In this case, we have a *multifactor stock model* in which the bond price $X_t$ and the stock prices $Y_{it}$ are determined by

$$\left\{ \begin{array}{l}
dX_t = rX_t dt \\
dY_{it} = e_i Y_{it} dt + \sum_{j=1}^n \sigma_{ij} Y_{it} dC_{jt}, \ i = 1, 2, \cdots, m 
\end{array} \right. \hspace{1cm} (15.56)$$

where $r$ is the riskless interest rate, $e_i$ are the log-drifts, $\sigma_{ij}$ are the log-diffusions, and $C_{jt}$ are independent Liu processes, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

**Portfolio Selection**

For the multifactor stock model (15.56), we have the choice of $m + 1$ different investments. At each time $t$ we may choose a portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ (i.e., the investment fractions meeting $\beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1$). Then the wealth $Z_t$ at time $t$ should follow the uncertain differential equation

$$dZ_t = r_t Z_t dt + \sum_{i=1}^m e_i \beta_{it} Z_t dt + \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij} \beta_{it} Z_t dC_{jt}. \hspace{1cm} (15.57)$$

That is,

$$Z_t = Z_0 \exp(rt) \exp\left( \int_0^t \sum_{i=1}^m (e_i - r) \beta_{is} ds + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij} \beta_{is} dC_{js} \right).$$

Portfolio selection problem is to find an optimal portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that the wealth $Z_s$ is maximized in the sense of expected value.
No-Arbitrage

The stock model (15.56) is said to be no-arbitrage if there is no portfolio \((\beta_t, \beta_{1t}, \cdots, \beta_{mt})\) such that for some time \(s > 0\), we have

\[
M\{\exp(-rs)Z_s \geq Z_0\} = 1 \tag{15.58}
\]

and

\[
M\{\exp(-rs)Z_s > Z_0\} > 0 \tag{15.59}
\]

where \(Z_t\) is determined by (15.57) and represents the wealth at time \(t\).

**Theorem 15.10** (Yao’s No-Arbitrage Theorem [191]) The multifactor stock model (15.56) is no-arbitrage if and only if the system of linear equations

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
=
\begin{pmatrix}
e_1 - r \\
e_2 - r \\
\vdots \\
e_m - r
\end{pmatrix} \tag{15.60}
\]

has a solution, i.e., \((e_1 - r, e_2 - r, \cdots, e_m - r)\) is a linear combination of column vectors \((\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1}), (\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2}), \cdots, (\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})\).

**Proof:** When the portfolio \((\beta_t, \beta_{1t}, \cdots, \beta_{mt})\) is accepted, the wealth at each time \(t\) is

\[
Z_t = Z_0 \exp(rt) \exp\left(\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is}ds + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is}dC_{js}\right).
\]

Thus

\[
\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^m (e_i - r)\beta_{is}ds + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is}dC_{js}
\]

is a normal uncertain variable with expected value

\[
\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is}ds
\]

and variance

\[
\left(\sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is}ds\right)^2.
\]

Assume the system (15.60) has a solution. The argument breaks down into two cases. Case I: for any given time \(t\) and portfolio \((\beta_t, \beta_{1t}, \cdots, \beta_{mt})\), suppose

\[
\sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is}ds = 0.
\]
Then
\[ \sum_{i=1}^{m} \sigma_{ij} \beta_{is} = 0, \quad j = 1, 2, \ldots, n, \ s \in (0, t]. \]

Since the system (15.60) has a solution, we have
\[ \sum_{i=1}^{m} (e_i - r) \beta_{is} = 0, \quad s \in (0, t] \]

and
\[ \int_0^t \sum_{i=1}^{m} (e_i - r) \beta_{is} \, ds = 0. \]

This fact implies that
\[ \ln(\exp(-rt)Z_t) - \ln Z_0 = 0 \]

and
\[ \mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 0. \]

That is, the stock model (15.56) is no-arbitrage. Case II: for any given time \( t \) and portfolio \((\beta_t, \beta_{1t}, \ldots, \beta_{mt})\), suppose
\[ \sum_{j=1}^{n} \int_0^t \left| \sum_{i=1}^{m} \sigma_{ij} \beta_{is} \right| \, ds \neq 0. \]

Then \( \ln(\exp(-rt)Z_t) - \ln Z_0 \) is a normal uncertain variable with nonzero variance and
\[ \mathcal{M}\{\ln(\exp(-rt)Z_t) - \ln Z_0 \geq 0\} < 1. \]

That is,
\[ \mathcal{M}\{\exp(-rt)Z_t \geq Z_0\} < 1 \]

and the multifactor stock model (15.56) is no-arbitrage.

Conversely, assume the system (15.60) has no solution. Then there exist real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_m \) such that
\[ \sum_{i=1}^{m} \sigma_{ij} \alpha_i = 0, \quad j = 1, 2, \ldots, n \]

and
\[ \sum_{i=1}^{m} (e_i - r) \alpha_i > 0. \]

Now we take a portfolio
\[ (\beta_t, \beta_{1t}, \ldots, \beta_{mt}) \equiv (1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_m), \alpha_1, \alpha_2, \ldots, \alpha_m). \]
Then
\[ \ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^m (e_i - r) \alpha_i \, ds > 0. \]
Thus we have
\[ M\{\exp(-rt)Z_t > Z_0\} = 1. \]
Hence the multifactor stock model (15.56) is arbitrage. The theorem is thus proved.

**Theorem 15.11** The multifactor stock model (15.56) is no-arbitrage if its log-diffusion matrix
\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn}
\end{pmatrix}
\]
has rank \( m \), i.e., the row vectors are linearly independent.

**Proof:** If the log-diffusion matrix (15.61) has rank \( m \), then the system of equations (15.60) has a solution. It follows from Theorem 15.10 that the multifactor stock model (15.56) is no-arbitrage.

**Theorem 15.12** The multifactor stock model (15.56) is no-arbitrage if its log-drifts are all equal to the interest rate \( r \), i.e.,
\[ e_i = r, \quad i = 1, 2, \cdots, m. \]

**Proof:** Since the log-drifts \( e_i = r \) for any \( i = 1, 2, \cdots, m \), we immediately have
\[ (e_1 - r, e_2 - r, \cdots, e_m - r) \equiv (0, 0, \cdots, 0) \]
that is a linear combination of \((\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1})\), \((\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2})\), \cdots, \((\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})\). It follows from Theorem 15.10 that the multifactor stock model (15.56) is no-arbitrage.

**15.7 Uncertain Interest Rate Model**

Real interest rates do not remain unchanged. Chen-Gao [15] assumed that the interest rate follows an uncertain differential equation and presented an uncertain interest rate model,
\[
dX_t = (m - aX_t)dt + \sigma dC_t
\]
where \( m, a, \sigma \) are positive numbers. Besides, Jiao-Yao [74] investigated the uncertain interest rate model,
\[
dX_t = (m - aX_t)dt + \sigma \sqrt{X_t}dC_t.
\]
More generally, we may assume the interest rate $X_t$ follows a general uncertain differential equation and obtain a general interest rate model,

$$dX_t = F(t, X_t)dt + G(t, X_t)dC_t$$  \hspace{1cm} (15.65)

where $F$ and $G$ are measurable functions, and $C_t$ is a Liu process.

**Zero-Coupon Bond**

A zero-coupon bond is a bond bought at a price lower than its face value that is the amount it promises to pay at the maturity date. For simplicity, we assume the face value is always 1 dollar.

Let $f$ represent the price of this zero-coupon bond. Then the investor pays $f$ for buying it at time 0, and receives 1 dollar at the maturity date $s$. Since the interest rate is $X_t$, the present value of 1 dollar is

$$\exp \left( - \int_0^s X_t dt \right). \hspace{1cm} (15.66)$$

Thus the net return of the investor at time 0 is

$$-f + \exp \left( - \int_0^s X_t dt \right). \hspace{1cm} (15.67)$$

On the other hand, the bank receives $f$ for selling the zero-coupon bond at time 0, and pays 1 dollar at the maturity date $s$. Thus the net return of the bank at the time 0 is

$$f - \exp \left( - \int_0^s X_t dt \right). \hspace{1cm} (15.68)$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f + E \left[ \exp \left( - \int_0^s X_t dt \right) \right] = f - E \left[ \exp \left( - \int_0^s X_t dt \right) \right] \hspace{1cm} (15.69)$$

Thus the price of the zero-coupon bond is just the expected present value of its face value.

**Definition 15.13** (Chen-Gao [15]) Let $X_t$ be the uncertain interest rate. Then the price of a zero-coupon bond with a maturity date $s$ is

$$f = E \left[ \exp \left( - \int_0^s X_t dt \right) \right]. \hspace{1cm} (15.70)$$

**Theorem 15.13** (Jiao-Yao [74]) Assume the uncertain interest rate $X_t$ follows the uncertain differential equation (15.65). Then the price of a zero-coupon bond with maturity date $s$ is

$$f = \int_0^1 \exp \left( - \int_0^s X_t^\alpha dt \right) d\alpha \hspace{1cm} (15.71)$$

where $X_t^\alpha$ is the $\alpha$-path of the corresponding uncertain differential equation.
Proof: It follows from the time integral of solution of uncertain differential equation that
\[ \int_0^s X_t dt \]
has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \int_0^s X_t^\alpha dt. \]
Thus
\[ \exp \left( - \int_0^s X_t dt \right) \]
has an inverse uncertainty distribution
\[ \Upsilon_s^{-1}(\alpha) = \exp \left( - \int_0^s X_t^{1-\alpha} dt \right). \]
By using (15.70), the expected value formula and the change of variables of integral, we get (15.71).

Interest Rate Ceiling
An interest rate ceiling is a derivative contract in which the borrower will not pay any more than a predetermined level of interest on his loan. Assume \( K \) is the maximum interest rate and \( s \) is the maturity date. For simplicity, we also assume the amount of loan is always 1 dollar.

Let \( f \) represent the price of this contract. Then the borrower pays \( f \) for buying the contract at time 0, and has a payoff
\[ \exp \left( \int_0^s X_t dt \right) - \exp \left( \int_0^s X_t \wedge K dt \right) \tag{15.72} \]
at the maturity date \( s \). Considering the time value of money, the present value of the payoff is
\[
\exp \left( - \int_0^s X_t dt \right) \left( \exp \left( \int_0^s X_t dt \right) - \exp \left( \int_0^s X_t \wedge K dt \right) \right)
= 1 - \exp \left( - \int_0^s X_t dt + \int_0^s X_t \wedge K dt \right)
= 1 - \exp \left( - \int_0^s (X_t - K)^+ dt \right). 
\]
Thus the net return of the borrower at time 0 is
\[ - f + 1 - \exp \left( - \int_0^s (X_t - K)^+ dt \right). \tag{15.73} \]
Similarly, we may verify that the net return of the bank at the time 0 is

$$f - 1 + \exp \left( - \int_0^s (X_t - K)^+ dt \right). \quad (15.74)$$

The fair price of this contract should make the borrower and the bank have an identical expected return, i.e.,

$$-f + 1 - E \left[ \exp \left( - \int_0^s (X_t - K)^+ dt \right) \right] = f - 1 + E \left[ \exp \left( - \int_0^s (X_t - K)^+ dt \right) \right].$$

Thus we have the following definition of the price of interest rate ceiling.

**Definition 15.14** (Zhang-Ralescu-Liu [224]) Assume an interest rate ceiling has a maximum interest rate $K$ and a maturity date $s$. Then the price of the interest rate ceiling is

$$f = 1 - E \left[ \exp \left( - \int_0^s (X_t - K)^+ dt \right) \right]. \quad (15.75)$$

**Theorem 15.14** (Zhang-Ralescu-Liu [224]) Assume the uncertain interest rate $X_t$ follows the uncertain differential equation (15.65). Then the price of the interest rate ceiling with a maximum interest rate $K$ and a maturity date $s$ is

$$f = 1 - \int_0^1 \exp \left( - \int_0^s (X_t^\alpha - K)^+ dt \right) d\alpha \quad (15.76)$$

where $X_t^\alpha$ is the $\alpha$-path of the corresponding uncertain differential equation.

**Proof:** It follows from the time integral of solution of uncertain differential equation that

$$\int_0^s (X_t - K)^+ dt$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s (X_t^\alpha - K)^+ dt.$$ 

Thus

$$\exp \left( - \int_0^s (X_t - K)^+ dt \right)$$

has an inverse uncertainty distribution

$$\Upsilon_s^{-1}(\alpha) = \exp \left( - \int_0^s (X_t^{1-\alpha} - K)^+ dt \right).$$

By using (15.75), the expected value formula and the change of variables of integral, we get (15.76).
Interest Rate Floor

An interest rate floor is a derivative contract in which the investor will not receive any less than a predetermined level of interest on his investment. Assume \( K \) is the minimum interest rate and \( s \) is the maturity date. For simplicity, we also assume the amount of investment is always 1 dollar.

Let \( f \) represent the price of this contract. Then the investor pays \( f \) for buying the contract at time 0, and has a payoff

\[
\exp \left( \int_0^s X_t \vee K \, dt \right) - \exp \left( \int_0^s X_t \, dt \right)
\]  

(15.77)

at the maturity date \( s \). Considering the time value of money, the present value of the payoff is

\[
\exp \left( -\int_0^s X_t \, dt \right) \left( \exp \left( \int_0^s X_t \vee K \, dt \right) - \exp \left( \int_0^s X_t \, dt \right) \right)
\]

\[
= \exp \left( -\int_0^s X_t \, dt + \int_0^s X_t \vee K \, dt \right) - 1
\]

\[
= \exp \left( \int_0^s (K - X_t)^+ \, dt \right) - 1.
\]

Thus the net return of the investor at time 0 is

\[
-f + \exp \left( \int_0^s (K - X_t)^+ \, dt \right) - 1.
\]  

(15.78)

Similarly, we may verify that the net return of the bank at the time 0 is

\[
f - \exp \left( \int_0^s (K - X_t)^+ \, dt \right) + 1.
\]  

(15.79)

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

\[-f + \mathbb{E} \left[ \exp \left( \int_0^s (K - X_t)^+ \, dt \right) \right] - 1 = f - \mathbb{E} \left[ \exp \left( \int_0^s (K - X_t)^+ \, dt \right) \right] + 1.\]

Thus we have the following definition of the price of interest rate floor.

**Definition 15.15** (Zhang-Ralescu-Liu [224]) Assume an interest rate floor has a minimum interest rate \( K \) and a maturity date \( s \). Then the price of the interest rate floor is

\[
f = \mathbb{E} \left[ \exp \left( \int_0^s (K - X_t)^+ \, dt \right) \right] - 1.
\]  

(15.80)
**Theorem 15.15** (Zhang-Ralescu-Liu [224]) Assume the uncertain interest rate $X_t$ follows the uncertain differential equation (15.65). Then the price of the interest rate floor with a minimum interest rate $K$ and a maturity date $s$ is

$$f = \int_0^1 \exp\left(\int_0^s (K - X_t^\alpha)^+ \, dt\right) \, d\alpha - 1 \quad (15.81)$$

where $X_t^\alpha$ is the $\alpha$-path of the corresponding uncertain differential equation.

**Proof:** It follows from the time integral of solution of uncertain differential equation that

$$\int_0^s (K - X_t)^+ \, dt$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s (K - X_t^{1-\alpha})^+ \, dt.$$ 

Thus

$$\exp\left(\int_0^s (K - X_t)^+ \, dt\right)$$

has an inverse uncertainty distribution

$$\Upsilon_s^{-1}(\alpha) = \exp\left(\int_0^s (K - X_t^{1-\alpha})^+ \, dt\right).$$

By using (15.80), the expected value formula and the change of variables of integral, we get (15.81).

### 15.8 Uncertain Currency Model

Liu-Chen-Ralescu [120] assumed that the exchange rate follows an uncertain differential equation and proposed an uncertain currency model,

$$\begin{cases} 
  dX_t = uX_t \, dt & \text{(Domestic Currency)} \\
  dY_t = vY_t \, dt & \text{(Foreign Currency)} \\
  dZ_t = eZ_t \, dt + \sigma Z_t \, dC_t & \text{(Exchange Rate)}
\end{cases} \quad (15.82)$$

where $X_t$ represents the domestic currency with domestic interest rate $u$, $Y_t$ represents the foreign currency with foreign interest rate $v$, and $Z_t$ represents the exchange rate that is domestic currency price of one unit of foreign currency at time $t$. Note that the domestic currency price is $X_t = X_0 \exp(ut)$, the foreign currency price is $Y_t = Y_0 \exp(\nu t)$, and the exchange rate is

$$Z_t = Z_0 \exp(et + \sigma C_t) \quad (15.83)$$

whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = Z_0 \exp\left(et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right). \quad (15.84)$$
Section 15.8 - Uncertain Currency Model

European Currency Option

Definition 15.16 A European currency option is a contract that gives the holder the right to exchange one unit of foreign currency at an expiration time \( s \) for \( K \) units of domestic currency.

Suppose that the price of this contract is \( f \) in domestic currency. Then the investor pays \( f \) for buying the contract at time 0, and receives \((Z_s - K)^+\) in domestic currency at the expiration time \( s \). Thus the net return of the investor at time 0 is

\[
-f + \exp(-us)(Z_s - K)^+. \tag{15.85}
\]

On the other hand, the bank receives \( f \) for selling the contract at time 0, and pays \((1 - K/Z_s)^+\) in foreign currency at the expiration time \( s \). Thus the net return of the bank at the time 0 is

\[
f - \exp(-vs)Z_0(1 - K/Z_s)^+. \tag{15.86}
\]

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

\[
-f + \exp(-us)E[(Z_s - K)^+] = f - \exp(-vs)Z_0E[(1 - K/Z_s)^+]. \tag{15.87}
\]

Thus the European currency option price is given by the definition below.

Definition 15.17 (Liu-Chen-Ralescu [120]) Assume a European currency option has a strike price \( K \) and an expiration time \( s \). Then the European currency option price is

\[
f = \frac{1}{2} \exp(-us)E[(Z_s - K)^+] + \frac{1}{2} \exp(-us)Z_0E[(1 - K/Z_s)^+]. \tag{15.88}
\]

Theorem 15.16 (Liu-Chen-Ralescu [120]) Assume a European currency option for the uncertain currency model (15.82) has a strike price \( K \) and an expiration time \( s \). Then the European currency option price is

\[
f = \frac{1}{2} \exp(-us) \int_0^1 \left(Z_0 \exp\left(es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right) - K\right)^+ d\alpha + \frac{1}{2} \exp(-vs) \int_0^1 \left(Z_0 - K/\exp\left(es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right)\right)^+ d\alpha.
\]

Proof: Note that the exchange rate \( Z_s \) in the uncertain currency model (15.82) has an inverse uncertainty distribution

\[
\Phi_s^{-1}(\alpha) = Z_0 \exp\left(es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right).
\]
Since \((Z_s - K)^+\) and \(Z_0(1 - K/Z_s)^+\) are increasing functions with respect to \(Z_s\), they have inverse uncertainty distributions

\[
\Psi_s^{-1}(\alpha) = \left( Z_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) - K \right)^+,
\]

\[
\Upsilon_s^{-1}(\alpha) = \left( Z_0 - K/ \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \right)^+,
\]

respectively. By using (15.88) and the expected value formula, we get the result.

**American Currency Option**

**Definition 15.18** An American currency option is a contract that gives the holder the right to exchange one unit of foreign currency at any time prior to an expiration time \(s\) for \(K\) units of domestic currency.

Suppose that the price of this contract is \(f\) in domestic currency. Then the net return of the investor at time 0 is

\[
-f + \sup_{0 \leq t \leq s} \exp(-ut)(Z_t - K)^+,
\]

(15.89)

and the net return of the bank at time 0 is

\[
f - \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t)^+.
\]

(15.90)

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

\[
-f + E \left[ \sup_{0 \leq t \leq s} \exp(-ut)(Z_t - K)^+ \right] = f - E \left[ \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t)^+ \right].
\]

(15.91)

Thus the American currency option price is given by the definition below.

**Definition 15.19** (Liu-Chen-Ralescu [120]) Assume an American currency option has a strike price \(K\) and an expiration time \(s\). Then the American currency option price is

\[
f = \frac{1}{2} E \left[ \sup_{0 \leq t \leq s} \exp(-ut)(Z_t - K)^+ \right] + \frac{1}{2} E \left[ \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t)^+ \right].
\]
**Theorem 15.17** (Liu-Chen-Ralescu [120]) Assume an American currency option for the uncertain currency model (15.82) has a strike price $K$ and an expiration time $s$. Then the American currency option price is

$$
f = \frac{1}{2} \int_0^1 \sup_{0 \leq t \leq s} \exp(-ut) \left( Z_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) - K \right)^+ d\alpha + \frac{1}{2} \int_0^1 \sup_{0 \leq t \leq s} \exp(-vt) \left( Z_0 - K \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \right)^+ d\alpha.
$$

**Proof:** Note that the exchange rate $Z_s$ in the uncertain currency model (15.82) has an inverse uncertainty distribution

$$
\Phi^{-1}_s(\alpha) = Z_0 \exp \left( es + \frac{\sigma s \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).
$$

It follows from the extreme value of solution of uncertain differential equation that

$$
\sup_{0 \leq t \leq s} \exp(-ut)(Z_t - K)^+ \quad \text{and} \quad \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t)^+
$$

have inverse uncertainty distributions

$$
\Psi^{-1}_s(\alpha) = \sup_{0 \leq t \leq s} \exp(-ut) \left( Z_0 \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) - K \right)^+,
$$

$$
\Upsilon^{-1}_s(\alpha) = \sup_{0 \leq t \leq s} \exp(-vt) \left( Z_0 - K \exp \left( et + \frac{\sigma t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) \right)^+,
$$

respectively. By using the expected value formula, we get the result.

**General Currency Model**

If the exchange rate follows a general uncertain differential equation, then we have a general currency model,

$$
\begin{cases}
  dX_t = u X_t dt \quad \text{(Domestic Currency)} \\
  dY_t = v Y_t dt \quad \text{(Foreign Currency)} \\
  dZ_t = F(t, Z_t) dt + G(t, Z_t) dC_t \quad \text{(Exchange Rate)}
\end{cases}
$$

(15.92)

where $u$ and $v$ are interest rates, $F$ and $G$ are measurable functions, and $C_t$ is a Liu process.
Theorem 15.18 (Liu [106]) Assume a European currency option for the uncertain currency model (15.92) has a strike price $K$ and an expiration time $s$. Then the European currency option price is

$$f = \frac{1}{2} \int_0^1 (\exp(-us)(Z_s^\alpha - K)^+ + \exp(-vs)Z_0(1 - K/Z_s^\alpha)^+) \, d\alpha$$

(15.93)

where $Z_t^\alpha$ is the $\alpha$-path of the corresponding uncertain differential equation.

**Proof:** On the one hand, it follows from the fair price principle that the European option price is

$$f = \frac{1}{2} \exp(-us)E[(Z_s - K)^+] + \frac{1}{2} \exp(-vs)Z_0E[(1 - K/Z_s)^+]$$

(15.94)

On the other hand, $(Z_s - K)^+$ and $Z_0(1 - K/Z_s)^+$ have inverse uncertainty distributions

$$\Psi_s^{-1}(\alpha) = (Z_s^\alpha - K)^+,$$

$$\Upsilon_s^{-1}(\alpha) = Z_0(1 - K/Z_s^\alpha)^+,$$

respectively. By using the expected value formula, we get the result.

Theorem 15.19 (Liu [106]) Assume an American currency option for the uncertain currency model (15.92) has a strike price $K$ and an expiration time $s$. Then the American currency option price is

$$f = \frac{1}{2} \int_0^1 \left( \sup_{0 \leq t \leq s} \exp(-ut)(Z_t^\alpha - K)^+ + \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t^\alpha)^+ \right) \, d\alpha$$

where $Z_t^\alpha$ is the $\alpha$-path of the corresponding uncertain differential equation.

**Proof:** On the one hand, it follows from the fair price principle that the American option price is

$$f = \frac{1}{2} E\left[ \sup_{0 \leq t \leq s} \exp(-ut)(Z_t - K)^+ \right] + \frac{1}{2} E\left[ \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t)^+ \right].$$

On the other hand, it follows from the extreme value of solution of uncertain differential equation that

$$\sup_{0 \leq t \leq s} \exp(-ut)(Z_t - K)^+ \text{ and } \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t)^+$$

have inverse uncertainty distributions

$$\Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} \exp(-ut)(Z_t^\alpha - K)^+,$$

$$\Upsilon_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} \exp(-vt)Z_0(1 - K/Z_t^\alpha)^+,$$

respectively. By using the expected value formula, we get the result.
15.9 Bibliographic Notes

The classical finance theory assumed that stock price, interest rate, and exchange rate follow stochastic differential equations. However, this preassumption was challenged among others by Liu [100] in which a convincing paradox was presented to show why the real stock price is impossible to follow any stochastic differential equations (see also Appendix B.7). As an alternative, Liu [100] suggested to develop a theory of uncertain finance.

Uncertain differential equations were first introduced into finance by Liu [91] in 2009 in which an uncertain stock model was proposed and European option price formulas were provided. Besides, Chen [7] derived American option price formulas, Sun-Chen [159] and Zhang-Liu [223] verified Asian option price formulas, and Yao [191] proved a no-arbitrage theorem for this type of uncertain stock model. It is emphasized that uncertain stock models were also actively investigated among others by Peng-Yao [135], Yu [208], Chen-Liu-Ralescu [13], Yao [196], and Ji-Zhou [69].

Uncertain differential equations were used to simulate floating interest rate by Chen-Gao [15] in 2013. Following that, Jiao-Yao [74] presented a price formula of zero-coupon bond, and Zhang-Ralescu-Liu [224] discussed the valuation of interest rate ceiling and floor.

Uncertain differential equations were employed to model currency exchange rate by Liu-Chen-Ralescu [120] in 2015 in which some currency option price formulas were derived for the uncertain currency markets. Afterwards, uncertain currency models were also actively investigated among others by Liu [106], Shen-Yao [151] and Wang-Ning [165].
Chapter 16

Uncertain Statistics

The study of uncertain statistics was started by Liu [95] in 2010. It is a methodology for collecting and interpreting expert’s experimental data by uncertainty theory. This chapter will design a questionnaire survey for collecting expert’s experimental data, and introduce linear interpolation method, principle of least squares, method of moments, and Delphi method for determining uncertainty distributions from the expert’s experimental data. Finally, uncertain regression analysis and uncertain time series analysis are documented in this chapter.

16.1 Expert’s Experimental Data

Uncertain statistics is based on expert’s experimental data rather than historical data. How do we obtain expert’s experimental data? Liu [95] proposed a questionnaire survey for collecting them. The starting point is to invite one or more domain experts who are asked to complete a questionnaire about the meaning of an uncertain variable $\xi$ like “how far from Beijing to Tianjin”.

The domain expert is first asked to choose a possible value $x$ (say 110km) that the uncertain variable $\xi$ may take, and then quiz him

“How likely is $\xi$ less than or equal to $x$?”  \hspace{1cm} (16.1)

Denote the expert’s belief degree by $\alpha$ (say 0.6). Note that the expert’s belief degree of $\xi$ greater than $x$ must be $1 - \alpha$ due to the duality axiom of uncertain measure. An expert’s experimental data

$(110, 0.6)$  \hspace{1cm} (16.2)

is thus acquired from the domain expert.

Repeating the above process, the following expert’s experimental data are obtained by the questionnaire,

$(x_1, \alpha_1), \ (x_2, \alpha_2), \ \cdots, \ (x_n, \alpha_n).$  \hspace{1cm} (16.3)
Remark 16.1: None of $x$, $\alpha$ and $n$ could be assigned a value in the questionnaire before asking the domain expert. Otherwise, the domain expert may have no knowledge or experiments enough to answer your questions.

Questionnaire Survey

Beijing is the capital of China, and Tianjin is a coastal city. Assume that the real distance between them is not exactly known for us, and is regarded as an uncertain variable. Chen-Ralescu [12] employed uncertain statistics to estimate the travel distance between Beijing and Tianjin. The consultation process is as follows:

Q1: May I ask you how far is from Beijing to Tianjin? What do you think is a likely distance?
A1: 130km.

Q2: To what degree do you think that the real distance is less than 130km?
A2: 60%. (*an expert’s experimental data (130, 0.6) is acquired*)

Q3: Is there another number this distance may be? If yes, what is it?
A3: 140km.

Q4: To what degree do you think that the real distance is greater than 140km?
A4: 10%. (*an expert’s experimental data (140, 0.9) is acquired*)

Q5: Is there another number this distance may be? If yes, what is it?
A5: 120km.

Q6: To what degree do you think that the real distance is less than 120km?
A6: 30%. (*an expert’s experimental data (120, 0.3) is acquired*)

Q7: Is there another number this distance may be? If yes, what is it?
A7: 100km.

Q8: To what degree do you think that the real distance is less than 100km?
A8: Impossible. (*an expert’s experimental data (100, 0) is acquired*)

Q9: Is there another number this distance may be? If yes, what is it?
A9: 150km.

Q10: To what degree do you think that the real distance is greater than 150km?
A10: Impossible. *(an expert’s experimental data (150, 1) is acquired)*

Q11: Is there another number this distance may be? If yes, what is it?

A11: No idea.

By using the questionnaire survey, five expert’s experimental data of the travel distance between Beijing and Tianjin are acquired from the domain expert (after a rearrangement),

\((100, 0), (120, 0.3), (130, 0.6), (140, 0.9), (150, 1).\) \((16.4)\)

**Exercise 16.1:** Please do a questionnaire survey on the height of some friend of yours.

### 16.2 Empirical Uncertainty Distribution

How do we determine the uncertainty distribution for an uncertain variable? Assume that we have obtained a set of expert’s experimental data

\[(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n)\] \((16.5)\)

de that meet the following consistence condition (perhaps after a rearrangement)

\[x_1 < x_2 < \cdots < x_n, \quad 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1.\] \((16.6)\)

Based on those expert’s experimental data, Liu [95] suggested an empirical uncertainty distribution,

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < x_1 \\
\alpha_i + \frac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1}, 1 \leq i < n \\
1, & \text{if } x > x_n.
\end{cases}
\] \((16.7)\)

Essentially, it is a type of linear interpolation method.

The empirical uncertainty distribution \(\Phi\) determined by (16.7) has an expected value

\[
E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} - \alpha_i}{2} x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right) x_n.
\] \((16.8)\)

If all \(x_i\)’s are nonnegative, then the \(k\)-th empirical moments are

\[
E[\xi^k] = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1 - \alpha_n) x_n^k.
\] \((16.9)\)
Example 16.1: Recall that the five experts’ experimental data \((100, 0), (120, 0.3), (130, 0.6), (140, 0.9), (150, 1)\) of the travel distance between Beijing and Tianjin have been acquired in Section 16.1. Based on those experts’ experimental data, an empirical uncertainty distribution of travel distance is shown in Figure 16.2.

16.3 Principle of Least Squares

Assume that an uncertainty distribution to be determined has a known functional form \(\Phi(x|\theta)\) with an unknown parameter \(\theta\). In order to estimate the
parameter \( \theta \), Liu [95] employed the principle of least squares that minimizes the sum of the squares of the distance of the expert’s experimental data to the uncertainty distribution. This minimization can be performed in either the vertical or horizontal direction. If the expert’s experimental data \((x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n)\) are obtained and the vertical direction is accepted, then we have

\[
\min_{\theta} \sum_{i=1}^{n} (\Phi(x_i|\theta) - \alpha_i)^2.
\]

The optimal solution \( \hat{\theta} \) of (16.11) is called the least squares estimate of \( \theta \), and then the least squares uncertainty distribution is \( \Phi(x|\hat{\theta}) \).

![Figure 16.3: Principle of Least Squares](image)

**Example 16.2:** Assume that an uncertainty distribution has a linear form with two unknown parameters \( a \) and \( b \), i.e.,

\[
\Phi(x|a, b) = \begin{cases} 
0, & \text{if } x \leq a \\
\frac{x - a}{b - a}, & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b.
\end{cases}
\]

We also assume the following expert’s experimental data,

\((1, 0.15), (2, 0.45), (3, 0.55), (4, 0.85), (5, 0.95)\).

The principle of least squares may yield that \( \hat{a} = 0.2273, \hat{b} = 4.7727 \) and the least squares uncertainty distribution is

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 0.2273 \\
(x - 0.2273)/4.5454, & \text{if } 0.2273 \leq x \leq 4.7727 \\
1, & \text{if } x \geq 4.7727.
\end{cases}
\]
Example 16.3: Assume that an uncertainty distribution has a lognormal form with two unknown parameters $e$ and $\sigma$, i.e.,

$$
\Phi(x|e, \sigma) = \left(1 + \exp \left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}.
$$

(16.15)

We also assume the following expert’s experimental data,

$$(0.6, 0.1), (1.0, 0.3), (1.5, 0.4), (2.0, 0.6), (2.8, 0.8), (3.6, 0.9).$$

(16.16)

The principle of least squares may yield that $\hat{e} = 0.4825$, $\hat{\sigma} = 0.7852$ and the least squares uncertainty distribution is

$$
\Phi(x) = \left(1 + \exp \left(\frac{0.4825 - \ln x}{0.4329}\right)\right)^{-1}.
$$

(16.17)

16.4 Method of Moments

Assume that a nonnegative uncertain variable has an uncertainty distribution

$$
\Phi(x|\theta_1, \theta_2, \ldots, \theta_p)
$$

(16.18)

with unknown parameters $\theta_1, \theta_2, \ldots, \theta_p$. Given a set of expert’s experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n)$$

(16.19)

with

$$0 \leq x_1 < x_2 < \cdots < x_n, \quad 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1,$$

(16.20)

Wang-Peng [169] proposed a method of moments to estimate the unknown parameters of uncertainty distribution. At first, the $k$th empirical moments of the expert’s experimental data are defined as that of the corresponding empirical uncertainty distribution, i.e.,

$$
\bar{\xi}_k = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1 - \alpha_n) x_n^k.
$$

(16.21)

The moment estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p$ are then obtained by equating the first $p$ moments of $\Phi(x|\theta_1, \theta_2, \ldots, \theta_p)$ to the corresponding first $p$ empirical moments. In other words, the moment estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p$ should solve the system of equations,

$$
\int_0^{+\infty} (1 - \Phi(\sqrt{x} | \theta_1, \theta_2, \ldots, \theta_p)) dx = \bar{\xi}_k, \quad k = 1, 2, \ldots, p
$$

(16.22)

where $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_p$ are empirical moments determined by (16.21).
Example 16.4: Assume that a questionnaire survey has successfully acquired the following expert’s experimental data,

$$(1.2, 0.1), (1.5, 0.3), (1.8, 0.4), (2.5, 0.6), (3.9, 0.8), (4.6, 0.9). \quad (16.23)$$

Then the first three empirical moments are $2.5100$, $7.7226$ and $29.4936$. We also assume that the uncertainty distribution to be determined has a zigzag form with three unknown parameters $a, b$ and $c$, i.e.,

$$\Phi(x|a, b, c) = \begin{cases} 
0, & \text{if } x \leq a \\
\frac{x - a}{2(b - a)}, & \text{if } a \leq x \leq b \\
\frac{x + c - 2b}{2(c - b)}, & \text{if } b \leq x \leq c \\
1, & \text{if } x \geq c.
\end{cases} \quad (16.24)$$

From the expert’s experimental data, we may believe that the unknown parameters must be positive numbers. Thus the first three moments of the zigzag uncertainty distribution $\Phi(x|a, b, c)$ are

$$\begin{align*}
a + 2b + c &= \frac{4}{4} \\
a^2 + ab + 2b^2 + bc + c^2 &= \frac{6}{6} \\
a^3 + a^2b + ab^2 + 2b^3 + b^2c + bc^2 + c^3 &= \frac{8}{8}.
\end{align*}$$

It follows from the method of moments that the unknown parameters $a, b, c$ should solve the system of equations,

$$\begin{align*}
a + 2b + c &= 4 \times 2.5100 \\
a^2 + ab + 2b^2 + bc + c^2 &= 6 \times 7.7226 \\
a^3 + a^2b + ab^2 + 2b^3 + b^2c + bc^2 + c^3 &= 8 \times 29.4936.
\end{align*} \quad (16.25)$$

The method of moment may yield that the moment estimates are $(\hat{a}, \hat{b}, \hat{c}) = (0.9804, 2.0303, 4.9991)$ and the corresponding uncertainty distribution is

$$\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 0.9804 \\
(x - 0.9804)/2.0998, & \text{if } 0.9804 \leq x \leq 2.0303 \\
(x + 0.9385)/5.9376, & \text{if } 2.0303 \leq x \leq 4.9991 \\
1, & \text{if } x \geq 4.9991.
\end{cases} \quad (16.26)$$

16.5 Multiple Domain Experts

Assume there are $m$ domain experts and each produces an uncertainty distribution. Then we may get $m$ uncertainty distributions $\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x)$. 
It was suggested by Liu [95] that the $m$ uncertainty distributions should be aggregated to an uncertainty distribution

$$
\Phi(x) = w_1 \Phi_1(x) + w_2 \Phi_2(x) + \cdots + w_m \Phi_m(x) \quad (16.27)
$$

where $w_1, w_2, \cdots, w_m$ are convex combination coefficients (i.e., they are non-negative numbers and $w_1 + w_2 + \cdots + w_n = 1$) representing weights of the domain experts. For example, we may set

$$
w_i = \frac{1}{m}, \quad \forall i = 1, 2, \cdots, n. \quad (16.28)
$$

Since $\Phi_1(x), \Phi_2(x), \cdots, \Phi_m(x)$ are uncertainty distributions, they are increasing functions taking values in $[0, 1]$ and are not identical to either 0 or 1. It is easy to verify that their convex combination $\Phi(x)$ is also an increasing function taking values in $[0, 1]$ and $\Phi(x) \not\equiv 0$, $\Phi(x) \not\equiv 1$. Hence $\Phi(x)$ is also an uncertainty distribution by Peng-Iwamura theorem.

### 16.6  Delphi Method

Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. This method asks the domain experts answer questionnaires in two or more rounds. After each round, a facilitator provides an anonymous summary of the answers from the previous round as well as the reasons that the domain experts provided for their opinions. Then the domain experts are encouraged to revise their earlier answers in light of the summary. It is believed that during this process the opinions of domain experts will converge to an appropriate answer. Wang-Gao-Guo [167] recast Delphi method as a process to determine uncertainty distributions. The main steps are listed as follows:

**Step 1.** The $m$ domain experts provide their expert’s experimental data,

$$(x_{ij}, \alpha_{ij}), \quad j = 1, 2, \cdots, n_i, \ i = 1, 2, \cdots, m. \quad (16.29)$$

**Step 2.** Use the $i$-th expert’s experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ to generate the uncertainty distributions $\Phi_i$ of the $i$-th domain experts, $i = 1, 2, \cdots, m$, respectively.

**Step 3.** Compute $\Phi(x) = w_1 \Phi_1(x) + w_2 \Phi_2(x) + \cdots + w_m \Phi_m(x)$ where $w_1, w_2, \cdots, w_m$ are convex combination coefficients representing weights of the domain experts.
Step 4. If \(|\alpha_{ij} - \Phi(x_{ij})|\) are less than a given level \(\varepsilon > 0\) for all \(i\) and \(j\), then go to Step 5. Otherwise, the \(i\)-th domain experts receive the summary (for example, the function \(\Phi\) obtained in the previous round and the reasons of other experts), and then provide a set of revised expert’s experimental data \((x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \ldots, (x_{in_i}, \alpha_{in_i})\) for \(i = 1, 2, \ldots, m\). Go to Step 2.

Step 5. The last function \(\Phi\) is the uncertainty distribution to be determined.

16.7 Uncertain Regression Analysis

Let \((x_1, x_2, \ldots, x_p)\) be a vector of explanatory variables, and let \(y\) be a response variable. Assume the functional relationship between \((x_1, x_2, \ldots, x_p)\) and \(y\) is expressed by a regression model

\[
y = f(x_1, x_2, \ldots, x_p|\beta) + \varepsilon
\]  

(16.30)

where \(\beta\) is an unknown vector of parameters, and \(\varepsilon\) is a disturbance term. Especially, we will call

\[
y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon
\]  

(16.31)

a linear regression model, and call

\[
y = \beta_0 - \beta_1 \exp(-\beta_2 x) + \varepsilon, \quad \beta_1 > 0, \beta_2 > 0
\]  

(16.32)

an asymptotic regression model.

Traditionally, it is assumed that \((x_1, x_2, \ldots, x_p, y)\) are able to be precisely observed. However, in many cases, the observations of those data are imprecise and characterized in terms of uncertain variables. It is thus assumed that we have a set of imprecisely observed data,

\[
(\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{ip}, \tilde{y}_{i}), \quad i = 1, 2, \ldots, n
\]  

(16.33)

where \(\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{ip}, \tilde{y}_i\) are uncertain variables with uncertainty distributions \(\Phi_{i1}, \Phi_{i2}, \ldots, \Phi_{ip}, \Psi_i, i = 1, 2, \ldots, n\), respectively.

Parameter Estimation

Based on the imprecisely observed data (16.33), Yao-Liu [204] suggested that the least squares estimate of \(\beta\) in the regression model

\[
y = f(x_1, x_2, \ldots, x_p|\beta) + \varepsilon
\]  

(16.34)

is the solution of the minimization problem,

\[
\min_{\beta} \sum_{i=1}^{n} E[(\tilde{y}_i - f(\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{ip}|\beta))^2].
\]  

(16.35)
If the minimization solution is $\beta^*$, then the fitted regression model is determined by
\[ y = f(x_1, x_2, \cdots, x_p|\beta^*). \tag{16.36} \]

**Theorem 16.1** (Yao-Liu [204] and Lio-Liu [85]) Let $(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i)$, $i = 1, 2, \cdots, n$ be a set of imprecisely observed data, where $\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i$ are independent uncertain variables with regular uncertainty distributions $\Phi_{i1}, \Phi_{i2}, \cdots, \Phi_{ip}, \Psi_i$, $i = 1, 2, \cdots, n$, respectively. Then the least squares estimate of $\beta_0, \beta_1, \cdots, \beta_p$ in the linear regression model
\[ y = \beta_0 + \sum_{j=1}^{p} \beta_j x_j + \varepsilon \tag{16.37} \]
solves the minimization problem,
\[ \min_{\beta_0, \beta_1, \cdots, \beta_p} \sum_{i=1}^{n} \int_0^1 \left( \Psi_i^{-1}(\alpha) - \beta_0 - \sum_{j=1}^{p} \beta_j \Upsilon_{ij}^{-1}(\alpha, \beta_j) \right)^2 d\alpha \tag{16.38} \]
where
\[ \Upsilon_{ij}^{-1}(\alpha, \beta_j) = \begin{cases} \Phi_{ij}^{-1}(1 - \alpha), & \text{if } \beta_j \geq 0 \\ \Phi_{ij}^{-1}(\alpha), & \text{if } \beta_j < 0 \end{cases} \tag{16.39} \]
for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, p$.

**Proof:** Note that the least squares estimate of $\beta_0, \beta_1, \cdots, \beta_p$ in the linear regression model is the solution of the minimization problem,
\[ \min_{\beta_0, \beta_1, \cdots, \beta_p} \sum_{i=1}^{n} \text{E} \left[ \left( \tilde{y}_i - \beta_0 - \sum_{j=1}^{p} \beta_j \tilde{x}_{ij} \right)^2 \right]. \tag{16.40} \]
For each index $i$, the inverse uncertainty distribution of the uncertain variable
\[ \tilde{y}_i - \beta_0 - \sum_{j=1}^{p} \beta_j \tilde{x}_{ij} \]
is just
\[ F_i^{-1}(\alpha) = \Psi_i^{-1}(\alpha) - \beta_0 - \sum_{j=1}^{p} \beta_j \Upsilon_{ij}^{-1}(\alpha, \beta_j). \]
It follows from Theorem 2.45 that
\[ \text{E} \left[ \left( \tilde{y}_i - \beta_0 - \sum_{j=1}^{p} \beta_j \tilde{x}_{ij} \right)^2 \right] = \int_0^1 \left( \Psi_i^{-1}(\alpha) - \beta_0 - \sum_{j=1}^{p} \beta_j \Upsilon_{ij}^{-1}(\alpha, \beta_j) \right)^2 d\alpha. \]
Hence the minimization problem (16.38) is equivalent to (16.40). The theorem is thus proved.

**Exercise 16.2:** (Lio-Liu [85]) Let \((\tilde{x}_i, \tilde{y}_i), i = 1, 2, \ldots, n\) be a set of imprecisely observed data, where \(\tilde{x}_i\) and \(\tilde{y}_i\) are independent uncertain variables with regular uncertainty distributions \(\Phi_i\) and \(\Psi_i\), \(i = 1, 2, \ldots, n\), respectively. Show that the least squares estimate of \(\beta_0, \beta_1, \beta_2\) in the asymptotic regression model
\[
y = \beta_0 - \beta_1 \exp(-\beta_2 x) + \epsilon, \quad \beta_1 > 0, \beta_2 > 0
\]  
(16.41)
solves the minimization problem,
\[
\min_{\beta_0, \beta_1 > 0, \beta_2 > 0} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi_i^{-1}(\alpha) - \beta_0 + \beta_1 \exp(-\beta_2 \Phi_i^{-1}(1-\alpha)) \right)^2 d\alpha. \quad (16.42)
\]

**Exercise 16.3:** (Lio-Liu [85]) Let \((\tilde{x}_i, \tilde{y}_i), i = 1, 2, \ldots, n\) be a set of imprecisely observed data, where \(\tilde{x}_i\) and \(\tilde{y}_i\) are independent and positive uncertain variables with regular uncertainty distributions \(\Phi_i\) and \(\Psi_i\), \(i = 1, 2, \ldots, n\), respectively. Show that the least squares estimate of \(\beta_1, \beta_2\) in the regression model
\[
y = \frac{\beta_1 x}{\beta_2 + x} + \epsilon, \quad \beta_1 > 0, \beta_2 > 0
\]  
(16.43)
solves the minimization problem,
\[
\min_{\beta_1 > 0, \beta_2 > 0} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi_i^{-1}(\alpha) - \frac{\beta_1 \Phi_i^{-1}(1-\alpha)}{\beta_2 + \Phi_i^{-1}(1-\alpha)} \right)^2 d\alpha. \quad (16.44)
\]

**Disturbance Term**

**Definition 16.1** (Lio-Liu [85]) Let \((\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{ip}, \tilde{y}_i), i = 1, 2, \ldots, n\) be a set of imprecisely observed data, and let the fitted regression model be
\[
y = f(x_1, x_2, \ldots, x_p|\beta^*). \quad (16.45)
\]

Then for each index \(i (i = 1, 2, \ldots, n)\), the term
\[
\tilde{\varepsilon}_i = \tilde{y}_i - f(\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{ip}|\beta^*)
\]  
(16.46)
is called the \(i\)-th residual.

If the disturbance term \(\varepsilon\) is assumed to be an uncertain variable, then its expected value can be estimated as the average of the expected values of residuals, i.e.,
\[
\hat{\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} E[\tilde{\varepsilon}_i] \quad (16.47)
\]
and the variance can be estimated as

\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} E[(\varepsilon_i - \hat{\varepsilon})^2] \] (16.48)

where \( \varepsilon_i \) are the i-th residuals, \( i = 1, 2, \cdots, n \), respectively.

**Theorem 16.2** (Lio-Liu [85]) Let \((\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i)\), \( i = 1, 2, \cdots, n \) be a set of imprecisely observed data, where \( \tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i \) are independent uncertain variables with regular uncertainty distributions \( \Phi_{i1}, \Phi_{i2}, \cdots, \Phi_{ip}, \Psi_i, i = 1, 2, \cdots, n \), respectively, and let the fitted linear regression model be

\[ y = \beta_0^* + \sum_{j=1}^{p} \beta_j^* x_j. \] (16.49)

Then the estimated expected value of disturbance term \( \varepsilon \) is

\[ \hat{\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi_{i}^{-1}(\alpha) - \beta_0^* - \sum_{j=1}^{p} \beta_j^* \Upsilon_{ij}^{-1}(\alpha, \beta_j^*) \right) d\alpha \] (16.50)

and the estimated variance is

\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi_{i}^{-1}(\alpha) - \beta_0^* - \sum_{j=1}^{p} \beta_j^* \Upsilon_{ij}^{-1}(\alpha, \beta_j^*) - \hat{\varepsilon} \right)^2 d\alpha \] (16.51)

where

\[ \Upsilon_{ij}^{-1}(\alpha, \beta_j^*) = \begin{cases} \Phi_{ij}^{-1}(1-\alpha), & \text{if } \beta_j^* \geq 0 \\ \Phi_{ij}^{-1}(\alpha), & \text{if } \beta_j^* < 0 \end{cases} \] (16.52)

for \( i = 1, 2, \cdots, n \) and \( j = 1, 2, \cdots, p \).

**Proof:** For each index \( i \), the inverse uncertainty distribution of the i-th residual

\[ \tilde{\varepsilon}_i = \tilde{y}_i - \beta_0^* - \sum_{j=1}^{p} \beta_j^* \tilde{x}_{ij} \]

is just

\[ F_i^{-1}(\alpha) = \Psi_i^{-1}(\alpha) - \beta_0^* - \sum_{j=1}^{p} \beta_j^* \Upsilon_{ij}^{-1}(\alpha, \beta_j^*). \]

It follows from Theorems 2.26 and 2.45 that (16.50) and (16.51) hold.

**Exercise 16.4:** (Lio-Liu [85]) Let \((\tilde{x}_i, \tilde{y}_i)\), \( i = 1, 2, \cdots, n \) be a set of imprecisely observed data, where \( \tilde{x}_i \) and \( \tilde{y}_i \) are independent uncertain variables
with regular uncertainty distributions $\Phi_i$ and $\Psi_i$, $i = 1, 2, \ldots, n$, respectively, and let the fitted asymptotic regression model be

$$y = \beta_0^* - \beta_1^* \exp(-\beta_2^* x), \quad \beta_1^* > 0, \beta_2^* > 0.$$ (16.53)

Show that the estimated expected value of disturbance term $\varepsilon$ is

$$\hat{e} = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi^{-1}_i(\alpha) - \beta_0^* + \beta_1^* \exp(-\beta_2^* \Phi^{-1}_i(1-\alpha)) \right) \, d\alpha$$ (16.54)

and the estimated variance is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi^{-1}_i(\alpha) - \beta_0^* + \beta_1^* \exp(-\beta_2^* \Phi^{-1}_i(1-\alpha)) - \hat{e} \right)^2 \, d\alpha.$$ (16.55)

Exercise 16.5: (Lio-Liu [85]) Let $(\tilde{x}_i, \tilde{y}_i), i = 1, 2, \ldots, n$ be a set of imprecisely observed data, where $\tilde{x}_i$ and $\tilde{y}_i$ are independent and positive uncertain variables with regular uncertainty distributions $\Phi_i$ and $\Psi_i$, $i = 1, 2, \ldots, n$, respectively, and let the fitted regression model be

$$y = \frac{\beta_1^* x}{\beta_2^* + x}, \quad \beta_1^* > 0, \beta_2^* > 0.$$ (16.56)

Show that the estimated expected value of disturbance term $\varepsilon$ is

$$\hat{e} = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi^{-1}_i(\alpha) - \beta_0^* \Phi^{-1}_i(1-\alpha) \right) \, d\alpha$$ (16.57)

and the estimated variance is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{1} \left( \Psi^{-1}_i(\alpha) - \beta_0^* \Phi^{-1}_i(1-\alpha) - \hat{e} \right)^2 \, d\alpha.$$ (16.58)

Forecast Value

Let $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p)$ be a new explanatory vector, where $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_p$, respectively. Assume (i) the fitted regression model is

$$y = f(x_1, x_2, \ldots, x_p | \beta^*),$$ (16.59)

and (ii) the estimated disturbance term $\hat{e}$ is an uncertain variable with expected value $\hat{e}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p$. Lio-Liu [85] suggested that the forecast uncertain variable of response variable $y$ with respect to $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p)$ is determined by

$$\hat{y} = f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p | \beta^*) + \hat{e},$$ (16.60)
and the **forecast value** is defined as the expected value of the forecast uncertain variable $\hat{y}$, i.e.,

$$\mu = E[f(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p|\beta^*)] + \hat{\epsilon}.$$  \hspace{1cm} (16.61)

**Exercise 16.6:** (Lio-Liu [85]) Let $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p)$ be a new explanatory vector, where $\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p$ are independent uncertain variables. Assume (i) the fitted linear regression model is

$$y = \beta_0^* + \sum_{j=1}^{p} \beta_j^* x_j,$$  \hspace{1cm} (16.62)

and (ii) the estimated disturbance term $\hat{\epsilon}$ is an uncertain variable with expected value $\hat{\epsilon}$ and variance $\hat{\sigma}^2$, and is independent of $\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p$. Show that the forecast value of response variable $y$ with respect to $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p)$ is

$$\mu = \beta_0^* + \sum_{j=1}^{p} \beta_j^* E[\hat{x}_j] + \hat{\epsilon}.$$  \hspace{1cm} (16.63)

**Exercise 16.7:** Let $\tilde{x}$ be a new explanatory variable with regular uncertainty distribution $\Phi$. Assume (i) the fitted asymptotic regression model is

$$y = \beta_0^* - \beta_1^* \exp(-\beta_2^* x), \quad \beta_1^* > 0, \beta_2^* > 0,$$  \hspace{1cm} (16.64)

and (ii) the estimated disturbance term $\hat{\epsilon}$ is an uncertain variable with expected value $\hat{\epsilon}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}$. Show that the forecast value of response variable $y$ with respect to $\tilde{x}$ is

$$\mu = \beta_0^* - \beta_1^* \int_{0}^{1} \exp(-\beta_2^* \Phi^{-1}(\alpha))d\alpha + \hat{\epsilon}.$$  \hspace{1cm} (16.65)

**Exercise 16.8:** Let $\tilde{x}$ be a new explanatory variable with regular uncertainty distribution $\Phi$. Assume (i) the fitted regression model is

$$y = \frac{\beta_1^* x}{\beta_2^* + x}, \quad \beta_1^* > 0, \beta_2^* > 0,$$  \hspace{1cm} (16.66)

and (ii) the estimated disturbance term $\hat{\epsilon}$ is an uncertain variable with expected value $\hat{\epsilon}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}$. Show that the forecast value of response variable $y$ with respect to $\tilde{x}$ is

$$\mu = \int_{0}^{1} \frac{\beta_1^* \Phi^{-1}(\alpha)}{\beta_2^* + \Phi^{-1}(\alpha)}d\alpha + \hat{\epsilon}.$$  \hspace{1cm} (16.67)
Confidence Interval

If we further suppose that the estimated disturbance term \( \hat{\varepsilon} \) follows normal uncertainty distribution with expected value \( \hat{e} \) and variance \( \hat{\sigma}^2 \), then the forecast uncertain variable

\[
\hat{y} = f(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p|\beta^*) + \hat{\varepsilon} \tag{16.68}
\]

has an inverse uncertainty distribution

\[
\hat{\Psi}^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \cdots, \Phi_p^{-1}(1-\alpha)|\beta^*) + G^{-1}(\alpha)
\]

provided that \( f(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_p|\beta^*) \) is continuous, strictly increasing with respect to \( \hat{x}_1, \cdots, \hat{x}_k \) and strictly decreasing with respect to \( \hat{x}_{k+1}, \cdots, \hat{x}_p \), where \( G^{-1}(\alpha) \) is the inverse uncertainty distribution of \( \mathcal{N}(\hat{e}, \hat{\sigma}) \), i.e.,

\[
G^{-1}(\alpha) = \hat{e} + \frac{\hat{\sigma} \sqrt{3}}{\ln \frac{\alpha}{1-\alpha}} \tag{16.69}
\]

From \( \hat{\Psi}^{-1} \), we may also derive the uncertainty distribution \( \hat{\Psi} \) of \( \hat{y} \). Take \( \alpha \) (e.g., 95%) as the confidence level, and find the minimum value \( b \) such that

\[
\hat{\Psi}(\mu + b) - \hat{\Psi}(\mu - b) \geq \alpha \tag{16.70}
\]

Since \( M\{\mu - b \leq \hat{y} \leq \mu + b\} \geq \hat{\Psi}(\mu + b) - \hat{\Psi}(\mu - b) \geq \alpha \), Lio-Liu [85] suggested that the \( \alpha \) confidence interval of response variable \( y \) is \([\mu - b, \mu + b]\), which is often abbreviated as \( \mu \pm b \).

Exercise 16.9: (Lio-Liu [85]) Let \( (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p) \) be a new explanatory vector, where \( \tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p \) are independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_p \), respectively. Assume (i) the fitted linear regression model is

\[
y = \beta_0^* + \sum_{j=1}^{p} \beta_j^* x_j, \tag{16.72}
\]

and (ii) the estimated disturbance term \( \hat{\varepsilon} \) follows normal uncertainty distribution with expected value \( \hat{e} \) and variance \( \hat{\sigma}^2 \), and is independent of \( \tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p \). What is the \( \alpha \) confidence interval of response variable \( y \)?

Exercise 16.10: (Lio-Liu [85]) Let \( \hat{x} \) be a new explanatory variable with regular uncertainty distribution \( \Phi \). Assume (i) the fitted asymptotic regression model is

\[
y = \beta_0^* - \beta_1^* \exp(-\beta_2^* x), \quad \beta_1^* > 0, \beta_2^* > 0, \tag{16.73}
\]

and (ii) the estimated disturbance term \( \hat{\varepsilon} \) follows normal uncertainty distribution with expected value \( \hat{e} \) and variance \( \hat{\sigma}^2 \), and is independent of \( \hat{x} \). What is the \( \alpha \) confidence interval of response variable \( y \)?
Exercise 16.11: (Lio-Liu [85]) Let $\tilde{x}$ be a new explanatory variable with regular uncertainty distribution $\Phi$. Assume (i) the fitted regression model is
\[ y = \frac{\beta_1^* x}{\beta_2^* + x}, \quad \beta_1^* > 0, \beta_2^* > 0, \] (16.74)
and (ii) the estimated disturbance term $\hat{\epsilon}$ follows normal uncertainty distribution with expected value $\hat{\epsilon}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}$. What is the $\alpha$ confidence interval of response variable $y$?

Precisely Observed Data

When the observed data $(x_{i1}, x_{i2}, \cdots, x_{ip}, y_i), i = 1, 2, \cdots, n$ become precise, uncertain regression analysis is simplified into the following procedure. The least squares estimate of $\beta$ in the regression model $y = f(x_1, x_2, \cdots, x_p | \beta) + \epsilon$ is the solution, $\beta^*$, of the minimization problem,
\[ \min_{\beta} \sum_{i=1}^{n} (y_i - f(x_{i1}, x_{i2}, \cdots, x_{ip} | \beta))^2. \] (16.75)

The estimated disturbance term $\hat{\epsilon}$ is an uncertain variable (not random one) whose expected value is
\[ \hat{\epsilon} = \frac{1}{n} \sum_{i=1}^{n} (y_i - f(x_{i1}, x_{i2}, \cdots, x_{ip} | \beta^*)) \] (16.76)
and variance is
\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - f(x_{i1}, x_{i2}, \cdots, x_{ip} | \beta^*)) - \hat{\epsilon})^2. \] (16.77)

Thus the forecast value of response variable $y$ with respect to the precise explanatory vector $(x_1, x_2, \cdots, x_p)$ is
\[ \mu = f(x_1, x_2, \cdots, x_p | \beta^*) + \hat{\epsilon}. \] (16.78)
If the estimated disturbance term $\hat{\epsilon}$ is further assumed to follow normal uncertainty distribution, then the $\alpha$ confidence interval of response variable $y$ is
\[ \mu \pm \frac{\hat{\sigma} \sqrt{3}}{\pi} \ln \frac{1 + \alpha}{1 - \alpha}. \] (16.79)

A Numerical Example

Suppose that there exist 24 imprecisely observed data $(\tilde{x}_{i1}, \tilde{x}_{i2}, \tilde{x}_{i3}, \tilde{y}_i), i = 1, 2, \cdots, 24$. For each $i$, $\tilde{x}_{i1}, \tilde{x}_{i2}, \tilde{x}_{i3}, \tilde{y}_i$ are independent linear uncertain variables. See Table 16.1. Let us show how the uncertain regression analysis is used to determine the functional relationship between $(x_1, x_2, x_3)$ and $y$. 
In order to determine it, we employ the uncertain linear regression model,

\[ y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon. \]  \hfill (16.80)

By solving the minimization problem (16.38), we get the least squares estimate

\( (\beta^*_0, \beta^*_1, \beta^*_2, \beta^*_3) = (21.5196, 0.8678, 0.3110, 1.0053). \)  \hfill (16.81)

Thus the fitted linear regression model is

\[ y = 21.5196 + 0.8678 x_1 + 0.3110 x_2 + 1.0053 x_3. \]  \hfill (16.82)

By using the formulas (16.50) and (16.51), we get the estimated expected value and variance of the disturbance term are

\[ \hat{\varepsilon} = 0.0000, \quad \hat{\sigma}^2 = 5.6305, \]  \hfill (16.83)
respectively. Now let
\[(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \sim (\mathcal{L}(5, 6), \mathcal{L}(28, 30), \mathcal{L}(6, 7))\] (16.84)
be a new uncertain explanatory vector. When the estimated disturbance term \(\hat{\varepsilon}\) and \(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\) are independent, by calculating the formula (16.63), we get the forecast value of response variable \(y\) is
\[\mu = 41.8460.\] (16.85)
Taking the confidence level \(\alpha = 95\%\), if the estimated disturbance term \(\hat{\varepsilon}\) is further assumed to follow normal uncertainty distribution, then
\[b = 5.9780\] (16.86)
is the minimum value such that (16.70) holds. Therefore, the 95\% confidence interval of response variable \(y\) is
\[41.8460 \pm 5.9780.\] (16.87)

## 16.8 Uncertain Time Series Analysis

An uncertain time series is a sequence of imprecisely observed values that are characterized in terms of uncertain variables. Mathematically, an uncertain time series is represented by
\[X = \{X_1, X_2, \cdots, X_n\}\] (16.88)
where \(X_t\) are imprecisely observed values (i.e., uncertain variables) at times \(t, t = 1, 2, \cdots, n\), respectively. A basic problem of uncertain time series analysis is to predict the value of \(X_{n+1}\) based on previously observed values \(X_1, X_2, \cdots, X_n\).

The simplest approach for modelling uncertain time series is the autoregressive model,
\[X_t = a_0 + \sum_{i=1}^{k} a_i X_{t-i} + \varepsilon_t\] (16.89)
where \(a_0, a_1, \cdots, a_k\) are unknown parameters, \(\varepsilon_t\) is a disturbance term, and \(k\) is called the order of the autoregressive model.

### Parameter Estimation

Based on the imprecisely observed values \(X_1, X_2, \cdots, X_n\), Yang-Liu [181] suggested that the least squares estimate of \(a_0, a_1, \cdots, a_k\) in the autoregressive model (16.89) is the solution of the minimization problem,
\[\min_{a_0, a_1, \cdots, a_k} \sum_{t=k+1}^{n} E \left[ \left( X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i} \right)^2 \right].\] (16.90)
If the minimization solution is \((a_0^*, a_1^*, \cdots, a_k^*)\), then the fitted autoregressive model is

\[
X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}.
\]  
(16.91)

**Theorem 16.3** (Yang-Liu [181]) Let \(X_1, X_2, \cdots, X_n\) be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions \(\Phi_1, \Phi_2, \cdots, \Phi_n\), respectively. Then the least squares estimate of \(a_0, a_1, \cdots, a_k\) in the autoregressive model

\[
X_t = a_0 + \sum_{i=1}^{k} a_i X_{t-i} + \varepsilon_t
\]  
(16.92)

solves the minimization problem,

\[
\min_{a_0, a_1, \cdots, a_k} \sum_{t=k+1}^{n} \int_{0}^{1} \left( \Phi_t^{-1}(\alpha) - a_0 - \sum_{i=1}^{k} a_i \Upsilon^{-1}_{t-i}(\alpha, a_i) \right)^2 d\alpha
\]  
(16.93)

where

\[
\Upsilon^{-1}_{t-i}(\alpha, a_i) = \begin{cases} 
\Phi_{t-i}^{-1}(1 - \alpha), & \text{if } a_i \geq 0 \\
\Phi_{t-i}^{-1}(\alpha), & \text{if } a_i < 0 
\end{cases}
\]  
(16.94)

for \(i = 1, 2, \cdots, k\).

**Proof:** For each index \(t\), the inverse uncertainty distribution of the uncertain variable

\[
X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i}
\]

is just

\[
F_t^{-1}(\alpha) = \Phi_t^{-1}(\alpha) - a_0 - \sum_{i=1}^{k} a_i \Upsilon^{-1}_{t-i}(\alpha, a_i).
\]

It follows from Theorem 2.45 that

\[
E \left[ \left( X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i} \right)^2 \right] = \int_{0}^{1} \left( \Phi_t^{-1}(\alpha) - a_0 - \sum_{i=1}^{k} a_i \Upsilon^{-1}_{t-i}(\alpha, a_i) \right)^2 d\alpha.
\]

Hence the minimization problem (16.93) is equivalent to (16.90). The theorem is thus proved.
Disturbance Term

**Definition 16.2** (Yang-Liu [181]) Let $X_1, X_2, \ldots, X_n$ be imprecisely observed values, and let the fitted autoregressive model be

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}. \quad (16.95)$$

Then for each index $t$ ($t = k+1, k+2, \ldots, n$), the difference between the actual observed value and the value predicted by the model,

$$\tilde{\epsilon}_t = X_t - a_0^* - \sum_{i=1}^{k} a_i^* X_{t-i} \quad (16.96)$$

is called the $t$-th residual.

If disturbance terms $\varepsilon_{k+1}, \varepsilon_{k+2}, \ldots, \varepsilon_n$ are assumed to be iid uncertain variables (hereafter called iid hypothesis), then the expected value of disturbance terms can be estimated as the average of the expected values of residuals, i.e.,

$$\hat{e} = \frac{1}{n-k} \sum_{t=k+1}^{n} \mathbb{E}[\tilde{\epsilon}_t] \quad (16.97)$$

and the variance can be estimated as

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{t=k+1}^{n} \mathbb{E}[(\tilde{\epsilon}_t - \hat{e})^2] \quad (16.98)$$

where $\tilde{\epsilon}_t$ are the $t$-th residuals, $t = k+1, k+2, \ldots, n$, respectively.

**Theorem 16.4** (Yang-Liu [181]) Let $X_1, X_2, \ldots, X_n$ be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and let the fitted autoregressive model be

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}. \quad (16.99)$$

Then the estimated expected value of disturbance terms under the iid hypothesis is

$$\hat{e} = \frac{1}{n-k} \sum_{t=k+1}^{n} \int_0^1 \left( \Phi_t^{-1}(\alpha) - a_0^* - \sum_{i=1}^{k} a_i^* Y_{t-i}^{-1}(\alpha, a_i^*) \right) \, d\alpha \quad (16.100)$$

and the estimated variance is

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{t=k+1}^{n} \int_0^1 \left( \Phi_t^{-1}(\alpha) - a_0^* - \sum_{i=1}^{k} a_i^* Y_{t-i}^{-1}(\alpha, a_i^*) - \hat{e} \right)^2 \, d\alpha \quad (16.101)$$
where
\[
\Upsilon^{-1}_{t-i}(\alpha, a^*_i) = \begin{cases} 
\Phi^{-1}_{t-i}(1 - \alpha), & \text{if } a^*_i \geq 0 \\
\Phi^{-1}_{t-i}(\alpha), & \text{if } a^*_i < 0 
\end{cases}
\] (16.102)
for \( i = 1, 2, \cdots, k \).

**Proof:** For each index \( t \), the inverse uncertainty distribution of the \( t \)-th residual
\[
\tilde{\varepsilon}_t = X_t - a^*_0 - \sum_{i=1}^{k} a^*_i X_{t-i}
\]
is just
\[
F^{-1}_t(\alpha) = \Phi^{-1}_t(\alpha) - a^*_0 - \sum_{i=1}^{k} a^*_i \Upsilon^{-1}_{t-i}(\alpha, a^*_i).
\]
It follows from Theorems 2.26 and 2.45 that (16.100) and (16.101) hold.

**Forecast Value**

Now let \( X_1, X_2, \cdots, X_n \) be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Assume (i) the fitted autoregressive model is
\[
X_t = a^*_0 + \sum_{i=1}^{k} a^*_i X_{t-i}, \quad (16.103)
\]
and (ii) the estimated disturbance term \( \hat{\varepsilon}_{n+1} \) is an uncertain variable with expected value \( \hat{\varepsilon} \) and variance \( \hat{\sigma}^2 \), and is independent of \( X_1, X_2, \cdots, X_n \). Yang-Liu [181] suggested that the forecast uncertain variable of \( X_{n+1} \) based on \( X_1, X_2, \cdots, X_n \) is determined by
\[
\hat{X}_{n+1} = a^*_0 + \sum_{i=1}^{k} a^*_i X_{n+1-i} + \hat{\varepsilon}_{n+1}, \quad (16.104)
\]
and the forecast value is defined as the expected value of the forecast uncertain variable \( \hat{X}_{n+1} \), i.e.,
\[
\mu = a^*_0 + \sum_{i=1}^{k} a^*_i E[X_{n+1-i}] + \hat{\varepsilon}. \quad (16.105)
\]

**Confidence Interval**

If we further suppose that the estimated disturbance term \( \hat{\varepsilon}_{n+1} \) follows normal uncertainty distribution with expected value \( \hat{\varepsilon} \) and variance \( \hat{\sigma}^2 \), then the inverse uncertainty distribution of forecast uncertain variable \( \hat{X}_{n+1} \) is
\[
\hat{\Phi}^{-1}_{n+1}(\alpha) = a^*_0 + \sum_{i=1}^{k} a^*_i \Upsilon^{-1}_{n+1-i}(\alpha, a^*_i) + G^{-1}(\alpha) \quad (16.106)
\]
where
\[
\Upsilon_{n+1-i}(\alpha, a_i^*) = \begin{cases} 
\Phi_{n+1-i}^{-1}(\alpha), & \text{if } a_i^* \geq 0 \\
\Phi_{n+1-i}(1-\alpha), & \text{if } a_i^* < 0 
\end{cases}
\]  

for \( i = 1, 2, \ldots, k \), and \( G^{-1}(\alpha) \) is the inverse uncertainty distribution of \( \mathcal{N}(\hat{e}, \hat{\sigma}) \), i.e.,
\[
G^{-1}(\alpha) = \hat{e} + \hat{\sigma} \sqrt{3} \pi \ln \frac{\alpha}{1-\alpha}.
\]  

From \( \Phi_{n+1}^{-1} \), we may also derive the uncertainty distribution \( \Phi_{n+1} \) of \( \hat{X}_{n+1} \). Take \( \alpha \) (e.g., 95%) as the confidence level, and find the minimum value \( b \) such that
\[
\hat{\Phi}_{n+1}(\mu + b) - \hat{\Phi}_{n+1}(\mu - b) \geq \alpha.
\]  

Since \( M\{\mu - b \leq \hat{X}_{n+1} \leq \mu + b\} \geq \Phi_{n+1}(\mu + b) - \Phi_{n+1}(\mu - b) \geq \alpha \), Yang-Liu [181] suggested that the \( \alpha \) confidence interval of \( X_{n+1} \) is \([\mu - b, \mu + b]\), which is often abbreviated as
\[
\mu \pm b.
\]  

Precisely Observed Values

When the observed values \( X_1, X_2, \ldots, X_n \) become precise, uncertain time series analysis is simplified into the following procedure. The least squares estimate of \( a_0, a_1, \ldots, a_k \) in the autoregressive model
\[
X_t = a_0 + \sum_{i=1}^{k} a_i X_{t-i} + \varepsilon_t
\]  

is the solution, \((a_0^*, a_1^*, \ldots, a_k^*)\), of the minimization problem,
\[
\min_{a_0, a_1, \ldots, a_k} \sum_{t=k+1}^{n} \left( X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i} \right)^2.
\]  

If disturbance terms are assumed to be iid uncertain variables (iid hypothesis), then the expected value of disturbance terms can be estimated as
\[
\hat{\varepsilon} = \frac{1}{n-k} \sum_{t=k+1}^{n} \left( X_t - a_0^* - \sum_{i=1}^{k} a_i^* X_{t-i} \right),
\]  

and the variance can be estimated as
\[
\hat{\sigma}^2 = \frac{1}{n-k} \sum_{t=k+1}^{n} \left( X_t - a_0^* - \sum_{i=1}^{k} a_i^* X_{t-i} - \hat{\varepsilon} \right)^2.
\]
Thus the forecast value of $X_{n+1}$ is

$$\mu = a_0^* + \sum_{i=1}^{k} a_i^* X_{n+1-i} + \hat{\epsilon}. \quad (16.115)$$

If the estimated disturbance term $\hat{\epsilon}_{n+1}$ is further assumed to follow normal uncertainty distribution, then the $\alpha$ confidence interval of $X_{n+1}$ is

$$\mu \pm \hat{\sigma} \sqrt{\frac{3}{\pi}} \ln \frac{1 + \alpha}{1 - \alpha}. \quad (16.116)$$

### A Numerical Example

Assume there exist 20 imprecisely observed carbon emissions $X_1, X_2, \cdots, X_{20}$ that are characterized in terms of independent linear uncertain variables. See Table 16.2. Let us show how the uncertain time series analysis is used to forecast the carbon emission in the 21st year.

#### Table 16.2: Imprecisely Observed Carbon Emissions over 20 Years

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}(330, 341)$</td>
<td>$\mathcal{L}(333, 346)$</td>
<td>$\mathcal{L}(335, 347)$</td>
<td>$\mathcal{L}(338, 350)$</td>
<td>$\mathcal{L}(340, 354)$</td>
</tr>
<tr>
<td>$\mathcal{L}(343, 359)$</td>
<td>$\mathcal{L}(344, 364)$</td>
<td>$\mathcal{L}(346, 366)$</td>
<td>$\mathcal{L}(350, 366)$</td>
<td>$\mathcal{L}(355, 369)$</td>
</tr>
<tr>
<td>$\mathcal{L}(360, 372)$</td>
<td>$\mathcal{L}(362, 376)$</td>
<td>$\mathcal{L}(365, 381)$</td>
<td>$\mathcal{L}(370, 384)$</td>
<td>$\mathcal{L}(373, 390)$</td>
</tr>
<tr>
<td>$\mathcal{L}(379, 391)$</td>
<td>$\mathcal{L}(380, 398)$</td>
<td>$\mathcal{L}(384, 402)$</td>
<td>$\mathcal{L}(388, 410)$</td>
<td>$\mathcal{L}(390, 415)$</td>
</tr>
</tbody>
</table>

In order to forecast it, we employ the 2-order uncertain autoregressive model,

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t. \quad (16.117)$$

By solving the minimization problem (16.93), we get the least squares estimate

$$(a_0^*, a_1^*, a_2^*) = (28.4715, 0.2367, 0.7018). \quad (16.118)$$

Thus the fitted autoregressive model is

$$X_t = 28.4715 + 0.2367 X_{t-1} + 0.7018 X_{t-2}. \quad (16.119)$$

By using the formulas (16.100) and (16.101), we get the estimated expected value and variance of disturbance term are

$$\hat{\epsilon} = 0.0000, \quad \hat{\sigma}^2 = 84.7422, \quad (16.120)$$
respectively. When the estimated disturbance term $\hat{\varepsilon}_{21}$ is assumed to be independent of $X_{20}$ and $X_{19}$, by calculating the formula (16.105), we get the forecast value of carbon emission in the 21st year (i.e., $X_{21}$) is

$$\mu = 403.7361.$$  \hfill (16.121)

Taking the confidence level $\alpha = 95\%$, if the estimated disturbance term $\hat{\varepsilon}_{21}$ is further assumed to follow normal uncertainty distribution, then

$$b = 28.7376$$  \hfill (16.122)

is the minimum value such that (16.109) holds. Therefore, the 95\% confidence interval of carbon emission in the 21st year (i.e., $X_{21}$) is

$$403.7361 \pm 28.7376.$$  \hfill (16.123)

### 16.9 Bibliographic Notes

The study of uncertain statistics was started by Liu [95] in 2010 in which a questionnaire survey for collecting expert’s experimental data was designed. It was showed among others by Chen-Ralescu [12] that the questionnaire survey may successfully acquire the expert’s experimental data.

Parametric uncertain statistics assumes that the uncertainty distribution to be determined has a known functional form but with unknown parameters. In order to estimate the unknown parameters, Liu [95] suggested the principle of least squares, and Wang-Peng [169] proposed the method of moments. Nonparametric uncertain statistics does not rely on the expert’s experimental data belonging to any particular uncertainty distribution. In order to determine the uncertainty distributions, Liu [95] introduced the linear interpolation method (i.e., empirical uncertainty distribution), and Chen-Ralescu [12] proposed a series of spline interpolation methods. When multiple domain experts are available, Wang-Gao-Guo [167] recast Delphi method as a process to determine uncertainty distributions.

In order to determine membership functions, a questionnaire survey for collecting expert’s experimental data was designed by Liu [96]. Based on expert’s experimental data, Liu [96] also suggested the linear interpolation method and the principle of least squares to determine membership functions. When multiple domain experts are available, Delphi method was introduced to uncertain statistics by Guo-Wang-Wang-Chen [59].

Uncertain regression analysis is used to model the relationship between explanatory variables and response variables when the imprecise observations are characterized in terms of uncertain variables. For that matter, Yao-Liu [204] suggested a point estimation for the unknown parameters in uncertain regression models by the principle of least squares, and Lio-Liu [85] proposed an interval estimation for predicting the response variables. After that, Liu-Jia [125] developed a cross-validation method for evaluating the performance of uncertain regression models.
Uncertain time series analysis was first presented by Yang-Liu [181] in order to predict the future values based on previously imprecise observations that are characterized in terms of uncertain variables.
Appendix A

Chance Theory

Uncertainty and randomness are two basic types of indeterminacy. Uncertain random variable was initialized by Liu [117] in 2013 for modelling complex systems with not only uncertainty but also randomness. This appendix will introduce the concepts of chance measure, uncertain random variable, chance distribution, operational law, expected value, variance, and law of large numbers. This appendix will also solve the choice problem in Ellsberg experiment by chance theory. Finally, uncertain random programming, uncertain random risk analysis, uncertain random reliability analysis, uncertain random graph, uncertain random network, and uncertain random process are documented.

A.1 Chance Measure

Let \((\Gamma, \mathcal{L}, \mathcal{M})\) be an uncertainty space and let \((\Omega, A, \Pr)\) be a probability space. Then the product \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, A, \Pr)\) is called a chance space. Essentially, it is another triplet, 
\[
(\Gamma \times \Omega, \mathcal{L} \times A, \mathcal{M} \times \Pr) \tag{A.1}
\]
where \(\Gamma \times \Omega\) is the universal set, \(\mathcal{L} \times A\) is the product \(\sigma\)-algebra, and \(\mathcal{M} \times \Pr\) is the product measure.

The universal set \(\Gamma \times \Omega\) is clearly the set of all ordered pairs of the form \((\gamma, \omega)\), where \(\gamma \in \Gamma\) and \(\omega \in \Omega\). That is,
\[
\Gamma \times \Omega = \{(\gamma, \omega) \mid \gamma \in \Gamma, \omega \in \Omega\}. \tag{A.2}
\]
Note that \(\Gamma \times \Omega\) can be understood as a rectangular coordinate system if \(\Gamma\) is understood as the horizontal axis and \(\Omega\) is understood as the vertical axis. The product \(\sigma\)-algebra \(\mathcal{L} \times A\) is the smallest \(\sigma\)-algebra containing measurable rectangles of the form \(\Lambda \times A\), where \(\Lambda \in \mathcal{L}\) and \(A \in A\). Each element in \(\mathcal{L} \times A\) is called an event in the chance space. What is the product measure \(\mathcal{M} \times \Pr\) for an event \(\Theta\)? We will call \(\mathcal{M} \times \Pr\) chance measure and represent it by \(\text{Ch}\{\Theta\}\).
Definition A.1 (Liu [117]) Let \((\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{A}, \Pr)\) be a chance space, and let \(\Theta \in \mathcal{L} \times \mathcal{A}\) be an event. Then the chance measure of \(\Theta\) is defined as

\[
\text{Ch}\{\Theta\} = \int_0^1 \Pr \{\omega \in \Omega | M\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} \, dx.
\] (A.3)

Remark A.1: Note that \(M\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\}\) is just the uncertain measure of cross section of \(\Theta\) at \(\omega\). Since \(M\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\}\) can be regarded as a function from the probability space \((\Omega, \mathcal{A}, \Pr)\) to \([0, 1]\), it is a random variable. Thus the chance measure \(\text{Ch}\{\Theta\}\) is just the expected value (i.e., average value) of this random variable.

Exercise A.1: Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \([0, 1]\) with Borel algebra and Lebesgue measure, and take a probability space \((\Omega, \mathcal{A}, \Pr)\) to be also \([0, 1]\) with Borel algebra and Lebesgue measure. Then

\[
\Theta = \{(\gamma, \omega) \in \Gamma \times \Omega | \gamma + \omega \leq 1\}
\] (A.4)

is an event in the chance space \((\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{A}, \Pr)\). Show that

\[
\text{Ch}\{\Theta\} = \frac{1}{2}.
\] (A.5)

Exercise A.2: Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \([0, 1]\) with Borel algebra and Lebesgue measure, and take a probability space \((\Omega, \mathcal{A}, \Pr)\) to be also \([0, 1]\) with Borel algebra and Lebesgue measure. Then

\[
\Theta = \{(\gamma, \omega) \in \Gamma \times \Omega | (\gamma - 0.5)^2 + (\omega - 0.5)^2 \leq 0.5^2\}
\] (A.6)

is an event in the chance space \((\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{A}, \Pr)\). Show that

\[
\text{Ch}\{\Theta\} = \frac{\pi}{4}.
\] (A.7)

Theorem A.1 (Liu [117]) Let \((\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{A}, \Pr)\) be a chance space. Then

\[
\text{Ch}\{\Lambda \times A\} = M\{\Lambda\} \times \Pr\{A\}
\] (A.8)

for any \(\Lambda \in \mathcal{L}\) and any \(A \in \mathcal{A}\). Furthermore, we have

\[
\text{Ch}\{\emptyset\} = 0, \quad \text{Ch}\{\Gamma \times \Omega\} = 1.
\] (A.9)

Proof: Let us first prove the identity (A.8). For any real number \(x \in (0, 1]\), if \(M\{\Lambda\} \geq x\), then

\[
\Pr \{\omega \in \Omega | M\{\gamma \in \Gamma | (\gamma, \omega) \in \Lambda \times A\} \geq x\} = \Pr\{A\}.
\]

If \(M\{\Lambda\} < x\), then

\[
\Pr \{\omega \in \Omega | M\{\gamma \in \Gamma | (\gamma, \omega) \in \Lambda \times A\} \geq x\} = \Pr\{\emptyset\} = 0.
\]
Thus

\[ \text{Ch}\{\Lambda \times A\} = \int_0^1 \Pr \{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Lambda \times A\} \geq x\} \, dx \]

\[ = \int_0^{\mathcal{M}\{\Lambda\}} \Pr\{A\} \, dx + \int_1^{\mathcal{M}\{\Lambda\}} 0 \, dx \]

\[ = \mathcal{M}\{\Lambda\} \times \Pr\{A\}. \]

Furthermore, it follows from (A.8) that

\[ \text{Ch}\{\emptyset\} = \mathcal{M}\{\emptyset\} \times \Pr\{\emptyset\} = 0, \]

\[ \text{Ch}\{\Gamma \times \Omega\} = \mathcal{M}\{\Gamma\} \times \Pr\{\Omega\} = 1. \]

The theorem is thus verified.

**Theorem A.2** (Liu [117], Monotonicity Theorem) The chance measure is a monotone increasing set function. That is, for any events \( \Theta_1 \) and \( \Theta_2 \) with \( \Theta_1 \subset \Theta_2 \), we have

\[ \text{Ch}\{\Theta_1\} \leq \text{Ch}\{\Theta_2\}. \quad (A.10) \]

**Proof:** Since \( \Theta_1 \) and \( \Theta_2 \) are two events with \( \Theta_1 \subset \Theta_2 \), for each \( \omega \), we immediately have

\[ \{\gamma \in \Gamma | (\gamma, \omega) \in \Theta_1\} \subset \{\gamma \in \Gamma | (\gamma, \omega) \in \Theta_2\} \]

and

\[ \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta_1\} \leq \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta_2\}. \]

Thus

\[ \text{Ch}\{\Theta_1\} = \int_0^1 \Pr \{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta_1\} \geq x\} \, dx \]

\[ \leq \int_0^1 \Pr \{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta_2\} \geq x\} \, dx \]

\[ = \text{Ch}\{\Theta_2\}. \]

That is, \( \text{Ch}\{\Theta\} \) is a monotone increasing function with respect to \( \Theta \). The theorem is thus verified.

**Theorem A.3** (Liu [117], Duality Theorem) The chance measure is self-dual. That is, for any event \( \Theta \), we have

\[ \text{Ch}\{\Theta\} + \text{Ch}\{\Theta^c\} = 1. \quad (A.11) \]
**Proof:** Since both uncertain measure and probability measure are self-dual, we have

\[
\text{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x\} \, dx
\]

\[
= \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta^c\} \leq 1 - x\} \, dx
\]

\[
= \int_0^1 (1 - \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta^c\} > 1 - x\}) \, dx
\]

\[
= 1 - \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta^c\} > x\} \, dx
\]

\[
= 1 - \text{Ch}\{\Theta^c\}.
\]

That is, \(\text{Ch}\{\Theta\} + \text{Ch}\{\Theta^c\} = 1\), i.e., the chance measure is self-dual.

**Theorem A.4** (*Hou [62], Subadditivity Theorem*) The chance measure is subadditive. That is, for any countable sequence of events \(\Theta_1, \Theta_2, \ldots\), we have

\[
\text{Ch} \left\{ \bigcup_{i=1}^{\infty} \Theta_i \right\} \leq \sum_{i=1}^{\infty} \text{Ch}\{\Theta_i\}. \tag{A.12}
\]

**Proof:** At first, it follows from the subadditivity of uncertain measure that

\[
M \left\{ \gamma \in \Gamma \mid (\gamma, \omega) \in \bigcup_{i=1}^{\infty} \Theta_i \right\} \leq \sum_{i=1}^{\infty} M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta_i\}.
\]

Thus

\[
\text{Ch} \left\{ \bigcup_{i=1}^{\infty} \Theta_i \right\} = \int_0^1 \Pr\left\{ \omega \in \Omega \mid M \left\{ \gamma \in \Gamma \mid (\gamma, \omega) \in \bigcup_{i=1}^{\infty} \Theta_i \right\} \geq x \right\} \, dx
\]

\[
\leq \int_0^{+\infty} \Pr\left\{ \omega \in \Omega \mid \sum_{i=1}^{\infty} M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta_i\} \geq x \right\} \, dx
\]

\[
= \sum_{i=1}^{\infty} \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta_i\} \geq x\} \, dx
\]

\[
= \sum_{i=1}^{\infty} \text{Ch}\{\Theta_i\}.
\]

That is, the chance measure is subadditive.
A.2 Uncertain Random Variable

Definition A.2 (Liu [117]) An uncertain random variable is a function $\xi$ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set $B$ of real numbers.

Remark A.2: An uncertain random variable $\xi(\gamma, \omega)$ degenerates to a random variable if it does not vary with $\gamma$. Thus a random variable is a special uncertain random variable.

Remark A.3: An uncertain random variable $\xi(\gamma, \omega)$ degenerates to an uncertain variable if it does not vary with $\omega$. Thus an uncertain variable is a special uncertain random variable.

Theorem A.5 Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain random variables on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$, and let $f$ be a measurable function. Then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{A.13}$$

is an uncertain random variable determined by

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega)) \tag{A.14}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain random variables, we know that they are measurable functions on the chance space, and $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is also a measurable function. Hence $\xi$ is an uncertain random variable.

Example A.1: A random variable $\eta$ plus an uncertain variable $\tau$ makes an uncertain random variable $\xi$, i.e.,

$$\xi(\gamma, \omega) = \eta(\omega) + \tau(\gamma) \tag{A.15}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Example A.2: A random variable $\eta$ times an uncertain variable $\tau$ makes an uncertain random variable $\xi$, i.e.,

$$\xi(\gamma, \omega) = \eta(\omega) \cdot \tau(\gamma) \tag{A.16}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Theorem A.6 (Liu [117]) Let $\xi$ be an uncertain random variable on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$, and let $B$ be a Borel set of real numbers. Then $\{\xi \in B\}$ is an uncertain random event with chance measure

$$\text{Ch}\{\xi \in B\} = \int_0^1 \text{Pr}\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | \xi(\gamma, \omega) \in B\} \geq x\} \, dx. \tag{A.17}$$
**Proof:** Since \( \{\xi \in B\} \) is an event in the chance space, the equation (A.17) follows from Definition A.1 immediately.

**Remark A.4:** If the uncertain random variable degenerates to a random variable \( \eta \), then \( \text{Ch}\{\eta \in B\} = \text{Ch}\{(\eta \in B) \times \Omega\} = \mathcal{M}\{\eta \in B\} \times \text{Pr}\{\eta \in B\} = \text{Pr}\{\eta \in B\} \). That is,
\[
\text{Ch}\{\eta \in B\} = \text{Pr}\{\eta \in B\}. \quad (A.18)
\]

**Remark A.5:** If the uncertain random variable degenerates to an uncertain variable \( \tau \), then \( \text{Ch}\{\tau \in B\} = \text{Ch}\{(\tau \in B) \times \Omega\} = \mathcal{M}\{\tau \in B\} \times \text{Pr}\{\Omega\} = \mathcal{M}\{\tau \in B\} \). That is,
\[
\text{Ch}\{\tau \in B\} = \mathcal{M}\{\tau \in B\}. \quad (A.19)
\]

**Theorem A.7** (Liu [117]) Let \( \xi \) be an uncertain random variable. Then the chance measure \( \text{Ch}\{\xi \in B\} \) is a monotone increasing function of \( B \) and
\[
\text{Ch}\{\xi \in \emptyset\} = 0, \quad \text{Ch}\{\xi \in \mathbb{R}\} = 1. \quad (A.20)
\]

**Proof:** Let \( B_1 \) and \( B_2 \) be Borel sets of real numbers with \( B_1 \subset B_2 \). Then we immediately have \( \{\xi \in B_1\} \subset \{\xi \in B_2\} \). It follows from the monotonicity of chance measure that
\[
\text{Ch}\{\xi \in B_1\} \leq \text{Ch}\{\xi \in B_2\}.
\]
Hence \( \text{Ch}\{\xi \in B\} \) is a monotone increasing function of \( B \). Furthermore, we have
\[
\text{Ch}\{\xi \in \emptyset\} = \text{Ch}\{\emptyset\} = 0,
\]
\[
\text{Ch}\{\xi \in \mathbb{R}\} = \text{Ch}\{\Gamma \times \Omega\} = 1.
\]
The theorem is verified.

**Theorem A.8** (Liu [117]) Let \( \xi \) be an uncertain random variable. Then for any Borel set \( B \) of real numbers, we have
\[
\text{Ch}\{\xi \in B\} + \text{Ch}\{\xi \in B^c\} = 1. \quad (A.21)
\]

**Proof:** It follows from \( \{\xi \in B\}^c = \{\xi \in B^c\} \) and the duality of chance measure immediately.

### A.3 Chance Distribution

**Definition A.3** (Liu [117]) Let \( \xi \) be an uncertain random variable. Then its chance distribution is defined by
\[
\Phi(x) = \text{Ch}\{\xi \leq x\} \quad (A.22)
\]
for any \( x \in \mathbb{R} \).
Example A.3: As a special uncertain random variable, the chance distribution of a random variable $\eta$ is just its probability distribution, that is,

$$\Phi(x) = \text{Ch}\{ \eta \leq x \} = \text{Pr}\{ \eta \leq x \}.$$  \hspace{1cm} (A.23)

Example A.4: As a special uncertain random variable, the chance distribution of an uncertain variable $\tau$ is just its uncertainty distribution, that is,

$$\Phi(x) = \text{Ch}\{ \tau \leq x \} = \mathcal{M}\{ \tau \leq x \}.$$  \hspace{1cm} (A.24)

Theorem A.9 \cite{Liu117, Sufficent and Necessary Condition for Chance Distribution} A function $\Phi : \mathbb{R} \to [0, 1]$ is a chance distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.

Proof: Assume $\Phi$ is a chance distribution of uncertain random variable $\xi$. Let $x_1$ and $x_2$ be two real numbers with $x_1 < x_2$. It follows from Theorem A.7 that

$$\Phi(x_1) = \text{Ch}\{ \xi \leq x_1 \} \leq \text{Ch}\{ \xi \leq x_2 \} = \Phi(x_2).$$

Hence the chance distribution $\Phi$ is a monotone increasing function. Furthermore, if $\Phi(x) \equiv 0$, then

$$\int_0^1 \text{Pr}\{ \omega \in \Omega \mid \mathcal{M}\{ \gamma \in \Gamma \mid \xi(\gamma, \omega) \leq x \} \geq r \} \, dr \equiv 0.$$  

Thus for almost all $\omega \in \Omega$, we have

$$\mathcal{M}\{ \gamma \in \Gamma \mid \xi(\gamma, \omega) \leq x \} \equiv 0, \quad \forall x \in \mathbb{R}$$

which is in contradiction to the asymptotic theorem, and then $\Phi(x) \not\equiv 0$ is verified. Similarly, if $\Phi(x) \equiv 1$, then

$$\int_0^1 \text{Pr}\{ \omega \in \Omega \mid \mathcal{M}\{ \gamma \in \Gamma \mid \xi(\gamma, \omega) \leq x \} \geq r \} \, dr \equiv 1.$$  

Thus for almost all $\omega \in \Omega$, we have

$$\mathcal{M}\{ \gamma \in \Gamma \mid \xi(\gamma, \omega) \leq x \} \equiv 1, \quad \forall x \in \mathbb{R}$$

which is also in contradiction to the asymptotic theorem, and then $\Phi(x) \not\equiv 1$ is proved.

Conversely, suppose $\Phi : \mathbb{R} \to [0, 1]$ is a monotone increasing function but $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$. It follows from Peng-Iwamura theorem that there is an uncertain variable whose uncertainty distribution is just $\Phi(x)$. Since an uncertain variable is a special uncertain random variable, we know that $\Phi$ is a chance distribution.
Theorem A.10 (Liu [117], Chance Inversion Theorem) Let \( \xi \) be an uncertain random variable with chance distribution \( \Phi \). Then for any real number \( x \), we have
\[
\text{Ch}\{\xi \leq x\} = \Phi(x), \quad \text{Ch}\{\xi > x\} = 1 - \Phi(x).
\] (A.25)

Proof: The equation \( \text{Ch}\{\xi \leq x\} = \Phi(x) \) follows from the definition of chance distribution immediately. By using the duality of chance measure, we get
\[
\text{Ch}\{\xi > x\} = 1 - \text{Ch}\{\xi \leq x\} = 1 - \Phi(x).
\]

Remark A.6: When the chance distribution \( \Phi \) is a continuous function, we also have
\[
\text{Ch}\{\xi < x\} = \Phi(x), \quad \text{Ch}\{\xi \geq x\} = 1 - \Phi(x).
\] (A.26)

A.4 Operational Law

Assume \( \eta_1, \eta_2, \ldots, \eta_m \) are independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and \( \tau_1, \tau_2, \ldots, \tau_n \) are independent uncertain variables with uncertainty distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. What is the chance distribution of the uncertain random variable
\[
\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)?
\] (A.27)

This section will provide an operational law to answer this question.

Theorem A.11 (Liu [118]) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent uncertain variables with uncertainty distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. If \( f \) is a measurable function, then the uncertain random variable
\[
\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)
\] (A.28)

has a chance distribution
\[
\Phi(x) = \int_{\mathbb{R}^m} F(x; y_1, y_2, \ldots, y_m) \, d\Psi_1(y_1) \, d\Psi_2(y_2) \cdots d\Psi_m(y_m)
\] (A.29)

where
\[
F(x; y_1, y_2, \ldots, y_m) = \mathcal{M}\{f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n) \leq x\}
\] (A.30)

is the uncertainty distribution of \( f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n) \) for any real numbers \( y_1, y_2, \ldots, y_m \), and is determined by \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \).
Proof: It follows from Theorem A.6 that the uncertain random variable $\xi$ has a chance distribution

$$
\Phi(x) = \int_0^1 \Pr \{ \omega \in \Omega | M \{ \gamma \in \Gamma | \xi(\gamma, \omega) \leq x \} \geq r \} \, dr
$$

$$
= \int_0^1 \Pr \{ \omega \in \Omega | M \{ f(\eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) \leq x \} \geq r \} \, dr
$$

$$
= \int_{\mathbb{R}^m} M \{ f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n) \leq x \} \, d\Psi_1(y_1) \cdots d\Psi_m(y_m)
$$

$$
= \int_{\mathbb{R}^m} F(x; y_1, y_2, \cdots, y_m) \, d\Psi_1(y_1) \cdots d\Psi_2(y_2) \cdots d\Psi_m(y_m).
$$

The theorem is thus verified.

Exercise A.3: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the sum

$$
\xi = \eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n \quad (A.31)
$$

has a chance distribution

$$
\Phi(x) = \int_{-\infty}^{+\infty} \Upsilon(x - y) \, d\Psi(y) \quad (A.32)
$$

where

$$
\Psi(y) = \int_{y_1 + y_2 + \cdots + y_m \leq y} d\Psi_1(y_1) \cdots d\Psi_m(y_m) \quad (A.33)
$$

is the probability distribution of $\eta_1 + \eta_2 + \cdots + \eta_m$, and

$$
\Upsilon(z) = \sup_{z_1 + z_2 + \cdots + z_n = z} \Upsilon_1(z_1) \wedge \Upsilon_2(z_2) \wedge \cdots \wedge \Upsilon_n(z_n) \quad (A.34)
$$

is the uncertainty distribution of $\tau_1 + \tau_2 + \cdots + \tau_n$.

Exercise A.4: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent positive random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent positive uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the product

$$
\xi = \eta_1 \eta_2 \cdots \eta_m \tau_1 \tau_2 \cdots \tau_n \quad (A.35)
$$

has a chance distribution

$$
\Phi(x) = \int_{0}^{+\infty} \Upsilon(x/y) \, d\Psi(y) \quad (A.36)
$$
where
\[
\Psi(y) = \int_{y_1 y_2 \cdots y_m \leq y} d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m) \quad (A.37)
\]
is the probability distribution of \(\eta_1 \eta_2 \cdots \eta_m\), and
\[
\Upsilon(z) = \sup_{z_1 z_2 \cdots z_n = z} \Upsilon_1(z_1) \wedge \Upsilon_2(z_2) \wedge \cdots \wedge \Upsilon_n(z_n) \quad (A.38)
\]
is the uncertainty distribution of \(\tau_1 \tau_2 \cdots \tau_n\).

**Exercise A.5:** Let \(\eta_1, \eta_2, \cdots, \eta_m\) be independent random variables with probability distributions \(\Psi_1, \Psi_2, \cdots, \Psi_m\), and let \(\tau_1, \tau_2, \cdots, \tau_n\) be independent uncertain variables with uncertainty distributions \(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n\), respectively. Show that the minimum
\[
\xi = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n \quad (A.39)
\]
has a chance distribution
\[
\Phi(x) = \Psi(x) + \Upsilon(x) - \Psi(x) \Upsilon(x) \quad (A.40)
\]
where
\[
\Psi(x) = 1 - (1 - \Psi_1(x))(1 - \Psi_2(x)) \cdots (1 - \Psi_m(x)) \quad (A.41)
\]
is the probability distribution of \(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m\), and
\[
\Upsilon(x) = \Upsilon_1(x) \vee \Upsilon_2(x) \vee \cdots \vee \Upsilon_n(x) \quad (A.42)
\]
is the uncertainty distribution of \(\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n\).

**Exercise A.6:** Let \(\eta_1, \eta_2, \cdots, \eta_m\) be independent random variables with probability distributions \(\Psi_1, \Psi_2, \cdots, \Psi_m\), and let \(\tau_1, \tau_2, \cdots, \tau_n\) be independent uncertain variables with uncertainty distributions \(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n\), respectively. Show that the maximum
\[
\xi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \quad (A.43)
\]
has a chance distribution
\[
\Phi(x) = \Psi(x) \Upsilon(x) \quad (A.44)
\]
where
\[
\Psi(x) = \Psi_1(x) \Psi_2(x) \cdots \Psi_m(x) \quad (A.45)
\]
is the probability distribution of \(\eta_1 \vee \eta_2 \vee \cdots \vee \eta_m\), and
\[
\Upsilon(x) = \Upsilon_1(x) \wedge \Upsilon_2(x) \wedge \cdots \wedge \Upsilon_n(x) \quad (A.46)
\]
is the uncertainty distribution of \(\tau_1 \vee \tau_2 \vee \cdots \vee \tau_n\).
Theorem A.12 (Liu [118]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is continuous, strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$. Then the uncertain random variable

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$  \hspace{1cm} (A.47)

has a chance distribution

$$\Phi(x) = \int_{\mathbb{R}^m} F(x; y_1, y_2, \cdots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m)$$  \hspace{1cm} (A.48)

where $F(x; y_1, y_2, \cdots, y_m)$ is the root $\alpha$ of the equation

$$f(y_1, y_2, \cdots, y_m, \Upsilon^{-1}_1(\alpha), \cdots, \Upsilon^{-1}_k(\alpha), \Upsilon^{-1}_{k+1}(1-\alpha), \cdots, \Upsilon^{-1}_n(1-\alpha)) = x.$$  \hspace{1cm} (A.49)

Proof: Since $F(x; y_1, y_2, \cdots, y_m) = M\{f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n) \leq x\}$ is just the root $\alpha$ of the equation

$$f(y_1, y_2, \cdots, y_m, \Upsilon^{-1}_1(\alpha), \cdots, \Upsilon^{-1}_k(\alpha), \Upsilon^{-1}_{k+1}(1-\alpha), \cdots, \Upsilon^{-1}_n(1-\alpha)) = x,$$

we get the result by Theorem A.11.

Order Statistics

Definition A.4 (Gao-Sun-Ralescu [43], Order Statistic) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain random variables, and let $k$ be an index with $1 \leq k \leq n$. Then

$$\xi = k\text{-min}[\xi_1, \xi_2, \cdots, \xi_n]$$  \hspace{1cm} (A.49)

is called the $k$th order statistic of $\xi_1, \xi_2, \cdots, \xi_n$.

Theorem A.13 (Gao-Sun-Ralescu [43]) Let $\eta_1, \eta_2, \cdots, \eta_n$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_n$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f_1, f_2, \cdots, f_n$ are continuous and strictly increasing functions, then the $k$th order statistic of $f_1(\eta_1, \tau_1), f_2(\eta_2, \tau_2), \cdots, f_n(\eta_n, \tau_n)$ has a chance distribution

$$\Phi(x) = \int_{\mathbb{R}^n} k\text{-max} \left[ \begin{array}{c} \sup_{f_1(y_1, z_1) = x} \Upsilon_1(z_1) \\ \sup_{f_2(y_2, z_2) = x} \Upsilon_2(z_2) \\ \vdots \\ \sup_{f_n(y_n, z_n) = x} \Upsilon_n(z_n) \end{array} \right] d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_n(y_n).$$
**Proof:** For each index \(i\) and each real number \(y_i\), since \(f_i\) is a strictly increasing function, the uncertain variable \(f_i(y_i, \tau_i)\) has an uncertainty distribution

\[
F_i(x; y_i) = \sup_{f_i(y_i, \tau_i) = x} \Upsilon_i(z_i).
\]

Theorem 2.18 states that the \(k\)th order statistic of \(f_1(y_1, \tau_1), f_2(y_2, \tau_2), \ldots, f_n(y_n, \tau_n)\) has an uncertainty distribution

\[
F(x; y_1, y_2, \ldots, y_n) = k-\max \left[ \sup_{f_1(y_1, \tau_1) = x} \Upsilon_1(z_1), \ldots, \sup_{f_n(y_n, \tau_n) = x} \Upsilon_n(z_n) \right].
\]

Thus the theorem follows from the operational law of uncertain random variables immediately.

**Exercise A.7:** Let \(\eta_1, \eta_2, \ldots, \eta_n\) be independent random variables with probability distributions \(\Psi_1, \Psi_2, \ldots, \Psi_n\), and let \(\tau_1, \tau_2, \ldots, \tau_n\) be independent uncertain variables with continuous uncertainty distributions \(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n\), respectively. Assume \(f_1, f_2, \ldots, f_n\) are continuous and strictly decreasing functions. Show that the \(k\)th order statistic of \(f_1(\eta_1, \tau_1), f_2(\eta_2, \tau_2), \ldots, f_n(\eta_n, \tau_n)\) has a chance distribution

\[
\Phi(x) = \int_{\mathbb{R}^n} k-\max \left[ \sup_{f_1(\eta_1, \tau_1) = x} (1 - \Upsilon_1(z_1)), \ldots, \sup_{f_n(\eta_n, \tau_n) = x} (1 - \Upsilon_n(z_n)) \right] d\Psi_1(\eta_1)d\Psi_2(\eta_2)\cdots d\Psi_n(\eta_n).
\]

**Exercise A.8:** Let \(\eta_1, \eta_2, \ldots, \eta_m\) be independent random variables with probability distributions \(\Psi_1, \Psi_2, \ldots, \Psi_m\), and let \(\tau_1, \tau_2, \ldots, \tau_n\) be independent uncertain variables with uncertainty distributions \(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n\), respectively. What is the \(k\)th order statistic of \(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n\)?

**Operational Law for Boolean System**

**Theorem A.14** (Liu [118]) Assume \(\eta_1, \eta_2, \ldots, \eta_m\) are independent Boolean random variables, i.e.,

\[
\eta_i = \begin{cases} 
1 & \text{with probability measure } a_i \\
0 & \text{with probability measure } 1 - a_i 
\end{cases} \quad (A.50)
\]

for \(i = 1, 2, \ldots, m\), and \(\tau_1, \tau_2, \ldots, \tau_n\) are independent Boolean uncertain variables, i.e.,

\[
\tau_j = \begin{cases} 
1 & \text{with uncertain measure } b_j \\
0 & \text{with uncertain measure } 1 - b_j 
\end{cases} \quad (A.51)
\]
for \( j = 1, 2, \cdots, n \). If \( f \) is a Boolean function, then

\[
\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)
\]

is a Boolean uncertain random variable such that

\[
\text{Ch}\{\xi = 1\} = \sum_{(x_1, \cdots, x_m) \in \{0, 1\}^m} \left( \prod_{i=1}^{m} \mu_i(x_i) \right) f^*(x_1, \cdots, x_m) \quad (A.53)
\]

where

\[
f^*(x_1, \cdots, x_m) = \begin{cases} 
\sup & \min_{1 \leq j \leq n} \nu_j(y_j), \\
 & \text{if } \sup_{f(x_1, \cdots, x_m, y_1, \cdots, y_n) = 1} \min_{1 \leq j \leq n} \nu_j(y_j) < 0.5 \\
1 - \sup & \min_{f(x_1, \cdots, x_m, y_1, \cdots, y_n) = 0} \min_{1 \leq j \leq n} \nu_j(y_j), \\
& \text{if } \sup_{f(x_1, \cdots, x_m, y_1, \cdots, y_n) = 1} \min_{1 \leq j \leq n} \nu_j(y_j) \geq 0.5,
\end{cases} \quad (A.54)
\]

\[
\mu_i(x_i) = \begin{cases} 
a_i, & \text{if } x_i = 1 \\
1 - a_i, & \text{if } x_i = 0
\end{cases} \quad (i = 1, 2, \cdots, m), \quad (A.55)
\]

\[
\nu_j(y_j) = \begin{cases} 
b_j, & \text{if } y_j = 1 \\
1 - b_j, & \text{if } y_j = 0
\end{cases} \quad (j = 1, 2, \cdots, n). \quad (A.56)
\]

**Proof:** At first, when \((x_1, \cdots, x_m)\) is given, \(f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n)\) is essentially a Boolean function of uncertain variables. It follows from the operational law of uncertain variables that

\[
\mathcal{M}\{f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n) = 1\} = f^*(x_1, \cdots, x_m)
\]

that is determined by (A.54). On the other hand, it follows from the operational law of uncertain random variables that

\[
\text{Ch}\{\xi = 1\} = \sum_{(x_1, \cdots, x_m) \in \{0, 1\}^m} \left( \prod_{i=1}^{m} \mu_i(x_i) \right) \mathcal{M}\{f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n) = 1\}.
\]

Thus (A.53) is verified.

**Remark A.7:** When the uncertain variables disappear, the operational law becomes

\[
\text{Pr}\{\xi = 1\} = \sum_{(x_1, x_2, \cdots, x_m) \in \{0, 1\}^m} \left( \prod_{i=1}^{m} \mu_i(x_i) \right) f(x_1, x_2, \cdots, x_m). \quad (A.57)
\]
Remark A.8: When the random variables disappear, the operational law becomes

\[
M\{\xi = 1\} = \begin{cases} 
\sup_{f(y_1,y_2,\cdots,y_n)=1} \min_{1\leq j\leq n} \nu_j(y_j), \\
\hbox{if } \sup_{f(y_1,y_2,\cdots,y_n)=1} \min_{1\leq j\leq n} \nu_j(y_j) < 0.5 \\
1 - \sup_{f(y_1,y_2,\cdots,y_n)=0} \min_{1\leq j\leq n} \nu_j(y_j), \\
\hbox{if } \sup_{f(y_1,y_2,\cdots,y_n)=1} \min_{1\leq j\leq n} \nu_j(y_j) \geq 0.5.
\end{cases}
\] (A.58)

Exercise A.9: Let \(\eta_1,\eta_2,\cdots,\eta_m\) be independent Boolean random variables defined by (A.50) and let \(\tau_1,\tau_2,\cdots,\tau_n\) be independent Boolean uncertain variables defined by (A.51). Then the minimum

\[
\xi = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n
\] (A.59)

is a Boolean uncertain random variable. Show that

\[
\text{Ch}\{\xi = 1\} = a_1 a_2 \cdots a_m (b_1 \wedge b_2 \wedge \cdots \wedge b_n).
\] (A.60)

Exercise A.10: Let \(\eta_1,\eta_2,\cdots,\eta_m\) be independent Boolean random variables defined by (A.50) and let \(\tau_1,\tau_2,\cdots,\tau_n\) be independent Boolean uncertain variables defined by (A.51). Then the maximum

\[
\xi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n
\] (A.61)

is a Boolean uncertain random variable. Show that

\[
\text{Ch}\{\xi = 1\} = 1 - (1 - a_1)(1 - a_2)\cdots(1 - a_m)(1 - b_1 \vee b_2 \vee \cdots \vee b_n).
\] (A.62)

Exercise A.11: Let \(\eta_1,\eta_2,\cdots,\eta_m\) be independent Boolean random variables defined by (A.50) and let \(\tau_1,\tau_2,\cdots,\tau_n\) be independent Boolean uncertain variables defined by (A.51). Then the \(k\)th order statistic

\[
\xi = k\min [\eta_1,\eta_2,\cdots,\eta_m,\tau_1,\tau_2,\cdots,\tau_n]
\] (A.63)

is a Boolean uncertain random variable. Show that

\[
\text{Ch}\{\xi = 1\} = \sum_{(x_1,\cdots,x_m) \in \{0,1\}^m} \left( \prod_{i=1}^m \mu_i(x_i) \right) k\min [x_1,\cdots,x_m,b_1,\cdots,b_n]
\]

where

\[
\mu_i(x_i) = \begin{cases} 
a_i, & \text{if } x_i = 1 \\
1 - a_i, & \text{if } x_i = 0
\end{cases} \quad (i = 1, 2, \cdots, m).
\] (A.64)
Exercise A.12: Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent Boolean random variables defined by (A.50) and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent Boolean uncertain variables defined by (A.51). Then
\[
\xi = k\text{-max} \left[ \eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n \right] \quad (A.65)
\]
is the \((n - k + 1)\)th order statistic. Show that
\[
\text{Ch}\{\xi = 1\} = \sum_{(x_1, \ldots, x_m) \in \{0,1\}^m} \prod_{i=1}^{m} \mu_i(x_i) \cdot k\text{-max} [x_1, \ldots, x_m, b_1, \ldots, b_n]
\]
where
\[
\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \ldots, m). \quad (A.66)
\]

A.5 Expected Value

Definition A.5 (Liu [117]) Let \( \xi \) be an uncertain random variable. Then its expected value is defined by
\[
E[\xi] = \int_{0}^{+\infty} \text{Ch}\{\xi \geq x\} \, dx - \int_{-\infty}^{0} \text{Ch}\{\xi \leq x\} \, dx \quad (A.67)
\]
provided that at least one of the two integrals is finite.

Theorem A.15 (Liu [117]) Let \( \xi \) be an uncertain random variable with chance distribution \( \Phi \). Then
\[
E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x)) \, dx - \int_{-\infty}^{0} \Phi(x) \, dx. \quad (A.68)
\]

Proof: It follows from the chance inversion theorem that for almost all numbers \( x \), we have \( \text{Ch}\{\xi \geq x\} = 1 - \Phi(x) \) and \( \text{Ch}\{\xi \leq x\} = \Phi(x) \). By using the definition of expected value operator, we obtain
\[
E[\xi] = \int_{0}^{+\infty} \text{Ch}\{\xi \geq x\} \, dx - \int_{-\infty}^{0} \text{Ch}\{\xi \leq x\} \, dx
\]
\[
= \int_{0}^{+\infty} (1 - \Phi(x)) \, dx - \int_{-\infty}^{0} \Phi(x) \, dx.
\]
Thus we obtain the equation (A.68).

Theorem A.16 Let \( \xi \) be an uncertain random variable with chance distribution \( \Phi \). Then
\[
E[\xi] = \int_{-\infty}^{+\infty} x \Phi(x) \, dx. \quad (A.69)
\]
Proof: It follows from the change of variables of integral and Theorem A.15 that the expected value is

\[ E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx \]
\[ = \int_{0}^{+\infty} x d\Phi(x) + \int_{-\infty}^{0} x d\Phi(x) = \int_{-\infty}^{+\infty} x d\Phi(x). \]

The theorem is proved.

Theorem A.17 Let \( \xi \) be an uncertain random variable with regular chance distribution \( \Phi \). Then

\[ E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha. \]  

(A.70)

Proof: It follows from the change of variables of integral and Theorem A.15 that the expected value is

\[ E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx \]
\[ = \int_{\Phi(0)}^{1} \Phi^{-1}(\alpha) d\alpha + \int_{0}^{\Phi(0)} \Phi^{-1}(\alpha) d\alpha = \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha. \]

The theorem is proved.

Theorem A.18 (Liu [118]) Let \( \eta_1, \eta_2, \cdots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \cdots, \Psi_m \), and let \( \tau_1, \tau_2, \cdots, \tau_n \) be independent uncertain variables with uncertainty distributions \( \Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n \), respectively. If \( f \) is a measurable function, then

\[ \xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \]  

(A.71)

has an expected value

\[ E[\xi] = \int_{\mathbb{R}^m} G(y_1, y_2, \cdots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m) \]  

(A.72)

where

\[ G(y_1, y_2, \cdots, y_m) = E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)] \]  

(A.73)

is the expected value of the uncertain variable \( f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n) \) for any real numbers \( y_1, y_2, \cdots, y_m \), and is determined by \( \Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n \).
**Proof:** For simplicity, we only prove the case $m = n = 2$. Write the uncertainty distribution of $f(y_1, y_2, \tau_1, \tau_2)$ by $F(x; y_1, y_2)$ for any real numbers $y_1$ and $y_2$. Then

$$E[f(y_1, y_2, \tau_1, \tau_2)] = \int_0^{+\infty} (1 - F(x; y_1, y_2)) dx - \int_{-\infty}^0 F(x; y_1, y_2) dx.$$

On the other hand, the uncertain random variable $\xi = f(\eta_1, \eta_2, \tau_1, \tau_2)$ has a chance distribution

$$\Phi(x) = \int_{\mathbb{R}^2} F(x; y_1, y_2) d\Psi_1(y_1) d\Psi_2(y_2).$$

It follows from Theorem A.15 and Fubini theorem that

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx$$

$$= \int_0^{+\infty} \left(1 - \int_{\mathbb{R}^2} F(x; y_1, y_2) d\Psi_1(y_1) d\Psi_2(y_2)\right) dx$$

$$- \int_{-\infty}^0 \int_{\mathbb{R}^2} F(x; y_1, y_2) d\Psi_1(y_1) d\Psi_2(y_2) dx$$

$$= \int_{\mathbb{R}^2} \left(\int_0^{+\infty} (1 - F(x; y_1, y_2)) dx - \int_{-\infty}^0 F(x; y_1, y_2) dx\right) d\Psi_1(y_1) d\Psi_2(y_2)$$

$$= \int_{\mathbb{R}^2} E[f(y_1, y_2, \tau_1, \tau_2)] d\Psi_1(y_1) d\Psi_2(y_2).$$

Thus the theorem is proved.

**Exercise A.13:** Let $\eta$ be a random variable and let $\tau$ be an uncertain variable. Show that

$$E[\eta + \tau] = E[\eta] + E[\tau]$$

and

$$E[\eta \tau] = E[\eta] E[\tau].$$

**Theorem A.19** (Liu [118]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a continuous and strictly increasing function (or strictly decreasing function) with respect to $\tau_1, \cdots, \tau_n$, then the expected function

$$E[f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)]$$

is equal to

$$\int_{\mathbb{R}^m} \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m).$$
Proof: Since \( f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \) is a continuous and strictly increasing function (or strictly decreasing function) with respect to \( \tau_1, \cdots, \tau_n \), we have
\[
E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)] = \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha))d\alpha.
\]
It follows from Theorem A.18 that the result holds.

Remark A.9: If \( f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \) is continuous, strictly increasing with respect to \( \tau_1, \cdots, \tau_k \) and strictly decreasing with respect to \( \tau_{k+1}, \cdots, \tau_n \), then the integrand in the formula of expected value should be replaced with
\[
f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)).
\]

Exercise A.14: Let \( \eta \) be a random variable with probability distribution \( \Psi \), and let \( \tau \) be an uncertain variable with regular uncertainty distribution \( \Upsilon \). Show that
\[
E[\eta \lor \tau] = \int_\mathbb{R} \int_0^1 (y \lor \Upsilon^{-1}(\alpha)) d\Psi(y)
\]  
(A.77)
and
\[
E[\eta \land \tau] = \int_\mathbb{R} \int_0^1 (y \land \Upsilon^{-1}(\alpha)) d\Psi(y).
\]  
(A.78)

Theorem A.20 (Liu [118], Linearity of Expected Value Operator) Assume \( \eta_1 \) and \( \eta_2 \) are random variables (not necessarily independent), \( \tau_1 \) and \( \tau_2 \) are independent uncertain variables, and \( f_1 \) and \( f_2 \) are measurable functions. Then
\[
E[f_1(\eta_1, \tau_1) + f_2(\eta_2, \tau_2)] = E[f_1(\eta_1, \tau_1)] + E[f_2(\eta_2, \tau_2)].
\]  
(A.79)

Proof: Since \( \tau_1 \) and \( \tau_2 \) are independent uncertain variables, for any real numbers \( y_1 \) and \( y_2 \), the functions \( f_1(y_1, \tau_1) \) and \( f_2(y_2, \tau_2) \) are also independent uncertain variables. Thus
\[
E[f_1(y_1, \tau_1) + f_2(y_2, \tau_2)] = E[f_1(y_1, \tau_1)] + E[f_2(y_2, \tau_2)].
\]
Let \( \Psi_1 \) and \( \Psi_2 \) be the probability distributions of random variables \( \eta_1 \) and \( \eta_2 \), respectively. Then we have
\[
E[f_1(\eta_1, \tau_1) + f_2(\eta_2, \tau_2)]
= \int_{\mathbb{R}^2} E[f_1(y_1, \tau_1) + f_2(y_2, \tau_2)]d\Psi_1(y_1)d\Psi_2(y_2)
= \int_{\mathbb{R}^2} (E[f_1(y_1, \tau_1)] + E[f_2(y_2, \tau_2)])d\Psi_1(y_1)d\Psi_2(y_2)
= \int_{\mathbb{R}} E[f_1(y_1, \tau_1)]d\Psi_1(y_1) + \int_{\mathbb{R}} E[f_2(y_2, \tau_2)]d\Psi_2(y_2)
= E[f_1(\eta_1, \tau_1)] + E[f_2(\eta_2, \tau_2)].
The theorem is proved.

**Exercise A.15:** Assume $\eta_1$ and $\eta_2$ are random variables, and $\tau_1$ and $\tau_2$ are independent uncertain variables. Show that

$$E[\eta_1 \lor \tau_1 + \eta_2 \land \tau_2] = E[\eta_1 \lor \tau_1] + E[\eta_2 \land \tau_2]. \quad (A.80)$$

**Theorem A.21** (Liu [117]) Let $\xi$ be an uncertain random variable, and let $f$ be a nonnegative even function. If $f$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$, then for any given number $t > 0$, we have

$$\text{Ch}\{|\xi| \geq t\} \leq \frac{E[f(\xi)]}{f(t)}. \quad (A.81)$$

**Proof:** It is clear that $\text{Ch}\{|\xi| \geq f^{-1}(r)\}$ is a monotone decreasing function of $r$ on $[0, \infty)$. It follows from the nonnegativity of $f(\xi)$ that

$$E[f(\xi)] = \int_0^{+\infty} \text{Ch}\{f(\xi) \geq x\}dx = \int_0^{+\infty} \text{Ch}\{|\xi| \geq f^{-1}(x)\}dx \geq \int_0^{f(t)} \text{Ch}\{|\xi| \geq f^{-1}(x)\}dx \geq \int_0^{f(t)} \text{Ch}\{|\xi| \geq f^{-1}(f(t))\}dx = \int_0^{f(t)} \text{Ch}\{|\xi| \geq t\}dx = f(t) \cdot \text{Ch}\{|\xi| \geq t\}$$

which proves the inequality.

**Theorem A.22** (Liu [117], Markov Inequality) Let $\xi$ be an uncertain random variable. Then for any given numbers $t > 0$ and $p > 0$, we have

$$\text{Ch}\{|\xi| \geq t\} \leq \frac{E[|\xi|^p]}{t^p}. \quad (A.82)$$

**Proof:** It is a special case of Theorem A.21 when $f(x) = |x|^p$.

### A.6 Variance

**Definition A.6** (Liu [117]) Let $\xi$ be an uncertain random variable with finite expected value $e$. Then the variance of $\xi$ is

$$V[\xi] = E[(\xi - e)^2]. \quad (A.83)$$

Since $(\xi - e)^2$ is a nonnegative uncertain random variable, we also have

$$V[\xi] = \int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}dx. \quad (A.84)$$
Theorem A.23 (Liu [117]) If $\xi$ is an uncertain random variable with finite expected value, $a$ and $b$ are real numbers, then

$$V[a\xi + b] = a^2V[\xi].$$ (A.85)

**Proof:** Let $e$ be the expected value of $\xi$. Then $a\xi + b$ has an expected value $ae + b$. Thus the variance is

$$V[a\xi + b] = E[(a\xi + b - (ae + b))^2] = E[a^2(\xi - e)^2] = a^2V[\xi].$$

The theorem is verified.

Theorem A.24 (Liu [117]) Let $\xi$ be an uncertain random variable with expected value $e$. Then $V[\xi] = 0$ if and only if $\text{Ch}\{\xi = e\} = 1$.

**Proof:** We first assume $V[\xi] = 0$. It follows from the equation (A.84) that

$$\int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}dx = 0$$

which implies $\text{Ch}\{(\xi - e)^2 \geq x\} = 0$ for any $x > 0$. Hence we have

$$\text{Ch}\{(\xi - e)^2 = 0\} = 1.$$

That is, $\text{Ch}\{\xi = e\} = 1$. Conversely, assume $\text{Ch}\{\xi = e\} = 1$. Then we immediately have $\text{Ch}\{(\xi - e)^2 = 0\} = 1$ and $\text{Ch}\{(\xi - e)^2 \geq x\} = 0$ for any $x > 0$. Thus

$$V[\xi] = \int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}dx = 0.$$

The theorem is proved.

Theorem A.25 (Liu [117], Chebyshev Inequality) Let $\xi$ be an uncertain random variable whose variance exists. Then for any given number $t > 0$, we have

$$\text{Ch}\{|\xi - E[\xi]| \geq t\} \leq \frac{V[\xi]}{t^2}. \quad (A.86)$$

**Proof:** It is a special case of Theorem A.21 when the uncertain random variable $\xi$ is replaced with $\xi - E[\xi]$, and $f(x) = x^2$. 
How to Obtain Variance from Distributions?

Let $\xi$ be an uncertain random variable with expected value $e$. If we only know its chance distribution $\Phi$, then the variance

$$V[\xi] = \int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}dx$$

$$= \int_0^{+\infty} \text{Ch}\{(\xi \geq e + \sqrt{x}) \cup (\xi \leq e - \sqrt{x})\}dx$$

$$\leq \int_0^{+\infty} (\text{Ch}\{\xi \geq e + \sqrt{x}\} + \text{Ch}\{\xi \leq e - \sqrt{x}\})dx$$

$$= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))dx.$$

Thus we have the following stipulation.

**Stipulation A.1** *(Guo-Wang [58])* Let $\xi$ be an uncertain random variable with chance distribution $\Phi$ and finite expected value $e$. Then

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{y}) + \Phi(e - \sqrt{y}))dy. \quad (A.87)$$

**Theorem A.26** *(Sheng-Yao [153])* Let $\xi$ be an uncertain random variable with chance distribution $\Phi$ and finite expected value $e$. Then

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2d\Phi(x). \quad (A.88)$$

**Proof:** This theorem is based on Stipulation A.1 that says the variance of $\xi$ is

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))dy + \int_0^{+\infty} \Phi(e - \sqrt{y})dy.$$ 

Substituting $e + \sqrt{y}$ with $x$ and $y$ with $(x - e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))dy = \int_e^{+\infty} (1 - \Phi(x))d(x - e)^2 = \int_e^{+\infty} (x - e)^2d\Phi(x).$$

Similarly, substituting $e - \sqrt{y}$ with $x$ and $y$ with $(x - e)^2$, we obtain

$$\int_0^{+\infty} \Phi(e - \sqrt{y})dy = \int_e^{-\infty} \Phi(x)d(x - e)^2 = \int_e^{+\infty} (x - e)^2d\Phi(x).$$

It follows that the variance is

$$V[\xi] = \int_e^{+\infty} (x - e)^2d\Phi(x) + \int_e^{+\infty} (x - e)^2d\Phi(x) = \int_{-\infty}^{+\infty} (x - e)^2d\Phi(x).$$

The theorem is verified.
Theorem A.27 (Sheng-Yao [153]) Let $\xi$ be an uncertain random variable with regular chance distribution $\Phi$ and finite expected value $e$. Then

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha. \quad (A.89)$$

**Proof:** Substituting $\Phi(x)$ with $\alpha$ and $x$ with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem A.26 that the variance is

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 d\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha.$$

The theorem is verified.

Theorem A.28 (Guo-Wang [58]) Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. Assume $f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$ is continuous, strictly increasing with respect to $\tau_1, \tau_2, \ldots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \ldots, \tau_n$. Then

$$\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \quad (A.90)$$

has a variance

$$V[\xi] = \int_{\mathbb{R}^m} \int_0^{+\infty} (1 - F(e + \sqrt{x}; y_1, y_2, \ldots, y_m) + F(e - \sqrt{x}; y_1, y_2, \ldots, y_m)) dxd\Psi_1(y_1)d\Psi_2(y_2)\ldots d\Psi_m(y_m)$$

where $F(x; y_1, y_2, \ldots, y_m)$ is the root $\alpha$ of the equation

$$f(y_1, y_2, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1 - \alpha), \ldots, \Upsilon_n^{-1}(1 - \alpha)) = x.$$

**Proof:** It follows from the operational law of uncertain random variables that $\xi$ has a chance distribution

$$\Phi(x) = \int_{\mathbb{R}^m} F(x; y_1, y_2, \ldots, y_m) d\Psi_1(y_1)d\Psi_2(y_2)\ldots d\Psi_m(y_m)$$

where $F(x; y_1, y_2, \ldots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$. Thus the theorem follows Stipulation A.1 immediately.

**Exercise A.16:** Let $\eta$ be a random variable with probability distribution $\Psi$, and let $\tau$ be an uncertain variable with uncertainty distribution $\Upsilon$. Show that the sum

$$\xi = \eta + \tau \quad (A.91)$$

has a variance

$$V[\xi] = \int_{-\infty}^{+\infty} \int_0^{+\infty} (1 - \Upsilon(e + \sqrt{x} - y) + \Upsilon(e - \sqrt{x} - y)) dxd\Psi(y). \quad (A.92)$$
A.7 Law of Large Numbers

**Theorem A.29** (Yao-Gao [198]. Law of Large Numbers) Let \( \eta_1, \eta_2, \cdots \) be iid random variables with a common probability distribution \( \Psi \), and let \( \tau_1, \tau_2, \cdots \) be iid uncertain variables. Assume \( f \) is a strictly monotone function. Then

\[
S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n) \tag{A.93}
\]

is a sequence of uncertain random variables and

\[
\frac{S_n}{n} \to \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \tag{A.94}
\]

in the sense of convergence in distribution as \( n \to \infty \).

**Proof:** According to the definition of convergence in distribution, it suffices to prove

\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \int_{-\infty}^{+\infty} f(y, z) d\Psi(y) \right\} = M \left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq \int_{-\infty}^{+\infty} f(y, z) d\Psi(y) \right\} \tag{A.95}
\]

for any real number \( z \) with

\[
\lim_{w \to z} M \left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq \int_{-\infty}^{+\infty} f(y, w) d\Psi(y) \right\} = M \left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq \int_{-\infty}^{+\infty} f(y, z) d\Psi(y) \right\}. 
\]

The argument breaks into two cases. Case 1: Assume \( f(y, z) \) is strictly increasing with respect to \( z \). Let \( \Upsilon \) denote the common uncertainty distribution of \( \tau_1, \tau_2, \cdots \). It is clear that

\[
M \{ f(y, \tau_1) \leq f(y, z) \} = \Upsilon(z)
\]

for any real numbers \( y \) and \( z \). Thus we have

\[
M \left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq \int_{-\infty}^{+\infty} f(y, z) d\Psi(y) \right\} = \Upsilon(z). \tag{A.96}
\]

In addition, since \( f(\eta_1, z), f(\eta_2, z), \cdots \) are a sequence of iid random variables, the law of large numbers for random variables tells us that

\[
\frac{f(\eta_1, z) + f(\eta_2, z) + \cdots + f(\eta_n, z)}{n} \to \int_{-\infty}^{+\infty} f(y, z) d\Psi(y), \text{ a.s.}
\]
as \( n \to \infty \). Thus
\[
\lim_{n \to \infty} \mathrm{Ch}\left\{ \frac{S_n}{n} \leq \int_{-\infty}^{+\infty} f(y, z) d\Psi(y) \right\} = \Upsilon(z).
\] (A.97)

It follows from (A.96) and (A.97) that (A.95) holds. Case 2: Assume \( f(y, z) \) is strictly decreasing with respect to \( z \). Then \( -f(y, z) \) is strictly increasing with respect to \( z \). By using Case 1, we obtain
\[
\lim_{n \to \infty} \mathrm{Ch}\left\{ -\frac{S_n}{n} < -z \right\} = \mathcal{M}\left\{ -\int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) < -z \right\}.
\]
That is,
\[
\lim_{n \to \infty} \mathrm{Ch}\left\{ \frac{S_n}{n} > z \right\} = \mathcal{M}\left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) > z \right\}.
\]
It follows from the duality property that
\[
\lim_{n \to \infty} \mathrm{Ch}\left\{ \frac{S_n}{n} \leq z \right\} = \mathcal{M}\left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq z \right\}.
\]
The theorem is thus proved.

**Exercise A.17:** Let \( \eta_1, \eta_2, \cdots \) be iid random variables, and let \( \tau_1, \tau_2, \cdots \) be iid uncertain variables. Define
\[
S_n = (\eta_1 + \tau_1) + (\eta_2 + \tau_2) + \cdots + (\eta_n + \tau_n).
\] (A.98)
Show that
\[
\frac{S_n}{n} \to E[\eta_1] + \tau_1
\] (A.99)
in the sense of convergence in distribution as \( n \to \infty \).

**Exercise A.18:** Let \( \eta_1, \eta_2, \cdots \) be iid positive random variables, and let \( \tau_1, \tau_2, \cdots \) be iid positive uncertain variables. Define
\[
S_n = \eta_1 \tau_1 + \eta_2 \tau_2 + \cdots + \eta_n \tau_n.
\] (A.100)
Show that
\[
\frac{S_n}{n} \to E[\eta_1] \tau_1
\] (A.101)
in the sense of convergence in distribution as \( n \to \infty \).

### A.8 Ellsberg Experiment

Assume an Ellsberg urn contains 30 red balls and 60 other balls that are either black or yellow in unknown proportion. One ball is randomly drawn from the urn. Consider the following two options:
A: You receive $30 if the drawn ball is red;
B: You receive $30 if the drawn ball is black\(^1\).

What is your choice between A and B? Through a lot of surveys, Ellsberg [32] showed that most people strictly prefer A to B since they prefer gambling on a known number of balls to gambling on an unknown number. However, the choice problem is clearly a scientific one. Does it make sense to vote on such a scientific problem? Definitely no! What we need is a scientific method rather than public opinion. Therefore, in order to solve the choice problem, Liu [110] pioneered a rigorous mathematical solution comprehensively by uncertainty theory, probability theory and chance theory, and concluded that we should be indifferent between the two options.

At first, all balls are virtually numbered from 1 to 90 in order of first black, then yellow and finally red. Take an uncertainty space \((\Gamma, \mathcal{L}, M)\) to be \(\{0, 1, 2, \cdots, 60\}\) with power set and uncertain measure

\[
M\{\Lambda\} = \frac{|\Lambda|}{61} \quad (A.102)
\]

where \(|\Lambda|\) represents the cardinality of \(\Lambda\). Since the composition of black and yellow balls is completely unknown and can be any integer pair among

\[(0, 60), (1, 59), (2, 58), \cdots, (58, 2), (59, 1), (60, 0)\]

with equal belief degrees, we can treat the number of black balls as an uncertain variable

\[
\xi(\gamma) = \gamma, \quad (A.103)
\]

and then the number of yellow balls is another uncertain variable

\[
\eta(\gamma) = 60 - \gamma. \quad (A.104)
\]

It is easy to verify that \(\xi\) and \(\eta\) are identically distributed uncertain variables with

\[
\xi = i \text{ with belief degree } \frac{1}{61}, \quad i = 0, 1, 2, \cdots, 60, \quad (A.105)
\]

\[
\eta = j \text{ with belief degree } \frac{1}{61}, \quad j = 0, 1, 2, \cdots, 60, \quad (A.106)
\]

and \(\xi + \eta = 60\). Thus the formulations (A.103) and (A.104) are indeed “fair” for both black and yellow balls. Therefore, the black balls are numbered from 1 to \(\gamma\), the yellow balls are numbered from \(\gamma + 1\) to 60, and the red balls are numbered from 61 to 90.

Take a probability space \((\Omega, \mathcal{A}, \Pr)\) to be \(\{1, 2, \cdots, 90\}\) with power set and probability measure

\[
\Pr\{\Lambda\} = \frac{|\Lambda|}{90}. \quad (A.107)
\]

\(^1\)The color is arbitrarily chosen by you from black and yellow.
Thus drawing one ball in an equally likely manner from the urn is equivalent to sampling one \( \omega \) from the probability space \((\Omega, \mathcal{A}, \Pr)\).

Since drawing a ball from Ellsberg urn is a mixture of uncertainty (unknown number of balls) and randomness (randomly drawing a ball), it has to be represented by an event in the chance space

\[
(\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{A}, \Pr) . \tag{A.108}
\]

Especially, a red ball is drawn if and only if \( \omega \geq 61 \). Thus drawing a red ball is represented by the event,

\[
"\text{red}\" = \{ (\gamma, \omega) \in \Gamma \times \Omega \mid \omega \geq 61 \} . \tag{A.109}
\]

A black ball is drawn if and only if \( \omega \leq \gamma \). Thus drawing a black ball is represented by the event,

\[
"\text{black}\" = \{ (\gamma, \omega) \in \Gamma \times \Omega \mid \omega \leq \gamma \} . \tag{A.110}
\]

A yellow ball is drawn if and only if \( \omega > \gamma \) and \( \omega \leq 60 \). Thus drawing a yellow ball is represented by the event,

\[
"\text{yellow}\" = \{ (\gamma, \omega) \in \Gamma \times \Omega \mid \gamma < \omega \leq 60 \} . \tag{A.111}
\]

See Figure A.1.

![Figure A.1: Three Events: “red”, “black” and “yellow”](image)

It follows from Definition A.1 that the chance measure of drawing a red
ball is

\[
\text{Ch}\{\text{red}\} = \int_0^1 \Pr \{ \omega \in \Omega \mid M\{ \gamma \in \Gamma \mid (\gamma, \omega) \in \text{"red"} \} \geq x \} \, dx
\]

\[
= \int_0^1 \Pr \{ \omega \in \{61, 62, \ldots, 90\} \mid M\{0, 1, \ldots, 60\} \geq x \} \, dx
\]

\[
= \int_0^1 \Pr \left\{ \omega \in \{61, 62, \ldots, 90\} \mid \frac{61}{61} \geq x \right\} \, dx
\]

\[
= \int_0^1 \Pr \{61, 62, \ldots, 90\} \, dx
\]

\[
= \int_0^1 \frac{30}{90} \, dx
\]

\[
= \frac{1}{3},
\]

the chance measure of drawing a black ball is

\[
\text{Ch}\{\text{black}\} = \int_0^1 \Pr \{ \omega \in \Omega \mid M\{ \gamma \in \Gamma \mid (\gamma, \omega) \in \text{"black"} \} \geq x \} \, dx
\]

\[
= \int_0^1 \Pr \{ \omega \in \{1, 2, \ldots, 60\} \mid M\{\omega, \omega + 1, \ldots, 60\} \geq x \} \, dx
\]

\[
= \int_0^1 \Pr \left\{ \omega \in \{1, 2, \ldots, 60\} \mid \frac{61 - \omega}{61} \geq x \right\} \, dx
\]

\[
= \sum_{k=0}^{60} \int_{\frac{k+1}{61}}^{\frac{k+1}{61}} \Pr \left\{ \omega \in \{1, 2, \ldots, 60\} \mid \frac{61 - \omega}{61} \geq x \right\} \, dx
\]

\[
= \sum_{k=0}^{60} \int_{\frac{k+1}{61}}^{\frac{k+1}{61}} \Pr \{1, 2, \ldots, 60 - k\} \, dx
\]

\[
= \sum_{k=0}^{60} \int_{\frac{k+1}{61}}^{\frac{k+1}{61}} \frac{60 - k}{90} \, dx
\]

\[
= \frac{1}{3},
\]
and the chance measure of drawing a yellow ball is

\[
    \text{Ch}\{\text{“yellow”}\} = \int_0^1 \Pr \{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \text{“yellow”}\} \geq x\} \, dx
\]

\[
    = \int_0^1 \Pr \{\omega \in \{1, 2, \cdots, 60\} \mid M\{0, 1, \cdots, \omega - 1\} \geq x\} \, dx
\]

\[
    = \int_0^1 \Pr \{\omega \in \{1, 2, \cdots, 60\} \mid \frac{\omega}{61} \geq x\} \, dx
\]

\[
    = \sum_{k=0}^{60} \int_{\frac{k}{61}}^{\frac{k+1}{61}} \Pr \{\omega \in \{1, 2, \cdots, 60\} \mid \frac{\omega}{61} \geq x\} \, dx
\]

\[
    = \sum_{k=0}^{60} \int_{\frac{k}{61}}^{\frac{k+1}{61}} \Pr \{k + 1, k + 2, \cdots, 60\} \, dx
\]

\[
    = \sum_{k=0}^{60} \int_{\frac{k}{61}}^{\frac{k+1}{61}} \frac{60 - k}{90} \, dx
\]

\[
    = \frac{1}{3}. \tag{A.112}
\]

Now we are ready to solve the choice problem. The income of A is an uncertain random variable

\[
    A(\gamma, \omega) = \begin{cases} 
        30, & \text{if } (\gamma, \omega) \in \text{“red”} \\
        0, & \text{otherwise}
    \end{cases} \tag{A.113}
\]

whose expected value is

\[
    E[A] = 30 \times \text{Ch}\{\text{“red”}\} + 0 \times (1 - \text{Ch}\{\text{“red”}\}) = 10. \tag{A.114}
\]

The income of B is an uncertain random variable

\[
    B(\gamma, \omega) = \begin{cases} 
        30, & \text{if } (\gamma, \omega) \in \text{“black”} \\
        0, & \text{otherwise}
    \end{cases} \tag{A.115}
\]

whose expected value is

\[
    E[B] = 30 \times \text{Ch}\{\text{“black”}\} + 0 \times (1 - \text{Ch}\{\text{“black”}\}) = 10. \tag{A.116}
\]

It follows that

\[
    E[A] = E[B]. \tag{A.117}
\]

Therefore, Liu [110] concluded that we should be indifferent between A and B. Simulation experiments also verified this conclusion. It is thus unreasonable to strictly prefer A to B.
A New Problem

In order to further explore this issue, Liu [110] revised the choice problem as follows: What is your choice if B is replaced with

C: You receive $31 if the drawn ball is black?

Through a lot of surveys, Liu [110] showed that most people continue to prefer A to C. However, the income of C is an uncertain random variable

\[ C(\gamma, \omega) = \begin{cases} 31, & \text{if } (\gamma, \omega) \in \text{“black”} \\ 0, & \text{otherwise} \end{cases} \]  \hspace{1cm} (A.117)

whose expected value is

\[ E[C] = 31 \times \text{Ch}\{\text{“black”}\} + 0 \times (1 - \text{Ch}\{\text{“black”}\}) = \frac{31}{3}. \]  \hspace{1cm} (A.118)

It follows that

\[ E[A] < E[C]. \]  \hspace{1cm} (A.119)

Therefore, Liu [110] concluded that we should prefer C to A. This conclusion was also confirmed through simulation experiments. It is thus unreasonable to prefer A to C.

**Exercise A.19:** An Ellsberg urn contains 30 red balls and 60 other balls that are either black or yellow in unknown proportion. One ball is randomly drawn from the urn. Consider the following three options:

A: You receive $30 if the drawn ball is black;

B: You receive $b$ if the drawn ball is black;

C: You receive $y$ if the drawn ball is black;

where $b$ and $y$ are the numbers of black and yellow balls in the urn, respectively. Through surveys, Eliaz-Ortoleva [30] showed that 36% of people bet on A, 52% bet on B, and 12% bet on C. What is your choice among them if uncertainty theory, probability theory and chance theory are comprehensively used? (Hint: The incomes of A, B and C are uncertain random variables,

\[ A(\gamma, \omega) = \begin{cases} 30, & \text{if } (\gamma, \omega) \in \text{“black”} \\ 0, & \text{otherwise}, \end{cases} \]  \hspace{1cm} (A.120)

\[ B(\gamma, \omega) = \begin{cases} \gamma, & \text{if } (\gamma, \omega) \in \text{“black”} \\ 0, & \text{otherwise}, \end{cases} \]  \hspace{1cm} (A.121)

\[ C(\gamma, \omega) = \begin{cases} 60 - \gamma, & \text{if } (\gamma, \omega) \in \text{“black”} \\ 0, & \text{otherwise}, \end{cases} \]  \hspace{1cm} (A.122)
where “black” = \{(γ, ω) ∈ Γ × Ω | ω ≤ γ\} on the chance space (A.108).)

**Exercise A.20:** An Ellsberg urn contains 30 red balls and 60 other balls that are either black or yellow in unknown proportion. Two balls are randomly drawn from the urn.

(i) How likely is it that the two drawn balls are red?

(ii) How likely is it that the two drawn balls are black?

Hint: Take an uncertainty space \((Γ, ℒ, ℳ)\) to be \(\{0, 1, 2, \cdots, 60\}\) with power set and uncertain measure

\[ ℳ\{Λ\} = \frac{|Λ|}{|Γ|}, \]

and take a probability space \((Ω, ℬ, Pr)\) to be \(\{(i, j) | i, j = 1, 2, \cdots, 90, i \neq j\}\) with power set and probability measure

\[ Pr\{Λ\} = \frac{|Λ|}{|Ω|}. \]

Then

“Two balls are red” = \(\{(γ, ω_1, ω_2) ∈ Γ × Ω | ω_1 ∧ ω_2 ≥ 61\}\),

“Two balls are black” = \(\{(γ, ω_1, ω_2) ∈ Γ × Ω | ω_1 ∨ ω_2 ≤ γ\}\).

**Exercise A.21:** An Ellsberg urn contains 30 red balls and 60 other balls that are either black or yellow in unknown proportion. Three balls are randomly drawn from the urn. What is the most probable color distribution among 3-0-0 (three drawn balls are of the same color), 2-1-0 (only two drawn balls are of the same color), and 1-1-1 (three drawn balls are of different colors)? Please justify your answer.

### A.9 Uncertain Random Programming

Assume that \(x\) is a decision vector, and \(ξ\) is an uncertain random vector. Since an uncertain random objective function \(f(x, ξ)\) cannot be directly minimized, we may minimize its expected value, i.e.,

\[ \min_x E[f(x, ξ)]. \quad (A.123) \]

Since the uncertain random constraints \(g_j(x, ξ) ≤ 0, j = 1, 2, \cdots, p\) do not make a crisp feasible set, it is naturally desired that the uncertain random constraints hold with confidence levels \(α_1, α_2, \cdots, α_p\). Then we have a set of chance constraints,

\[ Ch\{g_j(x, ξ) ≤ 0\} ≥ α_j, \quad j = 1, 2, \cdots, p. \quad (A.124) \]
In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu [118] proposed the following uncertain random programming model,

\[
\begin{aligned}
\min_{\mathbf{x}} & \quad E[f(\mathbf{x}, \xi)] \\
\text{subject to:} & \quad \text{Ch}\{g_j(\mathbf{x}, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \ldots, p.
\end{aligned}
\] (A.125)

**Definition A.7** (Liu [118]) A vector \( \mathbf{x} \) is called a feasible solution to the uncertain random programming model (A.125) if

\[
\text{Ch}\{g_j(\mathbf{x}, \xi) \leq 0\} \geq \alpha_j
\] (A.126)

for \( j = 1, 2, \ldots, p \).

**Definition A.8** (Liu [118]) A feasible solution \( \mathbf{x}^* \) is called an optimal solution to the uncertain random programming model (A.125) if

\[
E[f(\mathbf{x}^*, \xi)] \leq E[f(\mathbf{x}, \xi)]
\] (A.127)

for any feasible solution \( \mathbf{x} \).

**Theorem A.30** (Liu [118]) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent uncertain variables with regular uncertainty distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. Assume \( f(\mathbf{x}, \eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n) \) is a continuous and strictly increasing function (or strictly decreasing function) with respect to \( \tau_1, \ldots, \tau_n \). Then the expected function

\[
E[f(\mathbf{x}, \eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)]
\] (A.128)

is equal to

\[
\int_{\mathbb{R}^m} \int_0^1 f(\mathbf{x}, y_1, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha))d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m).
\]

**Proof:** It follows from Theorem A.19 immediately.

**Remark A.10:** If \( f(\mathbf{x}, \eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n) \) is continuous, strictly increasing with respect to \( \tau_1, \ldots, \tau_k \) and strictly decreasing with respect to \( \tau_{k+1}, \ldots, \tau_n \), then the integrand in the formula of expected value should be replaced with \( f(\mathbf{x}, y_1, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_{k-1}^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \ldots, \Upsilon_n^{-1}(1-\alpha)) \).

**Theorem A.31** (Liu [118]) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be
independent uncertain variables with regular uncertainty distributions \( \Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n \), respectively. If \( g_j(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \) is a continuous and strictly increasing function with respect to \( \tau_1, \cdots, \tau_n \), then the chance constraint
\[
\text{Ch}\{g_j(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \leq 0\} \geq \alpha_j \quad (A.129)
\]
holds if and only if
\[
\int_{\mathbb{R}^m} G_j(\mathbf{x}, y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \geq \alpha_j \quad (A.130)
\]
where \( G_j(\mathbf{x}, y_1, \cdots, y_m) \) is the root \( \alpha \) of the equation
\[
g_j(\mathbf{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0. \quad (A.131)
\]

**Proof:** It follows from Theorem A.6 that the left side of the chance constraint (A.129) is
\[
\text{Ch}\{g_j(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \leq 0\}
\]
\[
= \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{g_j(\mathbf{x}, \eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) \leq 0\} \geq r\} \, dr
\]
\[
= \int_{\mathbb{R}^m} \mathcal{M}\{g_j(\mathbf{x}, y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \leq 0\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[
= \int_{\mathbb{R}^m} G_j(\mathbf{x}, y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
where \( G_j(\mathbf{x}, y_1, \cdots, y_m) = \mathcal{M}\{g_j(\mathbf{x}, y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \leq 0\} \) is the root \( \alpha \) of the equation (A.131). Hence the chance constraint (A.129) holds if and only if (A.130) is true. The theorem is verified.

**Remark A.11:** If \( g_j(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \) is continuous, strictly increasing with respect to \( \tau_1, \cdots, \tau_k \) and strictly decreasing with respect to \( \tau_{k+1}, \cdots, \tau_n \), then the equation (A.131) becomes
\[
g_j(\mathbf{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = 0.
\]

**Theorem A.32 (Liu [118])** Let \( \eta_1, \eta_2, \cdots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \cdots, \Psi_m \), and let \( \tau_1, \tau_2, \cdots, \tau_n \) be independent uncertain variables with regular uncertainty distributions \( \Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n \), respectively. If the objective function \( f(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \) and the constraint functions \( g_j(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \) are continuous and strictly increasing functions with respect to \( \tau_1, \cdots, \tau_n \) for \( j = 1, 2, \cdots, p \), then the uncertain random programming
\[
\begin{align*}
\min_{\mathbf{x}} & \quad E[f(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)] \\
\text{subject to:} & \quad \text{Ch}\{g_j(\mathbf{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \leq 0\} \geq \alpha_j, \ j = 1, 2, \cdots, p
\end{align*}
\]
is equivalent to the crisp mathematical programming
\[
\min_{x} \int \mathbb{R}^m \int_0^1 f(x, y_1, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) d\Psi_1(y_1) \ldots d\Psi_m(y_m)
\]
subject to:
\[
\int \mathbb{R}^m G_j(x, y_1, \ldots, y_m) d\Psi_1(y_1) \ldots d\Psi_m(y_m) \geq \alpha_j, \ j = 1, 2, \ldots, p
\]
where \( G_j(x, y_1, \ldots, y_m) \) are the roots \( \alpha \) of the equations
\[
g_j(x, y_1, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) = 0 \quad (A.132)
\]
for \( j = 1, 2, \ldots, p \), respectively.

**Proof:** It follows from Theorems A.30 and A.31 immediately.

After an uncertain random programming is converted into a crisp mathematical programming, we may solve it by any classical numerical methods (e.g. iterative method) or intelligent algorithms (e.g. genetic algorithm).

**A.10 Uncertain Random Risk Analysis**

The study of uncertain random risk analysis was started by Liu-Ralescu [119] with the concept of risk index.

**Definition A.9** (Liu-Ralescu [119]) Assume that a system contains uncertain random factors \( \xi_1, \xi_2, \ldots, \xi_n \), and has a loss function \( f \). Then the risk index is the chance measure that the system is loss-positive, i.e.,
\[
Risk = \text{Ch}\{f(\xi_1, \xi_2, \ldots, \xi_n) > 0\}. \quad (A.133)
\]

If all uncertain random factors degenerate to random ones, then the risk index is the probability measure that the system is loss-positive (Roy [146]). If all uncertain random factors degenerate to uncertain ones, then the risk index is the uncertain measure that the system is loss-positive (Liu [94]).

**Theorem A.33** Assume that a system contains uncertain random factors \( \xi_1, \xi_2, \ldots, \xi_n \), and has a loss function \( f \). If \( f(\xi_1, \xi_2, \ldots, \xi_n) \) has a chance distribution \( \Phi \), then the risk index is
\[
Risk = 1 - \Phi(0). \quad (A.134)
\]

**Proof:** It follows from the definition of risk index and self-duality of chance measure that
\[
Risk = \text{Ch}\{f(\xi_1, \xi_2, \ldots, \xi_n) > 0\}
\]
\[
= 1 - \text{Ch}\{f(\xi_1, \xi_2, \ldots, \xi_n) \leq 0\}
\]
\[
= 1 - \Phi(0).
\]
The theorem is proved.
Theorem A.34 (Liu-Ralescu [119], Risk Index Theorem) Assume a system contains independent random factors $\eta_1, \eta_2, \cdots, \eta_m$ with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and independent uncertain factors $\tau_1, \tau_2, \cdots, \tau_n$ with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss function is $f$, then the risk index is

$$\text{Risk} = \int_{\mathbb{R}^m} G(y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$  \hspace{1cm} (A.135)

where

$$G(y_1, \cdots, y_m) = \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) > 0\}$$  \hspace{1cm} (A.136)

is determined by $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$.

**Proof:** It follows from the definition of risk index and Theorem A.6 that

$$\text{Risk} = \operatorname{Ch}\{f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) > 0\}$$

$$= \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{f(\eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) > 0\} \geq r\} \, dr$$

$$= \int_{\mathbb{R}^m} \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) > 0\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

$$= \int_{\mathbb{R}^m} G(y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m).$$

The theorem is thus verified.

**Exercise A.22:** (Series System) Consider a series system in which there are $m$ elements whose lifetimes are independent random variables $\eta_1, \eta_2, \cdots, \eta_m$ with continuous probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and $n$ elements whose lifetimes are independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with continuous uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time $T$, then the loss function is

$$f = T - \eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n.$$  \hspace{1cm} (A.137)

Show that the risk index is

$$\text{Risk} = a + b - ab$$  \hspace{1cm} (A.138)

where

$$a = 1 - (1 - \Psi_1(T))(1 - \Psi_2(T)) \cdots (1 - \Psi_m(T)),$$  \hspace{1cm} (A.139)

$$b = \Upsilon_1(T) \lor \Upsilon_2(T) \lor \cdots \lor \Upsilon_n(T).$$  \hspace{1cm} (A.140)
Exercise A.23: (Parallel System) Consider a parallel system in which there are \( m \) elements whose lifetimes are independent random variables \( \eta_1, \eta_2, \ldots, \eta_m \) with continuous probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \) and \( n \) elements whose lifetimes are independent uncertain variables \( \tau_1, \tau_2, \ldots, \tau_n \) with continuous uncertainty distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is

\[
f = T - \eta_1 \lor \eta_2 \lor \cdots \lor \eta_m \lor \tau_1 \lor \tau_2 \lor \cdots \lor \tau_n. \tag{A.141}
\]

Show that the risk index is

\[
Risk = ab \tag{A.142}
\]

where

\[
a = \Psi_1(T)\Psi_2(T)\cdots\Psi_m(T), \tag{A.143}
\]

\[
b = \Upsilon_1(T) \land \Upsilon_2(T) \land \cdots \land \Upsilon_n(T). \tag{A.144}
\]

Theorem A.35 (Liu-Ralescu [119], Risk Index Theorem) Assume a system contains independent random factors \( \eta_1, \eta_2, \ldots, \eta_m \) with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \) and independent uncertain factors \( \tau_1, \tau_2, \ldots, \tau_n \) with regular uncertainty distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. If the loss function \( f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n) \) is continuous, strictly increasing with respect to \( \tau_1, \ldots, \tau_k \) and strictly decreasing with respect to \( \tau_{k+1}, \ldots, \tau_n \), then the risk index is

\[
Risk = \int_{\mathbb{R}^m} G(y_1, \ldots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \tag{A.145}
\]

where \( G(y_1, \ldots, y_m) \) is the root \( \alpha \) of the equation

\[
f(y_1, \ldots, y_m, \Upsilon_1^{-1}(1-\alpha), \ldots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) = 0.
\]

Proof: Since \( G(y_1, \ldots, y_m) = M\{f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) > 0\} \) is just the root \( \alpha \) of the equation

\[
f(y_1, \ldots, y_m, \Upsilon_1^{-1}(1-\alpha), \ldots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) = 0,
\]

we get the result by Theorem A.34.

Exercise A.24: (\( k \)-out-of-(\( m + n \)) System) Consider a \( k \)-out-of-(\( m + n \)) system in which there are \( m \) elements whose lifetimes are independent random variables \( \eta_1, \eta_2, \ldots, \eta_m \) with continuous probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and \( n \) elements whose lifetimes are independent uncertain variables \( \tau_1, \tau_2, \ldots, \tau_n \) with regular uncertainty distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is

\[
f = T - k\max[\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n]. \tag{A.146}
\]
Show that the risk index is
\[
Risk = \int_{\mathbb{R}^m} G(y_1, y_2, \cdots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m) \quad (A.147)
\]
where \( G(y_1, y_2, \cdots, y_m) \) is the root \( \alpha \) of the equation
\[
k-\max\{y_1, y_2, \cdots, y_m, \mathcal{Y}_1^{-1}(\alpha), \mathcal{Y}_2^{-1}(\alpha), \cdots, \mathcal{Y}_n^{-1}(\alpha)\} = T. \quad (A.148)
\]

**Exercise A.25:** (Standby System) Consider a standby system in which there are \( m \) elements whose lifetimes are independent random variables \( \eta_1, \eta_2, \cdots, \eta_m \) with continuous probability distributions \( \Psi_1, \Psi_2, \cdots, \Psi_m \) and \( n \) elements whose lifetimes are independent uncertain variables \( \tau_1, \tau_2, \cdots, \tau_n \) with regular uncertainty distributions \( \mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_n \), respectively. If the loss is understood as the case that the system fails before the time \( T \), then the loss function is
\[
f = T - (\eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n). \quad (A.149)
\]
Show that the risk index is
\[
Risk = \int_{\mathbb{R}^m} G(y_1, y_2, \cdots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \cdots d\Psi_m(y_m) \quad (A.150)
\]
where \( G(y_1, y_2, \cdots, y_m) \) is the root \( \alpha \) of the equation
\[
\mathcal{Y}_1^{-1}(\alpha) + \mathcal{Y}_2^{-1}(\alpha) + \cdots + \mathcal{Y}_n^{-1}(\alpha) = T - (y_1 + y_2 + \cdots + y_m). \quad (A.151)
\]

**Remark A.12:** As a substitute of risk index, Liu-Ralescu [121] suggested a concept of value-at-risk,
\[
\text{VaR}(\alpha) = \sup\{x \mid \text{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \geq \alpha\}. \quad (A.152)
\]
Note that \( \text{VaR}(\alpha) \) represents the maximum possible loss when \( \alpha \) percent of the right tail distribution is ignored. In other words, the loss will exceed \( \text{VaR}(\alpha) \) with chance measure \( \alpha \). If the chance distribution \( \Phi(x) \) of \( f(\xi_1, \xi_2, \cdots, \xi_n) \) is continuous, then
\[
\text{VaR}(\alpha) = \sup \{x \mid \Phi(x) \leq 1 - \alpha\}. \quad (A.153)
\]
If its inverse chance distribution \( \Phi^{-1}(\alpha) \) exists, then
\[
\text{VaR}(\alpha) = \Phi^{-1}(1 - \alpha). \quad (A.154)
\]
It is also easy to show that \( \text{VaR}(\alpha) \) is a monotone decreasing function with respect to \( \alpha \). When the uncertain random variables degenerate to random variables, the value-at-risk becomes the one in Morgan [129].
uncertain random variables degenerate to uncertain variables, the value-at-risk becomes the one in Peng [136].

**Remark A.13:** Liu-Ralescu [123] proposed a concept of expected loss that is the expected value of the loss \( f(\xi_1, \xi_2, \cdots, \xi_n) \) given \( f(\xi_1, \xi_2, \cdots, \xi_n) > 0 \), i.e.,

\[
L = \int_0^{+\infty} \text{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \, dx.
\] (A.155)

If \( \Phi(x) \) is the chance distribution of the loss \( f(\xi_1, \xi_2, \cdots, \xi_n) \), then we immediately have

\[
L = \int_0^{+\infty} (1 - \Phi(x)) \, dx.
\] (A.156)

If its inverse chance distribution \( \Phi^{-1}(\alpha) \) exists, then the expected loss is

\[
L = \int_0^1 (\Phi^{-1}(\alpha))^+ \, d\alpha.
\] (A.157)

### A.11 Uncertain Random Reliability Analysis

The study of uncertain random reliability analysis was started by Wen-Kang [170] with the concept of reliability index.

**Definition A.10** (Wen-Kang [170]) Assume a Boolean system has uncertain random elements \( \xi_1, \xi_2, \cdots, \xi_n \) and a structure function \( f \). Then the reliability index is the chance measure that the system is working, i.e.,

\[
\text{Reliability} = \text{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}. \tag{A.158}
\]

If all uncertain random elements degenerate to random ones, then the reliability index is the probability measure that the system is working. If all uncertain random elements degenerate to uncertain ones, then the reliability index (Liu [94]) is the uncertain measure that the system is working.

**Theorem A.36** (Wen-Kang [170], Reliability Index Theorem) Assume that a system has a structure function \( f \) and contains independent random elements \( \eta_1, \eta_2, \cdots, \eta_m \) with reliabilities \( a_1, a_2, \cdots, a_m \), and independent uncertain elements \( \tau_1, \tau_2, \cdots, \tau_n \) with reliabilities \( b_1, b_2, \cdots, b_n \), respectively. Then the reliability index is

\[
\text{Reliability} = \sum_{(x_1, \cdots, x_m) \in \{0, 1\}^m} \left( \prod_{i=1}^m \mu_i(x_i) \right) f^*(x_1, \cdots, x_m). \tag{A.159}
\]
where

\[
f^*(x_1, \ldots, x_m) = \begin{cases} 
\sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{1 \leq j \leq n} \nu_j(y_j), & \text{if } \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{1 \leq j \leq n} \nu_j(y_j) < 0.5 \\
1 - \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0} \min_{1 \leq j \leq n} \nu_j(y_j), & \text{if } \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{1 \leq j \leq n} \nu_j(y_j) \geq 0.5,
\end{cases}
\]  
(A.160)

Exercise A.26: (Series System) Consider a series system in which there are \(m\) independent random elements \(\eta_1, \eta_2, \ldots, \eta_m\) with reliabilities \(a_1, a_2, \ldots, a_m\), and \(n\) independent uncertain elements \(\tau_1, \tau_2, \ldots, \tau_n\) with reliabilities \(b_1, b_2, \ldots, b_n\), respectively. Note that the structure function is

\[
f = \eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n.
\]  
(A.163)

Show that the reliability index is

\[
\text{Reliability} = a_1 a_2 \cdots a_m (b_1 \land b_2 \land \cdots \land b_n).
\]  
(A.164)

Exercise A.27: (Parallel System) Consider a parallel system in which there are \(m\) independent random elements \(\eta_1, \eta_2, \ldots, \eta_m\) with reliabilities \(a_1, a_2, \ldots, a_m\), and \(n\) independent uncertain elements \(\tau_1, \tau_2, \ldots, \tau_n\) with reliabilities \(b_1, b_2, \ldots, b_n\), respectively. Note that the structure function is

\[
f = \eta_1 \lor \eta_2 \lor \cdots \lor \eta_m \lor \tau_1 \lor \tau_2 \lor \cdots \lor \tau_n.
\]  
(A.165)

Show that the reliability index is

\[
\text{Reliability} = 1 - (1 - a_1)(1 - a_2) \cdots (1 - a_m)(1 - b_1 \lor b_2 \lor \cdots \lor b_n).
\]  
(A.166)

Exercise A.28: \((k\text{-out-of-}(m+n))\) System) Consider a \(k\)-out-of-\((m+n)\) system in which there are \(m\) independent random elements \(\eta_1, \eta_2, \ldots, \eta_m\) with reliabilities \(a_1, a_2, \ldots, a_m\), and \(n\) independent uncertain elements \(\tau_1, \tau_2, \ldots, \tau_n\)
with reliabilities \( b_1, b_2, \ldots, b_n \), respectively. Note that the structure function is
\[
f = k\max[\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n].
\]  
(A.167)

Show that the reliability index is
\[
\text{Reliability} = \sum_{(x_1, \ldots, x_m) \in \{0, 1\}^m} \left( \prod_{i=1}^{m} \mu_i(x_i) \right)^{k\max[x_1, \ldots, x_m, b_1, \ldots, b_n]}
\]
where
\[
\mu_i(x_i) = \begin{cases} 
  a_i, & \text{if } x_i = 1 \\
  1 - a_i, & \text{if } x_i = 0
\end{cases} \quad (i = 1, 2, \ldots, m). 
\]  
(A.168)

### A.12 Uncertain Random Graph

In classic graph theory, the edges and vertices are all deterministic, either exist or not. However, in practical applications, some indeterminate factors will no doubt appear in graphs. Thus it is reasonable to assume that in a graph some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure. In order to model this type of graph, Liu [104] presented a concept of uncertain random graph.

We say a graph is of order \( n \) if it has \( n \) vertices labeled by \( 1, 2, \ldots, n \). In this section, we assume the graph is always of order \( n \), and has a collection of vertices,
\[
\mathcal{V} = \{1, 2, \ldots, n\}. 
\]  
(A.169)

Let us define two collections of edges,
\[
\mathcal{U} = \{(i, j) | 1 \leq i < j \leq n \text{ and } (i, j) \text{ are uncertain edges}\}, \]  
(A.170)
\[
\mathcal{R} = \{(i, j) | 1 \leq i < j \leq n \text{ and } (i, j) \text{ are random edges}\}. 
\]  
(A.171)

Note that all deterministic edges are regarded as special uncertain ones. Then \( \mathcal{U} \cup \mathcal{R} = \{(i, j) | 1 \leq i < j \leq n\} \) that contains \( n(n-1)/2 \) edges. We will call

\[
\mathcal{J} = \begin{pmatrix} 
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{pmatrix}
\]  
(A.172)

an uncertain random adjacency matrix if \( \alpha_{ij} \) represent the truth values in uncertain measure or probability measure that the edges between vertices \( i \) and \( j \) exist, \( i, j = 1, 2, \ldots, n \), respectively. Note that \( \alpha_{ii} = 0 \) for \( i = 1, 2, \ldots, n \), and \( \mathcal{J} \) is a symmetric matrix, i.e., \( \alpha_{ij} = \alpha_{ji} \) for \( i, j = 1, 2, \ldots, n \).
Definition A.11 (Liu [104]) Assume $V$ is the collection of vertices, $U$ is the collection of uncertain edges, $R$ is the collection of random edges, and $T$ is the uncertain random adjacency matrix. Then the quartette $(V, U, R, T)$ is said to be an uncertain random graph.

Please note that the uncertain random graph becomes a random graph (Erdős-Rényi [33], Gilbert [57]) if the collection $U$ of uncertain edges vanishes; and becomes an uncertain graph (Gao-Gao [49]) if the collection $R$ of random edges vanishes.

In order to deal with uncertain random graph, let us introduce some symbols. Write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$  \hspace{1cm} (A.173)

and

$$X = \left\{ X \mid \begin{array}{l} x_{ij} = 0 \text{ or } 1, \text{ if } (i, j) \in R \\ x_{ij} = 0, \text{ if } (i, j) \in U \\ x_{ij} = x_{ji}, i, j = 1, 2, \ldots, n \\ x_{ii} = 0, i = 1, 2, \ldots, n \end{array} \right\}. \hspace{1cm} (A.174)$$

For each given matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}, \hspace{1cm} (A.175)$$

the extension class of $Y$ is defined by

$$Y^* = \left\{ X \mid \begin{array}{l} x_{ij} = y_{ij}, \text{ if } (i, j) \in R \\ x_{ij} = 0 \text{ or } 1, \text{ if } (i, j) \in U \\ x_{ij} = x_{ji}, i, j = 1, 2, \ldots, n \\ x_{ii} = 0, i = 1, 2, \ldots, n \end{array} \right\}. \hspace{1cm} (A.176)$$
Example A.5: (Liu [104], Connectivity Index) An uncertain random graph is connected for some realizations of uncertain and random edges, and disconnected for some other realizations. In order to show how likely an uncertain random graph is connected, a connectivity index of an uncertain random graph is defined as the chance measure that the uncertain random graph is connected. Let \((V, U, R, T)\) be an uncertain random graph. Liu [104] proved that the connectivity index is

\[
\rho = \sum_{Y \in \mathcal{X}} \left( \prod_{(i,j) \in \mathcal{R}} \nu_{ij}(Y) \right) f^*(Y) \tag{A.177}
\]

where

\[
f^*(Y) = \begin{cases} 
\sup_{X \in \mathcal{Y} : f(X) = 1} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X), & \text{if } \sup_{X \in \mathcal{Y} : f(X) = 1} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X) < 0.5 \\
1 - \sup_{X \in \mathcal{Y} : f(X) = 0} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X), & \text{if } \sup_{X \in \mathcal{Y} : f(X) = 1} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X) \geq 0.5,
\end{cases}
\]

\[
\nu_{ij}(X) = \begin{cases} 
\alpha_{ij}, & \text{if } x_{ij} = 1 \\
1 - \alpha_{ij}, & \text{if } x_{ij} = 0
\end{cases}, \quad (i,j) \in \mathcal{U}, \tag{A.178}
\]

\[
f(X) = \begin{cases} 
1, & \text{if } I + X + X^2 + \cdots + X^{n-1} > 0 \\
0, & \text{otherwise},
\end{cases} \tag{A.179}
\]

\(\mathcal{X}\) and \(Y^*\) are defined by (A.174) and (A.176), respectively.

Remark A.14: If the uncertain random graph becomes a random graph, then the connectivity index is

\[
\rho = \sum_{X \in \mathcal{X}} \left( \prod_{1 \leq i \leq j \leq n} \nu_{ij}(X) \right) f(X) \tag{A.180}
\]

where

\[
\mathcal{X} = \left\{ X \mid x_{ij} = 0 \text{ or } 1, i,j = 1,2,\cdots,n \right\}, \tag{A.181}
\]

Remark A.15: (Gao-Gao [49]) If the uncertain random graph becomes an uncertain graph, then the connectivity index is

\[
\rho = \begin{cases} 
\sup_{X \in \mathcal{X} : f(X) = 1} \min_{1 \leq i < j \leq n} \nu_{ij}(X), & \text{if } \sup_{X \in \mathcal{X} : f(X) = 1} \min_{1 \leq i < j \leq n} \nu_{ij}(X) < 0.5 \\
1 - \sup_{X \in \mathcal{X} : f(X) = 0} \min_{1 \leq i < j \leq n} \nu_{ij}(X), & \text{if } \sup_{X \in \mathcal{X} : f(X) = 1} \min_{1 \leq i < j \leq n} \nu_{ij}(X) \geq 0.5
\end{cases}
\]
where $X$ becomes

$$X = \left\{ \begin{array}{l}
  x_{ij} = 0 \text{ or } 1, \ i, j = 1, 2, \ldots, n \\
  x_{ij} = x_{ji}, \ i, j = 1, 2, \ldots, n \\
  x_{ii} = 0, \ i = 1, 2, \ldots, n
\end{array} \right\}. \quad (A.182)$$

**Exercise A.29:** (Zhang-Peng-Li [216]) An Euler circuit in the graph is a circuit that passes through each edge exactly once. In other words, a graph has an Euler circuit if it can be drawn on paper without ever lifting the pencil and without retracing over any edge. It has been proved that a graph has an Euler circuit if and only if it is connected and each vertex has an even degree (i.e., the number of edges that are adjacent to that vertex). In order to measure how likely an uncertain random graph has an Euler circuit, an Euler index is defined as the chance measure that the uncertain random graph has an Euler circuit. Please give a formula for calculating Euler index.

### A.13 Uncertain Random Network

The term network is a synonym for a weighted graph, where the weights may be understood as cost, distance or time consumed. Assume that in a network some weights are random variables and others are uncertain variables. In order to model this type of network, Liu [104] presented a concept of uncertain random network.

In this section, we assume the uncertain random network is always of order $n$, and has a collection of nodes,

$$\mathcal{N} = \{1, 2, \cdots, n\} \quad (A.183)$$

where “1” is always the source node, and “$n$” is always the destination node.

Let us define two collections of arcs,

$$\mathcal{U} = \{(i, j) \mid (i, j) \text{ are uncertain arcs}\}, \quad (A.184)$$

$$\mathcal{R} = \{(i, j) \mid (i, j) \text{ are random arcs}\}. \quad (A.185)$$

Note that all deterministic arcs are regarded as special uncertain ones. Let $w_{ij}$ denote the weights of arcs $(i, j)$, $(i, j) \in \mathcal{U} \cup \mathcal{R}$, respectively. Then $w_{ij}$ are uncertain variables if $(i, j) \in \mathcal{U}$, and random variables if $(i, j) \in \mathcal{R}$. Write

$$\mathcal{W} = \{w_{ij} \mid (i, j) \in \mathcal{U} \cup \mathcal{R}\}. \quad (A.186)$$

**Definition A.12** (Liu [104]) Assume $\mathcal{N}$ is the collection of nodes, $\mathcal{U}$ is the collection of uncertain arcs, $\mathcal{R}$ is the collection of random arcs, and $\mathcal{W}$ is the collection of uncertain and random weights. Then the quartette $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ is said to be an uncertain random network.
Figure A.3: An Uncertain Random Network

Please note that the uncertain random network becomes a random network (Frank-Hakimi [34]) if all weights are random variables; and becomes an uncertain network (Liu [95]) if all weights are uncertain variables.

Figure A.3 shows an uncertain random network \((N, U, R, W)\) of order 6 in which

\[
N = \{1, 2, 3, 4, 5, 6\},
\]

\[
U = \{(1,2), (1,3), (2,4), (2,5), (3,4), (3,5)\},
\]

\[
R = \{(4,6), (5,6)\},
\]

\[
W = \{w_{12}, w_{13}, w_{24}, w_{25}, w_{34}, w_{35}, w_{46}, w_{56}\}.
\]

**Example A.6:** (Liu [104], Shortest Path Distribution) Consider an uncertain random network \((N, U, R, W)\). Assume the uncertain weights \(w_{ij}\) have regular uncertainty distributions \(\Upsilon_{ij}\) for \((i, j) \in U\), and the random weights \(w_{ij}\) have probability distributions \(\Psi_{ij}\) for \((i, j) \in R\), respectively. Then the shortest path distribution from a source node to a destination node is

\[
\Phi(x) = \int_0^\infty \cdots \int_0^\infty F(x; y_{ij}, (i, j) \in R) \prod_{(i,j) \in R} d\Psi_{ij}(y_{ij})
\]

where \(F(x; y_{ij}, (i, j) \in R)\) is the root \(\alpha\) of the equation

\[
f(\Upsilon_{ij}^{-1}(\alpha), (i, j) \in U; y_{ij}, (i, j) \in R) = x
\]

and \(f\) is the length of the shortest path and may be calculated by the Dijkstra algorithm (Dijkstra [27]) when the weights are \(y_{ij}\) if \((i, j) \in R\) and \(\Upsilon_{ij}^{-1}(\alpha)\) if \((i, j) \in U\), respectively.

**Remark A.16:** If the uncertain random network becomes a random network, then the shortest path distribution is

\[
\Phi(x) = \int_{f(y_{ij}, (i, j) \in R) \leq x} \prod_{(i,j) \in R} d\Psi_{ij}(y_{ij}).
\]
Remark A.17:  (Gao [51]) If the uncertain random network becomes an uncertain network, then the inverse shortest path distribution is

$$\Phi^{-1}(\alpha) = f(Y_{ij}^{-1}(\alpha), (i, j) \in U).$$  \hspace{1cm} (A.194)

Exercise A.30:  (Sheng-Gao [154]) Maximum flow problem is to find a flow with maximum value from a source node to a destination node in an uncertain random network. What is the maximum flow distribution?

A.14 Uncertain Random Process

Uncertain random process is a sequence of uncertain random variables indexed by time. A formal definition is given below.

Definition A.13  (Gao-Yao [35]) Let $$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$$ be a chance space and let $T$ be a totally ordered set (e.g. time). An uncertain random process is a function $X_t(\gamma, \omega)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{X_t \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set $B$ of real numbers at each time $t$.

Example A.7: A stochastic process is a sequence of random variables indexed by time, and then is a special type of uncertain random process.

Example A.8: An uncertain process is a sequence of uncertain variables indexed by time, and then is a special type of uncertain random process.

Example A.9: Let $Y_t$ be a stochastic process, and let $Z_t$ be an uncertain process. If $f$ is a measurable function, then

$$X_t = f(Y_t, Z_t)$$  \hspace{1cm} (A.195)

is an uncertain random process.

Definition A.14  (Gao-Yao [35]) Let $\eta_1, \eta_2, \cdots$ be iid random variables, let $\tau_1, \tau_2, \cdots$ be iid uncertain variables, and let $f$ be a positive and strictly monotone function. Define $S_0 = 0$ and

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n)$$  \hspace{1cm} (A.196)

for $n \geq 1$. Then

$$N_t = \max \{n \mid S_n \leq t\}$$  \hspace{1cm} (A.197)

is called an uncertain random renewal process with interarrival times $f(\eta_1, \tau_1), f(\eta_2, \tau_2), \cdots$.
**Theorem A.37** (Gao-Yao [35]) Let $\eta_1, \eta_2, \cdots$ be iid random variables with a common probability distribution $\Psi$, let $\tau_1, \tau_2, \cdots$ be iid uncertain variables, and let $f$ be a positive and strictly monotone function. Assume $N_t$ is an uncertain random renewal process with interarrival times $f(\eta_1, \tau_1), f(\eta_2, \tau_2), \cdots$. Then the average renewal number

$$\frac{N_t}{t} \to \left( \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \right)^{-1}$$

(A.198)

in the sense of convergence in distribution as $t \to \infty$.

**Proof:** Write $S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n)$ for all $n \geq 1$. Let $x$ be a continuous point of the uncertainty distribution of

$$\left( \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \right)^{-1}.$$

It is clear that $1/x$ is a continuous point of the uncertainty distribution of

$$\int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y).$$

At first, it follows from the definition of uncertain random renewal process that

$$\text{Ch} \left\{ \frac{N_t}{t} \leq x \right\} = \text{Ch} \left\{ S_{\lfloor tx \rfloor + 1} > t \right\} = \text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{t}{\lfloor tx \rfloor + 1} \right\}$$

where $\lfloor tx \rfloor$ represents the maximal integer less than or equal to $tx$. Since $\lfloor tx \rfloor \leq tx < \lfloor tx \rfloor + 1$, we immediately have

$$\frac{\lfloor tx \rfloor}{\lfloor tx \rfloor + 1} \cdot \frac{1}{x} \leq \frac{t}{\lfloor tx \rfloor + 1} < \frac{1}{x}$$

and then

$$\text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{1}{x} \right\} \leq \text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{t}{\lfloor tx \rfloor + 1} \right\} \leq \text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor} > \frac{1}{x} \right\}.$$

It follows from the law of large numbers for uncertain random variables that

$$\lim_{t \to \infty} \frac{\text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{1}{x} \right\}}{\text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} \leq \frac{1}{x} \right\}} = 1 - \lim_{t \to \infty} \text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} \leq \frac{1}{x} \right\}$$

$$= 1 - \mathcal{M} \left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq \frac{1}{x} \right\}$$

$$= \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \right)^{-1} \leq x \right\}.$$
and
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor} > \frac{1}{x} \right\} = 1 - \lim_{t \to \infty} \text{Ch} \left\{ \frac{\lfloor tx \rfloor + 1}{\lfloor tx \rfloor} \cdot \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} \leq \frac{1}{x} \right\}
\]
\[
= 1 - \mathcal{M} \left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \leq \frac{1}{x} \right\}
\]
\[
= \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \right)^{-1} \leq x \right\}.
\]

From the above three relations we get
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{t}{\lfloor tx \rfloor + 1} \right\} = \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \right)^{-1} \leq x \right\}
\]
and then
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{N_t}{t} \leq x \right\} = \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(y, \tau_1) d\Psi(y) \right)^{-1} \leq x \right\}.
\]

The theorem is thus verified.

**Exercise A.31:** Let \( \eta_1, \eta_2, \cdots \) be iid positive random variables, and let \( \tau_1, \tau_2, \cdots \) be iid positive uncertain variables. Assume \( N_t \) is an uncertain random renewal process with interarrival times \( \eta_1 + \tau_1, \eta_2 + \tau_2, \cdots \). Show that
\[
\frac{N_t}{t} \to \frac{1}{E[\eta_1] + \tau_1}
\]
in the sense of convergence in distribution as \( t \to \infty \).

**Exercise A.32:** Let \( \eta_1, \eta_2, \cdots \) be iid positive random variables, and let \( \tau_1, \tau_2, \cdots \) be iid positive uncertain variables. Assume \( N_t \) is an uncertain random renewal process with interarrival times \( \eta_1 \tau_1, \eta_2 \tau_2, \cdots \). Show that
\[
\frac{N_t}{t} \to \frac{1}{E[\eta_1] \tau_1}
\]
in the sense of convergence in distribution as \( t \to \infty \).

**Theorem A.38 (Yao-Zhou [199]):** Let \( \eta_1, \eta_2, \cdots \) be iid random interarrival times, and let \( \tau_1, \tau_2, \cdots \) be iid uncertain rewards. Assume \( N_t \) is a stochastic renewal process with interarrival times \( \eta_1, \eta_2, \cdots \). Then
\[
R_t = \sum_{i=1}^{N_t} \tau_i
\]

(A.201)
is an uncertain random renewal reward process, and

\[ \frac{R_t}{t} \to \frac{\tau_1}{E[\eta_1]} \]  \hspace{1cm} (A.202)

in the sense of convergence in distribution as \( t \to \infty \).

**Proof:** Let \( \Upsilon \) denote the uncertainty distribution of \( \tau_1 \). Then for each realization of \( N_t \), the uncertain variable

\[ \frac{1}{N_t} \sum_{i=1}^{N_t} \tau_i \]

follows the uncertainty distribution \( \Upsilon \). In addition, by the definition of chance distribution, we have

\[
\text{Ch}\left\{ \frac{R_t}{t} \leq x \right\} = \int_0^1 \text{Pr}\left\{ M\left\{ \frac{R_t}{t} \leq x \right\} \geq r \right\} \, dr
\]

\[
= \int_0^1 \text{Pr}\left\{ \frac{1}{N_t} \sum_{i=1}^{N_t} \tau_i \leq \frac{tx}{N_t} \right\} \geq r \right\} \, dr
\]

\[
= \int_0^1 \text{Pr}\left\{ \Upsilon\left( \frac{tx}{N_t} \right) \geq r \right\} \, dr
\]

for any real number \( x \). Since \( N_t \) is a stochastic renewal process with iid interarrival times \( \eta_1, \eta_2, \cdots \), we have

\[ \frac{t}{N_t} \to E[\eta_1], \text{ a.s.} \]

as \( t \to \infty \). It follows from the Lebesgue domain convergence theorem that

\[
\lim_{t \to \infty} \text{Ch}\left\{ \frac{R_t}{t} \leq x \right\} = \lim_{t \to \infty} \int_0^1 \text{Pr}\left\{ \Upsilon\left( \frac{tx}{N_t} \right) \geq r \right\} \, dr
\]

\[
= \int_0^1 \text{Pr}\{ \Upsilon(E[\eta_1]|x) \geq r \} \, dr = \Upsilon(E[\eta_1]|x)
\]

that is just the uncertainty distribution of \( \tau_1/E[\eta_1] \). The theorem is thus proved.

**Theorem A.39** (Yao-Zhou [203]) Let \( \eta_1, \eta_2, \cdots \) be iid random rewards, and let \( \tau_1, \tau_2, \cdots \) be iid uncertain interarrival times. Assume \( N_t \) is an uncertain renewal process with interarrival times \( \tau_1, \tau_2, \cdots \). Then

\[ R_t = \sum_{i=1}^{N_t} \eta_i \]  \hspace{1cm} (A.203)
Appendix A - Chance Theory

is an uncertain random renewal reward process, and

\[
\frac{R_t}{t} \rightarrow \frac{E[\eta_1]}{\tau_1}
\]

(A.204)

in the sense of convergence in distribution as \( t \to \infty \).

**Proof:** Let \( Y \) denote the uncertainty distribution of \( \tau_1 \). It follows from the definition of chance distribution that for any real number \( x \), we have

\[
\text{Ch}\left\{ \frac{R_t}{t} \leq x \right\} = \int_0^1 \text{Pr}\left\{ \frac{1}{x} \cdot \frac{1}{N_t} \sum_{i=1}^{N_t} \eta_i \leq \frac{t}{N_t} \right\} \geq r \text{ dr}
\]

Since \( N_t \) is an uncertain renewal process with iid interarrival times \( \tau_1, \tau_2, \ldots \), by using Theorem 12.3, we have

\[
\frac{t}{N_t} \rightarrow \tau_1
\]

in the sense of convergence in distribution as \( t \to \infty \). In addition, for each realization of \( N_t \), the law of large numbers for random variables says

\[
\frac{1}{N_t} \sum_{i=1}^{N_t} \eta_i \rightarrow E[\eta_1], \text{ a.s.}
\]

as \( t \to \infty \) for each number \( x \). It follows from the Lebesgue domain convergence theorem that

\[
\lim_{t \to \infty} \text{Ch}\left\{ \frac{R_t}{t} \leq x \right\} = \int_0^1 \text{Pr}\left\{ 1 - Y \left( \frac{E[\eta_1]}{x} \right) \geq r \right\} \text{ dr} = 1 - Y \left( \frac{E[\eta_1]}{x} \right)
\]

that is just the uncertainty distribution of \( E[\eta_1]/\tau_1 \). The theorem is thus proved.

**Theorem A.40 (Yao-Gao [194])** Let \( \eta_1, \eta_2, \ldots \) be iid random on-times, and let \( \tau_1, \tau_2, \ldots \) be iid uncertain off-times. Assume \( N_t \) is an uncertain random renewal process with interarrival times \( \eta_1 + \tau_1, \eta_2 + \tau_2, \ldots \) Then

\[
A_t = \begin{cases} 
  t - \sum_{i=1}^{N_t} \tau_i, & \text{if } \sum_{i=1}^{N_t} (\eta_i + \tau_i) < t < \sum_{i=1}^{N_t} (\eta_i + \tau_i) + \eta_{N_t+1} \\
  \sum_{i=1}^{N_t+1} \eta_i, & \text{if } \sum_{i=1}^{N_t} (\eta_i + \tau_i) + \eta_{N_t+1} \leq t < \sum_{i=1}^{N_t+1} (\eta_i + \tau_i)
\end{cases}
\]

(A.205)
is an uncertain random alternating renewal process (i.e., the total time at which the system is on up to time $t$), and
\[
\frac{A_t}{t} \to \frac{E[\eta_1]}{E[\eta_1] + \tau_1}
\]
(A.206)
in the sense of convergence in distribution as $t \to \infty$.

**Proof:** Let $\Phi$ denote the uncertainty distribution of $\tau_1$, and let $\Upsilon$ be the uncertainty distribution of $E[\eta_1] / (E[\eta_1] + \tau_1)$. Then at each continuity point $x$ of $\Upsilon$, we have
\[
\Upsilon(x) = \mathcal{M} \left\{ \frac{E[\eta_1]}{E[\eta_1] + \tau_1} \leq x \right\} = \mathcal{M} \left\{ \tau_1 \geq \frac{E[\eta_1](1 - x)}{x} \right\}
= 1 - \mathcal{M} \left\{ \tau_1 < \frac{E[\eta_1](1 - x)}{x} \right\} = 1 - \Phi \left( \frac{E[\eta_1](1 - x)}{x} \right).
\]

On the one hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} = \lim_{t \to \infty} \int_0^1 \Pr \left\{ \mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} \geq r \right\} dr
= \int_0^1 \lim_{t \to \infty} \Pr \left\{ \mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} \geq r \right\} dr
= \int_0^1 \Pr \left\{ \lim_{t \to \infty} \mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} \geq r \right\} dr.
\]

Note that
\[
\mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} = \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( \frac{1}{t} \sum_{i=1}^{k} \eta_i \leq x \right) \cap (N_t = k) \right\}
\leq \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k} \eta_i \leq tx \right) \cap \left( \sum_{i=1}^{k+1} (\eta_i + \tau_i) > t \right) \right\}
\leq \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k} \eta_i \leq tx \right) \cap \left( tx + \eta_{k+1} + \sum_{i=1}^{k+1} \tau_i > t \right) \right\}
= \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} (k \leq N^*_t) \cap \left( \frac{\eta_{k+1}}{t} + \frac{1}{t} \sum_{i=1}^{k+1} \tau_i > 1 - x \right) \right\}
\]
where $N^*_t$ is a stochastic renewal process with random interarrival times $\eta_1, \eta_2, \cdots$. Since
\[
\frac{\eta_{k+1}}{t} \to 0 \text{ as } t \to \infty
\]
and
\[
\sum_{i=1}^{k+1} \tau_i \sim (k+1) \tau_1,
\]
we have
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} \leq \lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( k \leq N_{tx}^* \cap \left( \tau_1 > \frac{t - tx}{k + 1} \right) \right) \right\}
\]
\[
= \lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{N_{tx}^*} \left( \tau_1 > \frac{t - tx}{k + 1} \right) \right\}
\]
\[
= \lim_{t \to \infty} M \left\{ \tau_1 > \frac{t - tx}{N_{tx}^* + 1} \right\}
\]
\[
= 1 - \lim_{t \to \infty} \Phi \left( \frac{t - tx}{N_{tx}^* + 1} \right).
\]

By the elementary renewal theorem in probability, we have
\[
\frac{N_{tx}^*}{tx} \to \frac{1}{E[\eta_1]}, \quad \text{a.s.}
\]
as \( t \to \infty \), and then
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} \leq 1 - \Phi \left( \frac{E[\eta_1](1 - x)}{x} \right) = \Upsilon(x).
\]
Thus
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\} \leq \int_0^1 \text{Pr} \{ \Upsilon(x) \geq r \} \, dr = \Upsilon(x). \tag{A.207}
\]

On the other hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} = \lim_{t \to \infty} \int_0^1 \text{Pr} \left\{ M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \geq r \right\} \, dr
\]
\[
= \int_0^1 \lim_{t \to \infty} \text{Pr} \left\{ M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \geq r \right\} \, dr
\]
\[
= \int_0^1 \text{Pr} \left\{ \lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \geq r \right\} \, dr.
\]
Note that
\[
\mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \\
= \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( \frac{1}{t} \sum_{i=1}^{k+1} \eta_i > x \right) \cap (N_t = k) \right\} \\
\leq \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k+1} \eta_i > tx \right) \cap \left( \sum_{i=1}^{k} (\eta_i + \tau_i) \leq t \right) \right\} \\
\leq \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k+1} \eta_i > tx \right) \cap \left( tx - \eta_{k+1} + \sum_{i=1}^{k} \tau_i \leq t \right) \right\} \\
= \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( N_{tx}^* \leq k \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \tau_i - \frac{\eta_{k+1}}{t} \leq 1 - x \right) \right\}.
\]

Since
\[
\sum_{i=1}^{k} \tau_i \sim k \tau_1
\]
and
\[
\frac{\eta_{k+1}}{t} \to 0 \text{ as } t \to \infty,
\]
we have
\[
\lim_{t \to \infty} \mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \leq \lim_{t \to \infty} \mathcal{M} \left\{ \bigcup_{k=0}^{\infty} \left( N_{tx}^* \leq k \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \tau_i \leq 1 - x \right) \right\} \\
= \lim_{t \to \infty} \mathcal{M} \left\{ \bigcup_{k=N_{tx}^*}^{\infty} \left( \tau_1 \leq \frac{t - tx}{k} \right) \right\} \\
= \lim_{t \to \infty} \mathcal{M} \left\{ \tau_1 \leq \frac{t - tx}{N_{tx}^*} \right\} \\
= \lim_{t \to \infty} \Phi \left( \frac{t - tx}{N_{tx}^*} \right).
\]

By the elementary renewal theorem, we have
\[
\frac{N_{tx}^*}{tx} \to \frac{1}{E[\eta_1]}, \quad \text{a.s.}
\]
as \(t \to \infty\), and then
\[
\lim_{t \to \infty} \mathcal{M} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \leq \Phi \left( \frac{E[\eta_1](1-x)}{x} \right) = 1 - \Upsilon(x).
\]
Thus
\[
\lim_{t \to \infty} Ch \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i > x \right\} \leq \int_0^1 \Pr \{1 - \Upsilon(x) \geq r\} \, dr = 1 - \Upsilon(x).
\]

By using the duality property of chance measure, we get
\[
\lim_{t \to \infty} Ch \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i \leq x \right\} \geq \Upsilon(x).
\] (A.208)

Since
\[
\frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq \frac{A_t}{t} \leq \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i,
\]
we obtain
\[
Ch \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \eta_i \leq x \right\} \leq Ch \left\{ \frac{A_t}{t} \leq x \right\} \leq Ch \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \eta_i \leq x \right\}.
\]

It follows from (A.207) and (A.208) that
\[
\lim_{t \to \infty} Ch \left\{ \frac{A_t}{t} \leq x \right\} = \Upsilon(x).
\]

Hence the availability rate \(A_t/t\) converges in distribution to \(E[\eta_1]/(E[\eta_1] + \tau_1)\) as \(t \to \infty\). The theorem is proved.

**Theorem A.41 (Yao-Gao [194])** Let \(\tau_1, \tau_2, \cdots\) be iid uncertain on-times, and let \(\eta_1, \eta_2, \cdots\) be iid random off-times. Assume \(N_t\) is an uncertain random renewal process with interarrival times \(\tau_1 + \eta_1, \tau_2 + \eta_2, \cdots\) Then
\[
A_t = \begin{cases} 
  t - \sum_{i=1}^{N_t} \eta_i, & \text{if } \sum_{i=1}^{N_t} (\tau_i + \eta_i) \leq t < \sum_{i=1}^{N_t} (\tau_i + \eta_i) + \tau_{N_t+1} \\
  \sum_{i=1}^{N_t+1} \tau_i, & \text{if } \sum_{i=1}^{N_t} (\tau_i + \eta_i) + \tau_{N_t+1} \leq t < \sum_{i=1}^{N_t+1} (\tau_i + \eta_i)
\end{cases}
\] (A.209)

is an uncertain random alternating renewal process (i.e., the total time at which the system is on up to time \(t\)), and
\[
\frac{A_t}{t} \to \frac{\tau_1}{\tau_1 + E[\eta_1]}.
\] (A.210)

in the sense of convergence in distribution as \(t \to \infty\).
Proof: Let \( \Phi \) denote the uncertainty distribution of \( \tau_1 \), and let \( \Upsilon \) be the uncertainty distribution of \( \tau_1/(\tau_1 + E[\eta_1]) \). Then at each continuity point \( x \) of \( \Upsilon \), we have

\[
\Upsilon(x) = \mathcal{M}\left\{ \frac{\tau_1}{\tau_1 + E[\eta_1]} \leq x \right\} = \mathcal{M}\left\{ \tau_1 \leq \frac{E[\eta_1] x}{1 - x} \right\} = \Phi\left( \frac{E[\eta_1] x}{1 - x} \right).
\]

On the one hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have

\[
\begin{align*}
\lim_{t \to \infty} \text{Ch}\\left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} &= \lim_{t \to \infty} \int_0^1 \text{Pr}\left\{ \mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} \geq r \right\} \, dr \\
&= \int_0^1 \lim_{t \to \infty} \text{Pr}\left\{ \mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} \geq r \right\} \, dr \\
&= \int_0^1 \text{Pr}\left\{ \lim_{t \to \infty} \mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} \geq r \right\} \, dr.
\end{align*}
\]

Note that

\[
\begin{align*}
\mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} &= \mathcal{M}\left\{ \bigcup_{k=0}^{\infty} \left( \frac{1}{t} \sum_{i=1}^{k} \tau_i \leq x \right) \cap (N_t = k) \right\} \\
&\leq \mathcal{M}\left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k} \tau_i \leq tx \right) \cap \left( \sum_{i=1}^{k+1} (\tau_i + \eta_i) > t \right) \right\} \\
&\leq \mathcal{M}\left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k} \tau_i \leq tx \right) \cap \left( tx + \tau_{k+1} + \sum_{i=1}^{k+1} \eta_i > t \right) \right\} \\
&= \mathcal{M}\left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k} \tau_i \leq tx \right) \cap \left( \frac{\tau_{k+1}}{t} + \frac{1}{t} \sum_{i=1}^{k+1} \eta_i > 1 - x \right) \right\}.
\end{align*}
\]

Since

\[
\sum_{i=1}^{k} \tau_i \sim k\tau_1
\]

and

\[
\frac{\tau_{k+1}}{t} \to 0 \text{ as } t \to \infty,
\]

\[
\frac{1}{t} \sum_{i=1}^{k+1} \eta_i \to 0 \text{ as } t \to \infty,
\]

The desired result follows.
we have
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} \\
\leq \lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( \tau_1 \leq \frac{tx}{k} \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k+1} \eta_i > 1 - x \right) \right\} \\
= \lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( \tau_1 \leq \frac{tx}{k} \right) \cap \left( N_{t-tx}^* \leq k \right) \right\} \\
= \lim_{t \to \infty} \Phi \left( \frac{tx}{N_{t-tx}^*} \right)
\]

where \( N_t^* \) is a stochastic renewal process with random interarrival times \( \eta_1, \eta_2, \ldots \). By the elementary renewal theorem, we have
\[
\frac{N_{t-tx}}{t-tx} \to \frac{1}{E[\eta_1]}, \quad \text{a.s.}
\]
as \( t \to \infty \), and then
\[
\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} \leq \Phi \left( \frac{E[\eta_1]x}{1-x} \right) = \Upsilon(x).
\]
Thus
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\} \leq \int_0^1 \Pr \{ \Upsilon(x) \geq r \} \, dr = \Upsilon(x). \quad \text{(A.211)}
\]

On the other hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} = \lim_{t \to \infty} \int_0^1 \Pr \left\{ M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \geq r \right\} \, dr \\
= \int_0^1 \lim_{t \to \infty} \Pr \left\{ M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \geq r \right\} \, dr \\
= \int_0^1 \Pr \left\{ \lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \geq r \right\} \, dr.
\]
Note that
\[
\begin{align*}
M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \\
= M \left\{ \bigcup_{k=0}^{\infty} \left( \frac{1}{t} \sum_{i=1}^{k+1} \tau_i > x \right) \cap (N_t = k) \right\} \\
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k+1} \tau_i > tx \right) \cap \left( \sum_{i=1}^{k} (\tau_i + \eta_i) \leq t \right) \right\} \\
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k+1} \tau_i > tx \right) \cap \left( tx - \tau_{k+1} + \sum_{i=1}^{k} \eta_i \leq t \right) \right\} \\
\leq M \left\{ \bigcup_{k=0}^{\infty} \left( \sum_{i=1}^{k+1} \tau_i > tx \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \eta_i - \frac{\tau_{k+1}}{t} \leq 1 - x \right) \right\}.
\end{align*}
\]

Since
\[
\sum_{i=1}^{k+1} \tau_i \sim (k + 1)\tau_1
\]
and
\[
\frac{\tau_{k+1}}{t} \to 0 \text{ as } t \to \infty,
\]
we have
\[
\begin{align*}
&\lim_{t \to \infty} M \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \\
\leq &\lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( \tau_1 > tx \right) \cap \left( \frac{1}{t} \sum_{i=1}^{k} \tau_i \leq 1 - x \right) \right\} \\
= &\lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{\infty} \left( \tau_1 > tx \right) \cap \left( N_{t-tx}^* \geq k \right) \right\} \\
= &\lim_{t \to \infty} M \left\{ \bigcup_{k=0}^{N_{t-tx}^*} \left( \tau_1 > tx \right) \right\} \\
= &\lim_{t \to \infty} M \left\{ \tau_1 > \frac{tx}{N_{t-tx}^* + 1} \right\} \\
= &1 - \lim_{t \to \infty} \Phi \left( \frac{tx}{N_{t-tx}^* + 1} \right).
\end{align*}
\]

By the elementary renewal theorem, we have
\[
\frac{N_{t-tx}^*}{t-tx} \to \frac{1}{E[\eta_1]}, \text{ a.s.}
\]
as \( t \to \infty \), and then
\[
\lim_{t \to \infty} \mathcal{M}\left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \leq 1 - \Phi \left( \frac{E[\eta_1]x}{1-x} \right) = 1 - \Upsilon(x).
\]

Thus
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i > x \right\} \leq \int_0^1 \text{Pr} \left\{ 1 - \Upsilon(x) \geq r \right\} \, dr = 1 - \Upsilon(x).
\]

By using the duality property of chance measure, we get
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i \leq x \right\} \geq \Upsilon(x). \tag{A.212}
\]

Since
\[
\frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq \frac{A_t}{t} \leq \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i,
\]
we obtain
\[
\text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t+1} \tau_i \leq x \right\} \leq \text{Ch} \left\{ \frac{A_t}{t} \leq x \right\} \leq \text{Ch} \left\{ \frac{1}{t} \sum_{i=1}^{N_t} \tau_i \leq x \right\}.
\]

It follows from (A.211) and (A.212) that
\[
\lim_{t \to \infty} \text{Ch} \left\{ \frac{A_t}{t} \leq x \right\} = \Upsilon(x).
\]

Hence the availability rate \( A_t/t \) converges in distribution to \( \tau / (\tau_1 + E[\eta_1]) \) as \( t \to \infty \). The theorem is proved.

### A.15 Bibliographic Notes

Probability theory was developed by Kolmogorov [80] in 1933 for modelling frequencies, while uncertainty theory was founded by Liu [88] in 2007 for modelling belief degrees. However, in many cases, uncertainty and randomness simultaneously appear in a complex system. In order to describe this phenomenon, uncertain random variable was initialized by Liu [117] in 2013 with the concepts of chance measure and chance distribution. As an important contribution, Liu [118] presented an operational law of uncertain random variables. Furthermore, Yao-Gao [198], Gao-Sheng [38] and Gao-Ralescu [45] verified some laws of large numbers for uncertain random variables.

Stochastic programming was first studied by Dantzig [23] in 1965, while uncertain programming was first proposed by Liu [90] in 2009. In order to
model optimization problems with not only uncertainty but also randomness, uncertain random programming was founded by Liu [118] in 2013. As extensions, Zhou-Yang-Wang [226] proposed uncertain random multiobjective programming for optimizing multiple, noncommensurable and conflicting objectives, Qin [142] proposed uncertain random goal programming in order to satisfy as many goals as possible in the order specified, and Ke-Su-Ni [76] proposed uncertain random multilevel programming for studying decentralized decision systems in which the leader and followers may have their own decision variables and objective functions. After that, uncertain random programming was developed steadily and applied widely.

Probabilistic risk analysis was dated back to 1952 when Roy [146] proposed his safety-first criterion for portfolio selection. Another important contribution is the probabilistic value-at-risk methodology developed by Morgan [129] in 1996. On the other hand, uncertain risk analysis was proposed by Liu [94] in 2010 for evaluating the risk index that is the uncertain measure of an uncertain system being loss-positive. More generally, in order to quantify the risk of uncertain random systems, Liu-Ralescu [119] invented the tool of uncertain random risk analysis in 2014. Furthermore, the value-at-risk methodology was presented by Liu-Ralescu [121], and the expected loss methodology was investigated by Liu-Ralescu [123] for dealing with uncertain random systems.

Probabilistic reliability analysis was traced back to 1944 when Pugsley [140] first proposed structural accident rates for the aeronautics industry. Nowadays, probabilistic reliability analysis has become a widely used discipline. As a new methodology, uncertain reliability analysis was developed by Liu [94] in 2010 for evaluating the reliability index. More generally, for dealing with uncertain random systems, Wen-Kang [170] presented the tool of uncertain random reliability analysis and defined the reliability index in 2016. After that, uncertain random reliability analysis was studied by Gao-Yao [40] and Zhang-Kang-Wen [218].

Random graph was defined by Erdős-Rényi [33] in 1959 and independently by Gilbert [57] at nearly the same time. As an alternative, uncertain graph was proposed by Gao-Gao [49] in 2013 via uncertainty theory. Assuming some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure, Liu [104] defined the concept of uncertain random graph and analyzed the connectivity index in 2014. After that, Zhang-Peng-Li [216] and Chen-Peng-Rao-Rosyida [5] discussed the Euler index and cycle index of uncertain random graph, respectively.

Random network was first investigated by Frank-Hakimi [34] in 1965 for modelling communication network with random capacities. From then on, the random network was well developed and widely applied. As a breakthrough approach, uncertain network was first explored by Liu [95] in 2010 for modelling project scheduling problem with uncertain duration times. More generally, assuming some weights are random variables and others are uncer-
tain variables, Liu [104] initialized the concept of uncertain random network and discussed the shortest path problem in 2014. Following that, uncertain random network was explored by many researchers. For example, Sheng-Gao [154] investigated the maximum flow problem, and Sheng-Qin-Shi [157] dealt with the minimum spanning tree problem of uncertain random network.

Appendix B

Frequently Asked Questions

This appendix will answer some frequently asked questions related to probability theory and uncertainty theory as well as their applications. This appendix will also show that none of fuzzy set theory, interval analysis, rough set theory and grey system are consistent in mathematics. Finally, I will summarize the evolution history of the term uncertainty and clarify what uncertainty is.

B.1 What is the meaning that an object follows the laws of probability theory?

We say an object (e.g. frequency) follows the laws of probability theory if it meets not only the three axioms of probability theory but also the product probability theorem:

**Axiom 1 (Normality Axiom)** $\Pr\{\Omega\} = 1$ for the universal set $\Omega$;

**Axiom 2 (Nonnegativity Axiom)** $\Pr\{A\} \geq 0$ for any event $A$;

**Axiom 3 (Additivity Axiom)** For every countable sequence of mutually disjoint events $A_1, A_2, \cdots$, we have

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}; \quad (B.1)$$

**Product Probability Theorem:** Let $(\Omega_k, A_k, \Pr_k)$ be probability spaces for $k = 1, 2, \cdots$. Then there is a unique probability measure $\Pr$ such that

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\} \quad (B.2)$$
where $A_k$ are arbitrarily chosen events from $A_k$ for $k = 1, 2, \ldots$, respectively.

It is easy for us to understand why we need to justify that the object meets the three axioms. However, some readers may wonder why we also need to justify that the object meets the product probability theorem. The reason is that product probability theorem cannot be deduced from the three axioms of Kolmogorov except we presuppose that the product probability meets the three axioms. Logically, an object does not necessarily satisfy the product probability theorem if it is only justified to meet the three axioms. Would that surprise you?

Please keep in mind that “an object follows the laws of probability theory” is equivalent to “the object meets the three axioms plus the product probability theorem”. This assertion is stronger than “an object meets the three axioms of Kolmogorov”. In other words, the three axioms do not ensure that an object follows the laws of probability theory.

There exist two broad categories of interpretations of probability, one is frequency interpretation and the other is belief interpretation. The frequency interpretation takes the probability to be the frequency with which an event happens (Venn [161], Reichenbach [144], von Mises [162]), while the belief interpretation takes the probability to be the degree to which we believe an event will happen (Ramsey [143], de Finetti [24], Savage [148]).

The debate between the two interpretations has been lasting from the nineteenth century. Personally, I agree with the frequency interpretation, but strongly oppose the belief interpretation of probability because frequency follows the laws of probability theory but belief degree does not. The detailed reasons will be given in the following a few sections.

### B.2 Why does frequency follow the laws of probability theory?

In order to show that the frequency follows the laws of probability theory, we must verify that the frequency meets not only the three axioms of Kolmogorov but also the product probability theorem.

First, the frequency of the universal set takes value 1 because the universal set always happens. Thus the frequency meets the normality axiom. Second, it is obvious that the frequency is a number between 0 and 1. Thus the frequency of any event is nonnegative, and meets the nonnegativity axiom. Third, for any disjoint events $A$ and $B$, if $A$ happens $\alpha$ times and $B$ happens $\beta$ times (in percentage), it is clear that the union $A \cup B$ happens $\alpha + \beta$ times. This means the frequency is additive and then meets the additivity axiom. Finally, numerous experiments showed that if $A$ and $B$ are two events from different probability spaces (practically they come from two different experiments) and happen $\alpha$ and $\beta$ times (in percentage), respectively, then the product $A \times B$ happens $\alpha \times \beta$ times. See Figure B.1. Thus the frequency meets the product probability theorem. Hence the frequency does follow the
laws of probability theory. In fact, frequency is the only empirical basis for probability theory.

Figure B.1: Let $A$ and $B$ be two events from different probability spaces (practically they come from two different experiments). If $A$ happens $\alpha$ times and $B$ happens $\beta$ times, then the product $A \times B$ happens $\alpha \times \beta$ times, where $\alpha$ and $\beta$ are understood as percentage numbers.

### B.3 Why does Dutch book argument fail to prove that belief degree follows the laws of probability theory?

A belief degree represents the strength with which we believe an event will happen. For justifying whether probability theory is suitable for modelling belief degree or not, we must check if the belief degree follows the laws of probability theory.

Ramsey [143] suggested a Dutch book argument\(^1\) that says the belief degree is irrational if there exists a book that guarantees you a loss. For the moment, we are assumed to agree with it.

First, let $\Omega$ be a bet that offers $1$ if the sure event $\Omega$ (i.e., the universal set) happens. Assume the belief degree of $\Omega$ is $\alpha$. This means the price of $\Omega$ is $\$\alpha$. If $\alpha < 1$, then you simply sell $\Omega$, and you are guaranteed to lose $1 - \alpha > 0$. Thus there exists a Dutch book and the assumption $\alpha < 1$ is irrational. If $\alpha > 1$, then you simply buy $\Omega$, and you are guaranteed to lose $\alpha - 1 > 0$. Thus there exists a Dutch book and the assumption $\alpha > 1$ is irrational. Hence we have to assume $\alpha = 1$ and the belief degree meets the normality axiom of probability theory.

Second, let $A$ be a bet that offers $1$ if the event $A$ happens. Assume the belief degree of $A$ is $\alpha$. This means the price of $A$ is $\$\alpha$. If $\alpha < 0$, then you

---

\(^1\)A Dutch book in a betting market is a set of bets which guarantees a loss, regardless of the outcome of the gamble. For example, let $A$ be a bet that offers $1$ if $A$ happens, let $B$ be a bet that offers $1$ if $B$ happens, and let $A \lor B$ be a bet that offers $1$ if either $A$ or $B$ happens. If the prices of $A$, $B$ and $A \lor B$ are $30c$, $40c$ and $80c$, respectively, and you (i) sell $A$, (ii) sell $B$, and (iii) buy $A \lor B$, then you are guaranteed to lose $10c$ no matter what happens. Thus there exists a Dutch book, and the prices are considered to be irrational.
simply sell $A$, and you are guaranteed to lose $-\alpha > 0$ or $1 - \alpha > 0$. Thus there exists a Dutch book and the assumption $\alpha < 0$ is irrational. Hence we have to assume $\alpha \geq 0$ and the belief degree meets the nonnegativity axiom.

Third, let $A_1$ be a bet that offers $1$ if $A_1$ happens, and let $A_2$ be a bet that offers $1$ if $A_2$ happens (here $A_1$ and $A_2$ are disjoint events). Assume the belief degrees of $A_1$ and $A_2$ are $\alpha_1$ and $\alpha_2$, respectively. This means the prices of $A_1$ and $A_2$ are $\mathbb{D}_{\alpha_1}$ and $\mathbb{D}_{\alpha_2}$, respectively. Now we consider the bet $A_1 \lor A_2$ that offers $1$ if either $A_1$ or $A_2$ happens, and write the belief degree of $A_1 \lor A_2$ by $\alpha$. This means the price of $A_1 \lor A_2$ is $\mathbb{D}_\alpha$. If $\alpha > \alpha_1 + \alpha_2$, then you

(i) sell $A_1$, (ii) sell $A_2$, and (iii) buy $A_1 \lor A_2$.

It is clear that you are guaranteed to lose $\alpha - \alpha_1 - \alpha_2 > 0$ no matter what happens. Thus there exists a Dutch book and the assumption $\alpha > \alpha_1 + \alpha_2$ is irrational. If $\alpha < \alpha_1 + \alpha_2$, then you

(i) buy $A_1$, (ii) buy $A_2$, and (iii) sell $A_1 \lor A_2$.

It is clear that you are guaranteed to lose $\alpha_1 + \alpha_2 - \alpha > 0$ no matter what happens. Thus there exists a Dutch book and the assumption $\alpha < \alpha_1 + \alpha_2$ is irrational. Hence we have to assume $\alpha = \alpha_1 + \alpha_2$ and the belief degree meets the additivity axiom.

Up to now, Dutch book argument has verified that the belief degree meets the three axioms of probability theory. Almost all subjectivists stop here and think that belief degree follows the laws of probability theory.

Unfortunately, the evidence is not enough for this conclusion due to the following reasons:

(i) You cannot reverse “buy” and “sell” arbitrarily during a real consultation process.

(ii) Through a lot of surveys, Kahneman and Tversky [75] showed that human beings usually overweight unlikely events. From another side, Liu [106] showed that human beings usually estimate a much wider range of values than the object actually takes. This conservatism of human beings implies that the belief degree is not necessarily rational in the sense of Dutch book argument.

(iii) Even if a single person is rational in the sense of Dutch book argument, how do you ensure that the belief degree of compound event consisting of multiple independent events estimated by different people is rational, too?

(iv) Mathematically, Dutch book argument and the three axioms of probability theory are two equivalent propositions. Logically, a proposition is still unproven if it is proved by an unproven proposition. Since Dutch book argument has not been proved, we have not, in fact, proved that belief degree follows the laws of probability theory.
B.4 Why does Cox's theorem fail to prove that belief degree follows the laws of probability theory?

Some people affirm that probability theory is the only legitimate approach. Perhaps this misconception is rooted in Cox’s theorem [20] that any measure of belief is “isomorphic” to a probability measure. However, uncertain measure is considered coherent but not isomorphic to any probability measure. What goes wrong with Cox’s theorem? The root cause is that Cox’s theorem presumes the truth value of conjunction $P \land Q$ is a twice differentiable function $f$ of truth values of the two propositions $P$ and $Q$, i.e.,

$$T(P \land Q) = f(T(P), T(Q))$$  \hspace{1cm} (B.3)

and then excludes uncertain measure from its start because the function $f(x, y) = x \land y$ used in uncertainty theory is not differentiable with respect to $x$ and $y$. In fact, there does not exist any evidence that the truth value of conjunction is completely determined by the truth values of individual propositions, let alone a twice differentiable function.

On the one hand, it is recognized that probability theory is a legitimate approach to deal with the frequency. On the other hand, at any rate, it is impossible that probability theory is the unique one for modelling indeterminacy. In fact, it has been demonstrated in this book that uncertainty theory is successful to deal with belief degrees.

B.5 What is the difference between probability theory and uncertainty theory?

The difference between probability theory (Kolmogorov [80]) and uncertainty theory (Liu [88]) does not lie in whether the measures are additive or not, but how the product measures are defined. The product probability measure is the multiplication of the probability measures of the individual events, i.e.,

$$\Pr\{\Lambda_1 \times \Lambda_2\} = \Pr\{\Lambda_1\} \times \Pr\{\Lambda_2\},$$  \hspace{1cm} (B.4)

while the product uncertain measure is the minimum of the uncertain measures of the individual events, i.e.,

$$M\{\Lambda_1 \times \Lambda_2\} = M\{\Lambda_1\} \land M\{\Lambda_2\}.$$  \hspace{1cm} (B.5)

See Figure B.2.

Shortly, we may say that probability theory is a “multiplication” mathematical system, and uncertainty theory is a “minimum” mathematical system. This difference implies that random variables and uncertain variables obey different operational laws.

Probability theory and uncertainty theory are complementary mathematical systems that provide two acceptable mathematical models to deal with
B.6 How do we distinguish between randomness and uncertainty in practice?

There are two types of indeterminacy: randomness and uncertainty. Randomness is anything that follows the laws of probability theory (i.e., the three axioms of probability theory plus product probability theorem), and uncertainty is anything that follows the laws of uncertainty theory (i.e., the four axioms of uncertainty theory).

Of course, we can distinguish between randomness and uncertainty by the above definitions. However, in practice, we can quickly distinguish between them in the following way: For any given indeterminate quantity, we first produce a distribution function no matter what method is used. If we believe the distribution function is close enough to the frequency, then it can be treated as randomness. Otherwise, it has to be treated as uncertainty.

Most people believe that probability distribution is easy to obtain from the historical data, and then we should use probability theory. However, the distribution function obtained in most practical problems is, unfortunately, not close enough to the real frequency. In this case, we should regard it as an uncertainty distribution and then use uncertainty theory.

B.7 Why is stochastic differential equation not suitable for modelling stock price?

The origin of stochastic finance theory can be traced to Louis Bachelier’s doctoral dissertation *Théorie de la Speculation* in 1900. However, Bachelier’s work had little impact for more than a half century. After Kiyosi Ito invented stochastic calculus [63] in 1944 and stochastic differential equation [64] in 1951, stochastic finance theory was well developed among others by
Traditionally, stochastic finance theory presumes that the stock price (including interest rate and currency exchange rate) follows Ito’s stochastic differential equation. Is it really reasonable? In fact, this widely accepted presumption was challenged among others by Liu [100] via some paradoxes.

**First Paradox:** As an example, let us assume that the stock price $X_t$ follows the differential equation,

$$\frac{dX_t}{dt} = eX_t + \sigma X_t \cdot \text{“noise”} \quad (B.6)$$

where $e$ is the log-drift, $\sigma$ is the log-diffusion, and “noise” is a stochastic process. Now we take the mathematical interpretation of the “noise” term as

$$\text{“noise”} = \frac{dW_t}{dt} \quad (B.7)$$

where $W_t$ is a Wiener process. Thus the stock price $X_t$ follows the stochastic differential equation,

$$\frac{dX_t}{dt} = eX_t + \sigma X_t \frac{dW_t}{dt}. \quad (B.8)$$

Note that the “noise” term

$$\frac{dW_t}{dt} \sim \mathcal{N}(0, 1) \quad (B.9)$$

is a normal random variable whose expected value is 0 and variance tends to $\infty$. This setting is very different from other disciplines (e.g. statistics) that usually take $\mathcal{N}(0, 1)$ (whose variance is 1 rather than $\infty$) (B.10) as the “noise” term. In addition, since the right-hand part of (B.8) has an “infinite” variance at any time $t$, the left-hand part (i.e., the instantaneous growth rate $dX_t/dt$ of stock price) has to take an infinite variance at every time. However, the growth rate usually has a finite variance in practice, or at least, it is impossible to have infinite variance at every time. Thus it is impossible that the real stock price $X_t$ follows Ito’s stochastic differential equation.

**Second Paradox:** It follows from the stochastic differential equation (B.8) that $X_t$ is a geometric Wiener process, i.e.,

$$X_t = X_0 \exp((e - \sigma^2/2)t + \sigma W_t) \quad (B.11)$$

---

2A stochastic process $W_t$ is said to be a Wiener process if (i) $W_0 = 0$ and almost all sample paths are continuous (but non-Lipschitz), (ii) $W_t$ has stationary and independent increments, and (iii) every increment $W_{s+t} - W_s$ is a normal random variable with expected value 0 and variance $t$. Samuelson [147], Black-Scholes [3] and Merton [127] during the 1960s and 1970s.
from which we derive

$$W_t = \frac{\ln X_t - \ln X_0 - (e - \sigma^2/2)t}{\sigma} \quad (B.12)$$

whose increment is

$$\Delta W_t = \frac{\ln X_{t+\Delta t} - \ln X_t - (e - \sigma^2/2)\Delta t}{\sigma}. \quad (B.13)$$

Write

$$A = -\frac{(e - \sigma^2/2)\Delta t}{\sigma}. \quad (B.14)$$

Note that the stock price $X_t$ is actually a step function of time with a finite number of jumps although it looks like a curve. During a fixed period (e.g. one week), without loss of generality, we assume that $X_t$ is observed to have 100 jumps. Now we divide the period into 10,000 equal intervals. Then we may observe 10,000 samples of $X_t$. It follows from (B.13) that $\Delta W_t$ has 10,000 samples that consist of 9,900 $A$’s and 100 other numbers:

$$A, A, \ldots, A, \underbrace{B, C, \ldots, Z}_{9900 \ 100} \quad (B.15)$$

It is obvious that nobody can believe that those 10,000 samples follow a normal probability distribution with expected value 0 and variance $\Delta t$. This fact is in contradiction with the property of Wiener process that the increment $\Delta W_t$ is a normal random variable. Therefore, the real stock price $X_t$ does not follow the stochastic differential equation.

Figure B.3: There does not exist any continuous probability distribution (curve) that can approximate to the frequency (histogram) of $\Delta W_t$. Hence it is impossible that the real stock price $X_t$ follows any Ito’s stochastic differential equation.

Perhaps some people think that the stock price does behave like a geometric Wiener process (or Ornstein-Uhlenbeck process) in macroscopy although
they recognize the paradox in microscopy. However, as the very core of stochastic finance theory, Ito’s calculus is just built on the microscopic structure (i.e., the differential $dW_t$) of Wiener process rather than macroscopic structure.

On the basis of the above two paradoxes, personally I do not think Ito’s calculus can play the essential tool of finance theory because Ito’s stochastic differential equation is impossible to model stock price. As a substitute, uncertain calculus may be a potential mathematical foundation of finance theory. We will have a theory of uncertain finance if the stock price, interest rate and exchange rate are assumed to follow uncertain differential equations.

**B.8 In what situations should we use uncertainty theory?**

Uncertainty theory is a branch of mathematics for modelling belief degrees. In practice, I think we should use uncertainty theory in the following five situations.

(i) We should use uncertainty theory (here it refers to uncertain variable) to quantify the future when no samples are available. In this case, we have to invite some domain experts to evaluate the belief degree that each event will happen, and uncertainty theory is just the tool to deal with belief degrees.

(ii) We should use uncertainty theory (here it refers to uncertain variable) to quantify the future when an emergency arises, e.g., war, flood, earthquake, accident, and even rumour. In fact, in this case, all historical data are no longer valid to predict the future. Essentially, this situation equates to (i).

(iii) We should use uncertainty theory (here it refers to uncertain variable) to quantify the past when precise observations or measurements are impossible to perform, e.g., carbon emission, social benefit and oil reserves. In this case, we have to invite some domain experts to estimate them, thus obtaining their uncertainty distributions.

(iv) We should use uncertainty theory (here it refers to uncertain set) to model unsharp concepts, e.g., “young”, “tall”, “warm”, and “most” due to the ambiguity of human language.

(v) We should use uncertainty theory (here it refers to uncertain differential equation) to model dynamic systems with continuous-time noise, e.g., stock price, heat conduction, and population growth.

**B.9 Why do I think fuzzy set theory is bad mathematics?**

A fuzzy set is defined by its membership function $\mu$ which assigns to each element $x$ a real number $\mu(x)$ in the interval $[0, 1]$, where the value of $\mu(x)$ represents the grade of membership of $x$ in the fuzzy set. This definition was given by Zadeh [209] in 1965. Since then, fuzzy set theory has been spread
broadly. Although I strongly respect Professor Lotfi Zadeh’s achievements, I have to declare that fuzzy set theory is bad mathematics.

A very strange phenomenon in academia is that different people have different fuzzy set theories. Even so, we have to admit that every version of fuzzy set theory contains at least the following four items. The first one is a fuzzy set \( \xi \) with membership function \( \mu \). The next one is a complement set \( \xi^c \) with membership function

\[
\lambda(x) = 1 - \mu(x). \tag{B.16}
\]

The third one is a possibility measure defined by the three axioms,

\[
\text{Pos}\{\Omega\} = 1 \text{ for the universal set } \Omega, \tag{B.17}
\]

\[
\text{Pos}\{\emptyset\} = 0 \text{ for the empty set } \emptyset, \tag{B.18}
\]

\[
\text{Pos}\{A_1 \cup A_2\} = \text{Pos}\{A_1\} \lor \text{Pos}\{A_2\} \text{ for any events } A_1 \text{ and } A_2. \tag{B.19}
\]

And the fourth one is a relation between membership function and possibility measure (Zadeh [210]),

\[
\mu(x) = \text{Pos}\{x \in \xi\}. \tag{B.20}
\]

Now for any point \( x \), it is clear that \( \{x \in \xi\} \) and \( \{x \in \xi^c\} \) are opposite events\(^3\), and then

\[
\{x \in \xi\} \cup \{x \in \xi^c\} = \Omega. \tag{B.21}
\]

On the one hand, by using the possibility axioms, we have

\[
\text{Pos}\{x \in \xi\} \lor \text{Pos}\{x \in \xi^c\} = \text{Pos}\{\Omega\} = 1. \tag{B.22}
\]

On the other hand, by using the relation (B.20), we have

\[
\text{Pos}\{x \in \xi\} = \mu(x), \tag{B.23}
\]

\[
\text{Pos}\{x \in \xi^c\} = 1 - \mu(x). \tag{B.24}
\]

It follows from (B.22), (B.23) and (B.24) that

\[
\mu(x) \lor (1 - \mu(x)) = 1. \tag{B.25}
\]

Hence

\[
\mu(x) = 0 \text{ or } 1. \tag{B.26}
\]

This result shows that the membership function \( \mu \) can only be an indicator function of crisp set. In other words, only crisp sets can simultaneously satisfy (B.16)–(B.20). In this sense, fuzzy set theory collapses mathematically to

\(^3\)Perhaps some fuzzists insist that \( \{x \in \xi\} \) and \( \{x \in \xi^c\} \) are not opposite. Here I would like to advise them not to think so because it is in contradiction with \( \xi^c \) having the membership function \( \lambda(x) = 1 - \mu(x) \).
classical set theory. That is, fuzzy set theory is nothing but classical set theory.

Furthermore, it seems both in theory and practice that inclusion relation between fuzzy sets has to be needed. Thus fuzzy set theory also assumes a formula (Zadeh [210]),

$$\text{Pos}\{\xi \subset B\} = \sup_{x \in B} \mu(x) \quad (B.27)$$

for any crisp set $B$. Now consider two crisp intervals $[1, 2]$ and $[2, 3]$. It is completely unacceptable in mathematical community that $[1, 2]$ is included in $[2, 3]$, i.e., the inclusion relation

$$[1, 2] \subset [2, 3] \quad (B.28)$$

is 100% wrong. Note that $[1, 2]$ is a special fuzzy set whose membership function is

$$\mu(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases} \quad (B.29)$$

It follows from the formula (B.27) that

$$\text{Pos}\{[1, 2] \subset [2, 3]\} = \sup_{x \in [2, 3]} \mu(x) = 1. \quad (B.30)$$

That is, fuzzy set theory says that $[1, 2] \subset [2, 3]$ is 100% right. Are you willing to accept this result? If not, then fuzzy set theory is not acceptable.

Perhaps some fuzzists may argue that they never use possibility measure in fuzzy set theory. Here I would like to remind them that the membership degree $\mu(x)$ is just the possibility measure that the fuzzy set $\xi$ contains the point $x$ (i.e., $x$ belongs to $\xi$). Please also keep in mind that we cannot distinguish fuzzy set from random set (Robbins [145] and Matheron [126]) and uncertain set (Liu [93]) if the underlying measures are not available.

From the above discussion, we can see that fuzzy set theory is not consistent in mathematics and may lead to wrong results in practice. Therefore, I would like to conclude that "fuzzy set theory is bad mathematics." To express this more frankly, fuzzy set theory cannot be called mathematics. Can we improve fuzzy set theory? Yes, we can. But the change is so big that I have to give the revision a new name called uncertain set theory. See Chapter 8.

**B.10 Why is fuzzy variable not suitable for modelling indeterminate quantity?**

A fuzzy variable is a function from a possibility space to the set of real numbers (Nahmias [130]). Some people think that fuzzy variable is a suitable tool for modelling indeterminate quantity. Is it really true? Unfortunately, the answer is negative.
Appendix B - Frequently Asked Questions

Assume you are a fuzzist. For example, you think my height is a fuzzy variable $\xi$, and assign it a membership function,

$$
\mu(x) = \begin{cases} 
0, & \text{if } x \leq 1.60 \\
(x - 1.60)/0.04, & \text{if } 1.60 < x \leq 1.64 \\
1, & \text{if } 1.64 < x \leq 1.66 \\
(1.70 - x)/0.04, & \text{if } 1.66 < x \leq 1.70 \\
0, & \text{if } x > 1.70
\end{cases}
$$

that is just the trapezoidal fuzzy variable $(1.60, 1.64, 1.66, 1.70)$ in meters. Please do not argue why such a membership function is chosen because it is not important for the focus of debate. Based on the membership function $\mu$ and the definition of possibility measure

$$
\text{Pos}\{\xi \in B\} = \sup_{x \in B} \mu(x),
$$

it is easy for us to infer that

$$
\text{Pos}\{\text{“my height” = 1.65m}\} = 1
$$

and

$$
\text{Pos}\{\text{“my height” $\neq$ 1.65m}\} = 1
$$

by setting $B = \{1.65\}$ and $B = \{1.65\}^c$, respectively. Thus we immediately conclude the following three propositions:

(a) my height is “exactly 1.65m” with possibility measure 1,
(b) my height is “not 1.65m” with possibility measure 1,
(c) “exactly 1.65m” is as possible as “not 1.65m”.

The first proposition says you are 100% sure that my height is “exactly 1.65m”, neither less nor more. What a coincidence it should be! It is doubtless that the belief degree of “exactly 1.65m” is almost zero, and nobody is so naive to expect that “exactly 1.65m” is the true value of my height. The second proposition sounds good. The third proposition says “exactly 1.65m” and “not 1.65m” have the same possibility measure. Thus you have to regard them “equally likely”. Consider a bet:

You get $100 if my height is exactly 1.65m, and pay $100 otherwise (i.e., my height is not 1.65m).

Do you think the bet is fair? It seems that no one thinks so. Hence the conclusion (c) is unacceptable because “exactly 1.65m” is almost impossible compared with “not 1.65m”. This paradox shows that those indeterminate quantities like my height cannot be quantified by possibility measure and then they are not fuzzy concepts.
B.11 What is the difference between uncertainty theory and possibility theory?

The essential difference between uncertainty theory (Liu [88]) and possibility theory (Zadeh [210]) is that the former holds

$$\mathcal{M}\{\Lambda_1 \cup \Lambda_2\} = \mathcal{M}\{\Lambda_1\} \lor \mathcal{M}\{\Lambda_2\}$$

(B.35)

only for independent events $\Lambda_1$ and $\Lambda_2$, and the latter holds

$$\text{Pos}\{\Lambda_1 \cup \Lambda_2\} = \text{Pos}\{\Lambda_1\} \lor \text{Pos}\{\Lambda_2\}$$

(B.36)

for any events $\Lambda_1$ and $\Lambda_2$ no matter if they are independent or not. A lot of surveys showed that the measure of a union of events is usually greater than the maximum of the measures of individual events when they are not independent. This fact states that human brains do not behave fuzziness.

Both uncertainty theory and possibility theory attempt to model belief degrees, where the former uses the tool of uncertain measure and the latter uses the tool of possibility measure. Thus they are complete competitors.

B.12 How do we handle interval numbers by uncertainty theory?

In practice, information is sometimes only given by lower and upper bounds due to the imprecise observations or estimations by human beings. For example, “I think your height is between 1.6 and 1.7 meters”. From this statement, we may infer the following conclusions:

(i) Your height is not exactly known to us;

(ii) The true value of your height is on the interval $[1.6, 1.7]$;

(iii) All numbers on the interval $[1.6, 1.7]$ are equally likely.

This type of information is called interval-valued. In order to describe interval-valued information, we define an interval number as a number equally distributed on a specified interval. Using this concept, my statement becomes “I think your height is an interval number $[1.6, 1.7]$”. Hence how to rationally handle interval numbers is an important topic in science and engineering.

In uncertainty theory, an interval number $[a, b]$ is regarded as a linear uncertain variable (written as $L(a, b)$ in this book) with uncertainty distribution

$$\Phi(x) = \frac{x - a}{b - a}, \quad \text{if } a \leq x \leq b$$

(B.37)

and inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b, \quad 0 < \alpha < 1.$$  

(B.38)
Operational Law: Let \([a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\) be independent interval numbers. Assume \(f(x_1, x_2, \ldots, x_n)\) is strictly increasing with respect to \(x_1, x_2, \ldots, x_m\) and strictly decreasing with \(x_{m+1}, x_{m+2}, \ldots, x_n\). It follows from Theorem 2.16 (i.e., operational law of uncertain variables) that

\[
\xi = f([a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n])
\]

(B.39)

has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = f(\Phi^{-1}_1(\alpha), \ldots, \Phi^{-1}_m(\alpha), \Phi^{-1}_{m+1}(1-\alpha), \ldots, \Phi^{-1}_n(1-\alpha))
\]

(B.40)

where

\[
\Phi^{-1}_i(\alpha) = (1-\alpha)a_i + \alpha b_i
\]

(B.41)

for \(i = 1, 2, \ldots, n\). Note that \(\xi\) determined by (B.39) is an uncertain variable, but not necessarily an interval number.

Addition: Let \([a_1, b_1]\) and \([a_2, b_2]\) be independent interval numbers. It follows from the operational law that the addition \([a_1, b_1] + [a_2, b_2]\) has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = ((1-\alpha)a_1 + \alpha b_1) + ((1-\alpha)a_2 + \alpha b_2)
\]

\[
= (1-\alpha)(a_1 + a_2) + \alpha(b_1 + b_2)
\]

(B.42)

that happens to be an interval number \([a_1 + a_2, b_1 + b_2]\), i.e.,

\[
[a_1, b_1] + [a_2, b_2] = [a_1 + b_1, a_2 + b_2].
\]

(B.43)

Subtraction: Let \([a_1, b_1]\) and \([a_2, b_2]\) be independent interval numbers. It follows from the operational law that the subtraction \([a_1, b_1] - [a_2, b_2]\) has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = ((1-\alpha)a_1 + \alpha b_1) - (\alpha a_2 + (1-\alpha)b_2)
\]

\[
= (1-\alpha)(a_1 - b_2) + \alpha(b_1 - a_2)
\]

(B.44)

that happens to be an interval number \([a_1 - b_2, b_1 - a_2]\), i.e.,

\[
[a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2].
\]

(B.45)

Scalar Multiplication: Let \([a, b]\) be an interval number, and let \(k\) be a scalar number. It follows from the operational law that the scalar product \(k \cdot [a, b]\) has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = \begin{cases} 
(1-\alpha)(ka) + \alpha(kb), & \text{if } k \geq 0 \\ 
(1-\alpha)(kb) + \alpha(ka), & \text{if } k < 0 
\end{cases}
\]

(B.46)
that happens to be an interval number, and
\[
k \cdot [a, b] = \begin{cases} 
[ka, kb], & \text{if } k \geq 0 \\
[kb, ka], & \text{if } k < 0.
\end{cases} \tag{B.47}
\]

**Linear Function:** Let \([a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\) be independent interval numbers. It follows from addition, subtraction and scalar multiplication of interval numbers that the linear function
\[
k_1 \cdot [a_1, b_1] + k_2 \cdot [a_2, b_2] + \cdots + k_n \cdot [a_n, b_n] \tag{B.48}
\]
happens to be an interval number.

**Multiplication:** Let \([a_1, b_1]\) and \([a_2, b_2]\) be independent interval numbers with \(a_1 > 0\) and \(a_2 > 0\). It follows from the operational law that the multiplication \([a_1, b_1] \times [a_2, b_2]\) has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = ((1 - \alpha)a_1 + \alpha b_1) \times ((1 - \alpha)a_2 + \alpha b_2) \tag{B.49}
\]
that is no longer an interval number, i.e.,
\[
[a_1, b_1] \times [a_2, b_2] \neq [a_1 \times a_2, b_1 \times b_2]. \tag{B.50}
\]

**Division:** Let \([a_1, b_1]\) and \([a_2, b_2]\) be independent interval numbers with \(a_1 > 0\) and \(a_2 > 0\). It follows from the operational law that the division \([a_1, b_1] \div [a_2, b_2]\) has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = ((1 - \alpha)a_1 + \alpha b_1) \div (\alpha a_2 + (1 - \alpha)b_2) \tag{B.51}
\]
that is no longer an interval number, i.e.,
\[
[a_1, b_1] \div [a_2, b_2] \neq [a_1 \div a_2, b_1 \div b_2]. \tag{B.52}
\]

**Ranking Method:** Let \([a, b]\) be an interval number, and let \(c\) be a constant. It follows from Theorem 2.3 (i.e., measure inversion theorem) that the belief degree of \([a, b]\) being less than or equal to \(c\) is
\[
\mathcal{M}\{[a, b] \leq c\} = \begin{cases} 
0, & \text{if } c < a \\
\frac{c - a}{b - a}, & \text{if } a \leq c \leq b \\
1, & \text{if } c > b,
\end{cases} \tag{B.53}
\]
and the belief degree of \([a, b]\) being greater than or equal to \(c\) is
\[
\mathcal{M}\{[a, b] \geq c\} = \begin{cases} 
0, & \text{if } b < c \\
\frac{b - c}{b - a}, & \text{if } a \leq c \leq b \\
1, & \text{if } a > c.
\end{cases} \tag{B.54}
\]
For any independent interval numbers \([a_1, b_1]\) and \([a_2, b_2]\), it follows from the subtraction of interval numbers that

\[
M\{[a_1, b_1] \leq [a_2, b_2]\} = M\{[a_1 - b_2, b_1 - a_2] \leq 0\}. \tag{B.55}
\]

By using (B.53), we get

\[
M\{[a_1, b_1] \leq [a_2, b_2]\} = \begin{cases} 
0, & \text{if } a_1 > b_2 \\
1, & \text{if } a_2 > b_1 \\
b_2 - a_1, & \text{otherwise.}
\end{cases} \tag{B.56}
\]

**Expected Value:** It follows from Theorem 2.26 (i.e., expected value operator) that the interval number \([a, b]\) has an expected value

\[
E[a, b] = \frac{a + b}{2}. \tag{B.57}
\]

**Variance:** It follows from Theorem 2.42 that the interval number \([a, b]\) has a variance

\[
V[a, b] = \frac{(b - a)^2}{12}. \tag{B.58}
\]

**Second Moment:** It follows from Theorem 2.45 that the interval number \([a, b]\) has a second moment

\[
E[a, b]^2 = \frac{a^2 + ab + b^2}{3}. \tag{B.59}
\]

**Distance:** It follows from Theorem 2.49 that the distance between two independent interval numbers \([a_1, b_1]\) and \([a_2, b_2]\) is

\[
d([a_1, b_1], [a_2, b_2]) = \begin{cases} 
\frac{|(a_1 - b_2) + (b_1 - a_2)|}{2}, & \text{if } [a_1, b_1] \cap [a_2, b_2] = \emptyset \\
\frac{(a_1 - b_2)^2 + (b_1 - a_2)^2}{2(b_1 - a_2 - a_1 + b_2)}, & \text{if } [a_1, b_1] \cap [a_2, b_2] \neq \emptyset.
\end{cases}
\]

**B.13 Why do I think none of interval analysis, rough set theory and grey system are consistent?**

Interval analysis (Moore [128]), rough set theory (Pawlak [134]) and grey system (Deng [26]) declare that they are also able to handle interval numbers, and each of them contains the following five assumptions:

(i) \([a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]\),
Section B.14 - What is Uncertainty?

(ii) \([a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]\),

(iii) \([a_1, b_1] \times [a_2, b_2] = [a_1 \times a_2, b_1 \times b_2]\), if \(a_1 > 0, a_2 > 0\),

(iv) \([a_1, b_1] \div [a_2, b_2] = [a_1 \div b_2, b_1 \div a_2]\), if \(a_1 > 0, a_2 > 0\),

(v) \(\pi\{[a, b] \leq c\} = (c - a)/(b - a)\), if \(a \leq c \leq b\)

where \(\pi\{[a, b] \leq c\}\) represents the degree that the interval number \([a, b]\) is less than or equal to a constant \(c\). Although engineers like this type of mathematical system very much, unfortunately, there does not exist any mathematical system that simultaneously contains (i), (iii) and (v) since the three items are inconsistent with each other. For this reason, none of interval analysis, rough set theory and grey system are consistent.

In order to show the inconsistence, let us consider two interval numbers \([0, 1]\) and \([0, 1]\). It follows from items (i) and (v) that

\([0, 1] + [0, 1] = [0, 2]\)

and

\[0.5 = \pi\{[0, 2] \leq 1\} = \pi\{0, 1] + [0, 1] \leq 1\} = \pi\{(x,y) | x + y \leq 1, x \geq 0, y \geq 0\}.

On the other hand, it follows from (iii) and (v) that

\([0, 1] \times [0, 1] = [0, 1]\)

and

\[0.4 = \pi\{[0, 1] \leq 0.4\} = \pi\{0, 1] \times [0, 1] \leq 0.4\} = \pi\{(x,y) | xy \leq 0.4, 0 \leq x \leq 1, 0 \leq y \leq 1\}.

As a summary, from (i), (iii) and (v) we derive the following two equations:

\[\pi\{(x,y) | x + y \leq 1, x \geq 0, y \geq 0\} = 0.5, \quad \text{(B.60)}\]

\[\pi\{(x,y) | xy \leq 0.4, 0 \leq x \leq 1, 0 \leq y \leq 1\} = 0.4. \quad \text{(B.61)}\]

That is, \(\pi\{\Lambda\} > \pi\{\Delta\}\). However, unfortunately, \(\Lambda \subset \Delta\). See Figure B.4. This contradiction shows that a mathematical system is not consistent if it simultaneously contains items (i), (iii) and (v). Hence none of interval analysis, rough set theory and grey system are consistent in mathematics.
B.14 How did “uncertainty” evolve over the past 100 years?

After the word “randomness” was used to represent probabilistic phenomena, Knight (1921) and Keynes (1936) started to use the word “uncertainty” to represent any non-probabilistic phenomena. The academic community also calls it Knightian uncertainty, Keynesian uncertainty, or true uncertainty. Unfortunately, it seems impossible for us to develop a mathematical theory to deal with such a broad class of uncertainty because “non-probability” represents too many things. This disadvantage makes uncertainty in the sense of Knight and Keynes not able to become a scientific terminology. Despite that, we have to recognize that they made a great process to break the monopoly of probability theory.

However, there existed two major retrogressions on this issue. The first retrogression arose from Ramsey (1931) with the Dutch book argument that “proves” belief degree follows the laws of probability theory. Mathematically, Dutch book argument and the three axioms of probability theory are two equivalent propositions. Logically, a proposition is still unproven if it is proved by an unproven proposition. Since Dutch book argument has not been proved, the belief degree has not been proved to follow the axioms of probability theory. The second retrogression arose from Cox’s theorem (1946) that belief degree is isomorphic to a probability measure. Most people do not notice that Cox’s theorem is based on an unreasonable assumption, and then mistakenly believe that uncertainty and probability are synonymous. This idea remains alive today under the name of subjective probability (de Finetti, 1937). Yet numerous experiments demonstrated that belief degree does not follow the laws of probability theory.

An influential exploration by Zadeh (1965) was the fuzzy set theory that
was widely said to be successfully applied in many areas of our life. However, fuzzy set theory has neither evolved as a mathematical system nor become a suitable tool for rationally modelling belief degrees. The main mistake of fuzzy set theory is based on the wrong assumption that the belief degree of a union of events is the maximum of the belief degrees of the individual events no matter if they are independent or not. The extensions of fuzzy set theory fail to become a consistent mathematical system, too.

In addition, interval analysis (Moore, 1966), rough set theory (Pawlak, 1982) and grey system (Deng, 1982) each have also a significant effect on engineering and management. However, unfortunately, none of them are consistent in mathematics.

The latest development was uncertainty theory founded by Liu (2007). Nowadays, uncertainty theory has become a branch of mathematics that is not only a formal study of an abstract structure (i.e., uncertainty space) but also applicable to modelling belief degrees. Uncertainty is defined as anything that follows the laws of uncertainty theory. From then on, “uncertainty” became a scientific terminology on the basis of uncertainty theory.
Bibliography


List of Frequently Used Symbols

- $\mathcal{M}$ uncertain measure
- $(\Gamma, \mathcal{L}, \mathcal{M})$ uncertainty space
- $\xi, \eta, \tau$ uncertain variables
- $\Phi, \Psi, \Upsilon$ uncertainty distributions
- $\Phi^{-1}, \Psi^{-1}, \Upsilon^{-1}$ inverse uncertainty distributions
- $\mu, \nu, \lambda$ membership functions
- $\mu^{-1}, \nu^{-1}, \lambda^{-1}$ inverse membership functions
- $\mathcal{L}(a, b)$ linear uncertain variable
- $\mathcal{Z}(a, b, c)$ zigzag uncertain variable
- $\mathcal{N}(e, \sigma)$ normal uncertain variable
- $\text{LOGN}(e, \sigma)$ lognormal uncertain variable
- $(a, b, c)$ triangular uncertain set
- $(a, b, c, d)$ trapezoidal uncertain set
- $E$ expected value
- $V$ variance
- $H$ entropy
- $X_t, Y_t, Z_t$ uncertain processes
- $C_t$ Liu process
- $N_t$ renewal process
- $\mathcal{Q}$ uncertain quantifier
- $(\mathcal{Q}, S, P)$ uncertain proposition
- $\forall$ universal quantifier
- $\exists$ existential quantifier
- $\lor$ maximum operator
- $\land$ minimum operator
- $\neg$ negation symbol
- $\Pr$ probability measure
- $(\Omega, \mathcal{A}, \Pr)$ probability space
- $Ch$ chance measure
- $k$-$\text{max}$ the $k$th largest value
- $k$-$\text{min}$ the $k$th smallest value
- $\emptyset$ the empty set
- $\mathbb{R}$ the set of real numbers
- $|A|$ cardinality of set $A$
Index

age replacement policy, 295
algebra, 9
α-path, 333
alternating renewal process, 300
American option, 349
Asian option, 351
asymptotic theorem, 15
autoregressive model, 390
belief degree, 3
betting ratio, 3
bisection method, 130
block replacement policy, 287
Boolean function, 65
Boolean uncertain variable, 66
Borel algebra, 10
Borel set, 10
bridge system, 144
chain rule, 317
chance distribution, 404
chance inversion theorem, 406
chance measure, 400
change of variables, 317
Chebyshev inequality, 82
Chen-Ralescu theorem, 151
comonotonic function, 76
complement of uncertain set, 171, 196
complete uncertainty space, 16
conditional uncertainty, 28, 95, 217
confidence interval, 387, 394
containment, 203
distance, 87, 214
disturbance term, 381, 390
drift, 309, 314
dual quantifier, 229
duality axiom, 11
Dutch book argument, 459
Ellsberg experiment, 422
Ellsberg urn, 4
empirical uncertainty distribution, 41
entropy, 89, 215
Euler method, 342
European option, 345
event, 11
expected loss, 137, 435
expected value, 70, 206, 413
expert’s experimental data, 373
extreme value theorem, 62, 269
fair price principle, 346
feasible solution, 107
first hitting time, 271, 339
forecast value, 386, 393
frequency, 2
fundamental theorem of calculus, 315
fuzzy set, 465
goal programming, 124
Hölder’s inequality, 79
hypothetical syllogism, 164
imaginary inclusion, 206
inclusion, 203
independence, 23, 47, 189
independent increment, 266
indeterminacy, 1
individual feature data, 223
inference rule, 247
integration by parts, 318
interest rate ceiling, 363
interest rate floor, 365
intersection of uncertain sets, 171, 194
interval number, 469
inverse membership function, 187
inverse uncertainty distribution, 43
investment risk analysis, 138
inverted pendulum, 254
Jensen’s inequality, 80
k-out-of-n system, 128
Laplace criterion, 4
law of contradiction, xiv, 172
law of excluded middle, xiv, 172
law of large numbers, 421
law of truth conservation, xiv
Lebesgue measure, 13
linear uncertain variable, 38
linguistic summarizer, 242
Liu integral, 310
Liu process, 305
logical equivalence theorem, 236
lognormal uncertain variable, 41
loss function, 127
machine scheduling problem, 112
Markov inequality, 79
maximum entropy principle, 93
maximum flow problem, 442
maximum uncertainty principle, xiv
measurable function, 31
measurable set, 10
measure inversion formula, 174
measure inversion theorem, 41
median, 39
membership function, 174
method of moments, 378
Minkowski inequality, 80
modus ponens, 162
modus tollens, 163
moment, 84
monotone quantifier, 227
monotonicity theorem, 14
multilevel programming, 125
multiobjective programming, 122
Nash equilibrium, 125
negated quantifier, 228
nonempty uncertain set, 171
normal uncertain variable, 40
normality axiom, 11
operational law, 49, 192, 266, 406
optimal solution, 108
option pricing, 345
order statistic, 61, 409
parallel system, 128
Pareto solution, 123
Peng-Iwamura theorem, 37
Pólya urn, 4
polyrectangular theorem, 26
portfolio selection, 358
possibility measure, 466
power set, 10
principle of least squares, 376
product axiom, 17
product probability theorem, 457
product uncertain measure, 17
project scheduling problem, 119
randomness, definition of, 462
regression model, 381
regular membership function, 187
regular uncertainty distribution, 43
reliability index, 143, 435
renewal process, 283, 442
renewal reward process, 288
residual, 383, 392
risk index, 129, 431
ruin index, 292
ruin time, 293
rule-base, 251
Runge-Kutta method, 343
sample path, 260
series system, 127
shortest path problem, 441
σ-algebra, 9
stability, 331
Stackelberg-Nash equilibrium, 126
standby system, 128
stationary increment, 276
strictly decreasing function, 55
strictly increasing function, 49
strictly monotone function, 56
structural risk analysis, 132
structure function, 141
subadditivity axiom, 11
subjective probability, 474
time integral, 273, 341
totally ordered uncertain set, 181
trapezoidal uncertain set, 179
triangular uncertain set, 179
truth value, 149, 236
uncertain calculus, 305
uncertain control, 254
uncertain currency model, 366
uncertain differential equation, 321
uncertain entailment, 160
uncertain finance, 345
uncertain graph, 437
uncertain inference, 247
uncertain insurance model, 291
uncertain integral, 310
uncertain interest rate model, 361
uncertain logic, 223
uncertain measure, 12
uncertain network, 440
uncertain process, 259
uncertain programming, 107
uncertain proposition, 147, 235
uncertain quantifier, 224
uncertain random process, 442
uncertain random programming, 428
uncertain random variable, 403
uncertain regression analysis, 381
uncertain reliability analysis, 142
uncertain renewal process, 283
uncertain risk analysis, 127
uncertain sequence, 98
uncertain set, 167
uncertain statistics, 373
uncertain stock model, 345
uncertain system, 251
uncertain time series analysis, 390
uncertain variable, 31
uncertain vector, 103
uncertainty, definition of, 462
uncertainty distribution, 34, 260
uncertainty space, 16
unimodal quantifier, 227
union of uncertain sets, 171, 192
urn problem, 4, 422
value-at-risk, 136, 434
variance, 81, 212, 417
vehicle routing problem, 115
Wiener process, 463
Yao-Chen formula, 334
zero-coupon bond, 362
zigzag uncertain variable, 39
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Uncertainty Theory

When no samples are available to estimate distribution functions, or some emergency (e.g., war, flood, earthquake, accident, and even rumour) arises, we have to invite some domain experts to evaluate belief degree that each event will happen. Perhaps some people think that belief degree should be modeled by subjective probability or fuzzy set theory. However, it is usually inappropriate because both of them may lead to counterintuitive results in this case. In order to rationally deal with personal belief degrees, uncertainty theory was founded in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of mathematics.

This is an introductory textbook on uncertainty theory, uncertain programming, uncertain risk analysis, uncertain reliability analysis, uncertain set, uncertain logic, uncertain inference, uncertain process, uncertain calculus, uncertain differential equation, and uncertain statistics. This textbook also shows applications of uncertainty theory to scheduling, logistics, network optimization, data mining, control, and finance.

Axiom 1. (Normality Axiom) $M\{\Gamma\} = 1$ for the universal set $\Gamma$.
Axiom 2. (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$.
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1$, $\Lambda_2, \cdots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$  

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $M$ is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Probability theory is a branch of mathematics for modelling frequencies, while uncertainty theory is a branch of mathematics for modelling belief degrees.