Fuzzy Optimal Control with Application to Portfolio Selection

Yuanguo Zhu
Department of Applied Mathematics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu
People’s Republic of China
(ygzhu@mail.njust.edu.cn)

Abstract

Optimal control is a very important field of study not only in theory but in applications, and stochastic optimal control is also a significant branch of research in theory and applications. Based on the concept of fuzzy process introduced by Liu, we present a fuzzy optimal control problem. Applying Bellman’s optimal principle, we obtain the principle of optimality for fuzzy optimal control, and then give a fundamental result called the equation of optimality in fuzzy optimal control. Finally, as application, we use equation of optimality to solve a portfolio selection problem.

Keywords: fuzzy optimal control, fuzzy process, principle of optimality, equation of optimality, portfolio selection

1 Introduction

Since 50s of last century, optimal control theory has been an important branch of modern control theory. The study of optimal control greatly attracted the attention of many mathematician because of the necessary of strict expression form in optimal control theory. With the more use of methods and results on mathematics and computer science, optimal control theory has greatly achieved development, and been applied to many fields such as production engineering, programming, economy and management.

The study of stochastic optimal control initiated in 1970s such as in Merton [7] for finance. Some researches on optimal control of Brownian motion process or stochastic differential equations and applications in finance may refer to some books such as Fleming and Rishel [3], Harrison [5] and Karatzas [6]. One of the main methods to study optimal control dues to dynamic programming. The use of dynamic programming in optimization over Ito’s process is discussed in Dixit and Pindyck [1].

The complexity of the world makes the events we face uncertain in various forms. Besides randomness, fuzziness is also an important uncertainty, which plays an essential role in the real world. Fuzzy set theory has been developed very fast since it was introduced by scientist on cybernetics Zadeh [18] in 1965. A fuzzy set was characterized with its membership function by Zadeh. For the purpose of measuring fuzzy events, Zadeh [19] presented the concept of possibility measure and the term of fuzzy variable in 1978. In order to give a self-dual measure for fuzzy events, Liu and Liu [13] introduced the concept of credibility measure in 2002. Based on credibility
measure, credibility theory was founded by Liu [9] in 2004 and refined by Liu [11] as a branch of mathematics for dealing with the behavior of fuzzy phenomena. A fuzzy variable may be redefined as a function from a credibility space to the set of real numbers. As fuzzy counterpart of stochastic process and Brownian motion, fuzzy process and \( C \) process was introduced by Liu [12] recently. We may call \( C \) process to be Liu process.

In order to handle an optimal control problem with fuzzy process, now we will introduce and investigate fuzzy optimal control. We will deal with fuzzy optimal control by using dynamic programming. In next section, we will review some concepts such as expected value of fuzzy variable, fuzzy process, Liu process, and fuzzy deferential equation. In Section 3, we will introduce fuzzy optimal control problem, and present the principle of optimality based on Bellman’s principle of optimality in dynamic programming. In Section 4, we will obtain a fundamental result called the equation of optimality in fuzzy optimal control. In the last section, we will solve a portfolio selection problem by using the equation of optimality.

2 Preliminary

If a fuzzy variable \( \xi \) is given by a membership function \( \mu \), then we may get the credibility value via
\[
\text{Cr} \{ \xi \in B \} = \frac{1}{2} \left( \sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x) \right), \quad B \subset \mathbb{R}.
\]
Membership function represents the degree of possibility that the fuzzy variable \( \xi \) takes some prescribed value.

**Definition 2.1.** (Liu and Liu [13]) Let \( \xi \) be a fuzzy variable. Then the expected value of \( \xi \) is defined by
\[
E[\xi] = \int_{-\infty}^{+\infty} \text{Cr} \{ \xi \geq r \} dr - \int_{-\infty}^{0} \text{Cr} \{ \xi \leq r \} dr
\] (2.1)
provided that at least one of two integrals is finite.

**Definition 2.2.** (Liu and Liu [13]) Let \( \xi \) be a fuzzy variable with finite expected value \( e \). Then the variance of \( \xi \) is defined by \( V[\xi] = E[(\xi - e)^2] \).

**Theorem 2.1.** (Liu and Liu [14]) Let \( \xi \) and \( \eta \) be independent fuzzy variables with finite expected values. Then for any numbers \( a \) and \( b \), we have
\[
E[a\xi + b\eta] = aE[\xi] + bE[\eta].
\]

Based on the credibility space, Liu introduced the concepts of fuzzy process, Liu process, fuzzy differential equation, and etc.

**Definition 2.3.** (Liu [12]) Let \( T \) be an index set and let \((\Theta, \mathcal{F}, \text{Cr})\) be a credibility space. A fuzzy process is a function from \( T \times (\Theta, \mathcal{F}, \text{Cr}) \) to the set of real numbers.

**Definition 2.4.** (Liu [12]) A fuzzy process, simply denoted by \( X_t \), is said to have independent increments if \( X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}} \) are independent fuzzy variables for any times \( t_0 < t_1 < \cdots < t_k \). A fuzzy process \( X_t \) is said to have stationary increments if, for any given \( t > 0 \), the increments \( X_{s+t} - X_s \) are identically distributed fuzzy variables for all \( s > 0 \).
Definition 2.5. (Liu [12]) A fuzzy process $C_t$ is said to be Liu process if
(i) $C_0 = 0$,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value $e t$ and variance $\sigma^2 t^2$, whose membership function is
$$
\mu(x) = 2 \left(1 + \exp\left(\frac{\pi |x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad x \in \mathbb{R}.
$$

The parameters $e$ and $\sigma$ are called the drift and diffusion coefficients, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$. The Liu process plays the role of Brownian motion or Wiener process.

Based on Liu process, a new kind of fuzzy differential introduced by Liu [12]. As the inverse of fuzzy differential, a kind of integral, called Liu integral, was also introduced by Liu [12]. Liu integral is different from the fuzzy integral based on fuzzy measure given by Sugeno [17]. The following concept of fuzzy differential equation is important in theory and applications, and essential in the study of this paper.

Definition 2.6. (Liu [12]) Suppose $C_t$ is a standard Liu process, and $f$ and $g$ are some given functions. then
$$
dX_t = f(t, X_t)dt + g(t, X_t)dC_t \quad (2.2)
$$
is called a fuzzy differential equation. A solution is a fuzzy process $X_t$ that satisfies (2.2) identically in $t$.


3 Problem of Fuzzy Optimal Control

Fuzzy optimal control is to choose the best decision such that some objective function related to a fuzzy process provided by a fuzzy differential equation is optimized. Unless stated otherwise, we assume that $C_t$ is a standard Liu process. We consider the following fuzzy expected value optimal control problem

$$
\begin{align*}
J(0, x_0) \equiv & \sup_{D} E \left[ \int_0^T f(X_s, D, s)ds + G(X_T, T) \right] \\
\text{subject to} & \\
dX_s = & \mu(X_s, D, s)ds + \sigma(X_s, D, s)dC_s \quad \text{and} \quad X_0 = x_0.
\end{align*} \quad (3.1)
$$

In the above problem, $X_s$ is the state variable, $D$ the decision variable (represents the function $D(t, x)$ of time $t$ and state $x$), $f$ the objective function, and $G$ the function of terminal reward. For a given $D$, $dX_s$ is defined by the fuzzy differential equation

$$
dX_s = \mu(X_s, D, s)ds + \sigma(X_s, D, s)dC_s \quad \text{and} \quad X_0 = x_0 \quad (3.2)
$$

where $\mu$ and $\sigma$ are two functions of $X_s$, $D$ and time $s$. The function $J(0, x_0)$ is the expected optimal reward obtainable in $[0, T]$ with the initial condition that at time 0 we are in state $x_0$. For
any $0 < t < T$, $J(t, x)$ is the expected optimal reward obtainable in $[t, T]$ with the condition that at time $t$ we are in state $X_t = x$. That is, we have

$$J(t, x) \equiv \sup_D E \left[ \int_t^T f(X_s, D, s)ds + G(X_T, T) \right]$$

subject to

$$dX_s = \mu(X_s, D, s)ds + \sigma(X_s, D, s)dC_s \quad \text{and} \quad X_t = x.$$

**Remark 3.1.** Of course, we can consider the fuzzy $\alpha$-optimistic optimal control problem and $\alpha$-pessimistic optimal control problem. These will be our next work.

Now we present the following principle of optimality.

**Theorem 3.1** (Principle of optimality). For any $(t, x) \in [0, T) \times R$, and $\Delta t > 0$ with $t + \Delta t < T$, we have

$$J(t, x) = \sup_D E \left[ \int_t^{t+\Delta t} f(X_s, D, s)ds + J(t + \Delta t, x + \Delta X_t) \right],$$

where $x + \Delta X_t = X_{t+\Delta t}$.

**Proof:** We denote the right side of (3.4) by $\tilde{J}(t, x)$. It follows from the definition of $J(t, x)$ that

$$J(t, x) \geq E \left[ \int_t^{t+\Delta t} f(X_s, D_{|[t,t+\Delta t]}, s)ds + \int_{t+\Delta t}^T f(X_s, D_{|[t+\Delta t,T]}, s)ds + G(X_T, T) \right]$$

for any $D$, where $D_{|[t,t+\Delta t]}$ and $D_{|[t+\Delta t,T]}$ are the variables of decision variable $D$ restrictedly on $[t, t + \Delta t)$ and $[t + \Delta t, T]$, respectively. Since the fuzzy processes $dC_s (s \in [t, t + \Delta t))$ and $dC_s (s \in [t + \Delta t, T])$ are independent, we know that

$$\int_t^{t+\Delta t} f(X_s, D_{|[t,t+\Delta t]}, s)ds \quad \text{and} \quad \int_{t+\Delta t}^T f(X_s, D_{|[t+\Delta t,T]}, s)ds$$

are independent. Thus

$$J(t, x) \geq E \left[ \int_t^{t+\Delta t} f(X_s, D_{|[t,t+\Delta t]}, s)ds + \int_{t+\Delta t}^T f(X_s, D_{|[t+\Delta t,T]}, s)ds + G(X_T, T) \right]$$

by Theorem 2.1. Taking the supremum with respect to $D_{|[t+\Delta t,T]}$ first, and then $D_{|[t,t+\Delta t]}$ in (3.6), we get $J(t, x) \geq \tilde{J}(t, x)$.

On the other hand, for all $D$, we have

$$E \left[ \int_t^T f(X_s, D, s)ds + G(X_T, T) \right]$$

$$= E \left\{ \int_t^{t+\Delta t} f(X_s, D, s)ds + \int_{t+\Delta t}^T f(X_s, D_{|[t+\Delta t,T]}, s)ds + G(X_T, T) \right\}$$

$$\leq E \left[ \int_t^{t+\Delta t} f(X_s, D, s)ds + J(t + \Delta t, x + \Delta X_t) \right]$$

$$\leq \tilde{J}(t, x).$$

Hence, $J(t, x) \leq \tilde{J}(t, x)$, and then $J(t, x) = \tilde{J}(t, x)$. The theorem is proved.
4 Equation of Optimality

Also, we assume $C_t$ is a standard Liu process. Consider the following fuzzy optimal control problem

\[
\begin{aligned}
J(t, x) &= \sup_D E \left[ \int_t^T f(X_s, D, s)ds + G(X_T, T) \right] \\
\text{subject to}
\end{aligned}
\tag{4.1}
\]

\[
dX_s = \mu(X_s, D, s)ds + \sigma(X_s, D, s)dC_s \quad \text{and} \quad X_t = x.
\]

Substituting Equations (4.3) and (4.4) into Equation (3.4) yields

By using Taylor series expansion, we get

\[
J(t + \Delta t, x + \Delta X_t) = J(t, x) + J_t(t, x)\Delta t + J_x(t, x)\Delta X_t + \frac{1}{2} J_{xx}(t, x)\Delta X_t^2 + \frac{1}{2} J_{tx}(t, x)\Delta t\Delta X_t + o(\Delta t)
\tag{4.4}
\]

Substituting Equations (4.3) and (4.4) into Equation (3.4) yields

\[
0 = \sup_D \{ f(x, D, t)\Delta t + J_t(t, x)\Delta t + E[J_x(t, x)\Delta X_t] + \frac{1}{2} J_{xx}(t, x)\Delta X_t^2 + J_{tx}(t, x)\Delta t\Delta X_t + o(\Delta t) \}
\tag{4.5}
\]

Let $\xi$ be a fuzzy variable such that $\Delta X_t = \xi + \mu(x, D, t)\Delta t$. It follows from (4.5) that

\[
0 = \sup_D \{ f(x, D, t)\Delta t + J_t(t, x)\Delta t + J_x(t, x)\mu(x, D, t)\Delta t + E[J_x(t, x)\Delta X_t] + \frac{1}{2} J_{xx}(t, x)\xi^2 + J_{tx}(t, x)\Delta t\xi + o(\Delta t) \}
\tag{4.6}
\]

where $a \equiv J_x(t, x) + J_{xx}(t, x)\mu(x, D, t)\Delta t + J_{tx}(t, x)\Delta t$, and $b \equiv \frac{1}{2} J_{xx}(t, x)$. It follows from the fuzzy differential equation

\[
dX_s = \mu(X_s, D, s)ds + \sigma(X_s, D, s)dC_t \quad \text{and} \quad X_t = x
\]

that $\xi = \Delta X_t - \mu(x, D, t)\Delta t$ is a normally distributed fuzzy variable with expected value 0 and variance $\sigma^2(x, D, t)\Delta t^2$. Simply denote $\sigma = \sigma(x, D, t)$ in sequel. Let

\[
x_0 = \frac{|a|}{2|b|}, \quad z = \frac{\pi x}{\sqrt{6\sigma \Delta t}}, \quad z_0 = \frac{\pi |a|}{2\sqrt{6}|b|\sigma \Delta t}.
\]
Formula (6.1) in the Appendix implies that
\[
E[a\xi + b\xi^2] = \int_0^{+\infty} (a + 2bx) \left( 1 + \exp \left( \frac{\pi x}{\sqrt{6}\sigma\Delta t} \right) \right)^{-1} dx \\
- \int_0^{x_0} (a - 2bx) \left( 1 + \exp \left( \frac{\pi x}{\sqrt{6}\sigma\Delta t} \right) \right)^{-1} dx \\
= \frac{\sqrt{6}\sigma\Delta t^2}{\pi} \int_{x_0}^{+\infty} \frac{1}{1 + e^z} dz + \frac{12b\sigma^2\Delta t^2}{\pi^2} \int_0^{z_0} \frac{z}{1 + e^z} dz + \frac{12b\sigma^2\Delta t^2}{\pi^2} \int_0^{+\infty} \frac{z}{1 + e^z} dz \\
= o(\Delta t).
\]

Formula (6.2) in the Appendix implies that
\[
E[a\xi + b\xi^2] = \int_0^{x_0} (a + 2bx) \left( 1 + \exp \left( \frac{\pi x}{\sqrt{6}\sigma\Delta t} \right) \right)^{-1} dx \\
- \int_0^{+\infty} (a - 2bx) \left( 1 + \exp \left( \frac{\pi x}{\sqrt{6}\sigma\Delta t} \right) \right)^{-1} dx \\
= -\frac{\sqrt{6}\sigma\Delta t^2}{\pi} \int_{x_0}^{+\infty} \frac{1}{1 + e^z} dz + \frac{12b\sigma^2\Delta t^2}{\pi^2} \int_0^{z_0} \frac{z}{1 + e^z} dz + \frac{12b\sigma^2\Delta t^2}{\pi^2} \int_0^{+\infty} \frac{z}{1 + e^z} dz \\
= o(\Delta t).
\]

Hence
\[
E[a\xi + b\xi^2] = \pm \frac{\sqrt{6}\sigma\Delta t}{\pi} \int_{x_0}^{+\infty} \frac{1}{1 + e^z} dz + \frac{12b\sigma^2\Delta t^2}{\pi^2} \int_0^{z_0} \frac{z}{1 + e^z} dz \\
+ \frac{12b\sigma^2\Delta t^2}{\pi^2} \int_0^{+\infty} \frac{z}{1 + e^z} dz \\
= o(\Delta t). \quad (4.7)
\]

Substituting Equation (4.7) into Equation (4.6) yields
\[
-J_t(t, x)\Delta t = \sup_D \left\{ f(x, D, t)\Delta t + J_x(t, x)\mu(x, D, t)\Delta t + o(\Delta t) \right\}. \quad (4.8)
\]

Dividing Equation (4.8) by \(\Delta t\), and letting \(\Delta t \to 0\), we can obtain the result (4.2).

**Remark 4.1.** The equation of optimality gives a necessary condition for an extremum. If the equation has solutions, then the optimal decision and optimal value of objective function are determined. If function \(f\) is convex in its arguments, then the equation will produce a minimum, and if \(f\) is concave in its arguments, then it will produce a maximum. We note that the boundary condition for the equation is \(J(T, X_T) = G(X_T, T)\).

**Remark 4.2.** We note that in the equation of optimality for stochastic optimal control (called Hamilton-Jacobi-Bellman equation), there is an extra term \(\frac{1}{2} J_{xx}(t, x)\sigma^2(x, D, t)\).
**Example 4.1.** Consider the following fuzzy optimization problem:

\[
\begin{align*}
J(t, x) &\equiv \min_{D} E \left[ \int_{0}^{T} e^{-\beta s}(aX_s^2 + bX_s D + cD^2) ds \right] \\
\text{subject to} & \quad dX_s = (\mu D + \alpha X_s) ds + \sigma X_s dC_s,
\end{align*}
\]

where \( a > 0, c > 0, \sigma > 0, b^2 - 4ac \leq 0, \mu \neq 0, \alpha \in \mathbb{R} \), \( \beta \) is a discount factor, \( T \) is the terminal time.

We see that \( f(X_s, D, s) = e^{-\beta s}(aX_s^2 + bX_s D + cD^2) \). It follows from Equation (4.2) that

\[
-J_t = \min_{D} \left\{ e^{-\beta t}(ax^2 + bx D + cD^2) + J_x(\mu D + \alpha x) \right\} = \min_{D} L(D),
\]

where \( L(D) \) denotes the term in the braces. The optimal decision \( D \) satisfies

\[
\frac{dL(D)}{dD} = bx e^{-\beta t} + 2c D e^{-\beta t} + J_x \mu = 0 \quad \text{or} \quad D = -\frac{J_x e^{\beta t} \mu + bx}{2c}.
\]

Equation (4.9) becomes

\[
-J_t e^{\beta t} = \left( a - \frac{b^2}{4c} \right) x^2 - \frac{J_x^2 e^{\beta t} \mu^2}{2c} + \left( \alpha - \frac{b \mu}{2c} \right) x J_x e^{\beta t}.
\]

That is

\[
-J_t e^{\beta t} = \left( a - \frac{b^2}{4c} \right) x^2 - \frac{J_x^2 e^{\beta t} \mu^2}{2c} + \left( \alpha - \frac{b \mu}{2c} \right) x J_x e^{\beta t}.
\]

We conjecture that \( J(t, x) = k x^2 e^{-\beta t} \). Thus, we have

\[
J_t = -\beta k x^2 e^{-\beta t}, \quad J_x = 2 k x e^{-\beta t}.
\]

Substituting them into Equation (4.10) yields

\[
\beta k x^2 = \left( a - \frac{b^2}{4c} \right) x^2 - \frac{\mu^2}{4c} (2kx)^2 - \frac{b \mu}{2c} (2kx^2) + \alpha (2kx^2),
\]

or

\[
\frac{\mu^2}{c} k^2 + \left( \beta - 2 \alpha + \frac{b \mu}{c} \right) k + \left( \frac{b^2}{4c} - a \right) = 0.
\]

The solution of the parameter \( k \) (selecting \( k \) such that \( J(t, x) \geq 0 \)) is

\[
k = \frac{\left( 2 \alpha c - c \beta - b \mu \right) + \sqrt{(2 \alpha c - c \beta - b \mu)^2 + \mu^2 (4ac - b^2)}}{2 \mu^2}.
\]

Hence, the optimal decision and the optimal value of the objective function are given as follows, respectively

\[
D = -\frac{(2k \mu + b)x}{2c}, \quad J(t, x) = k x^2 e^{-\beta t}.
\]
5 Portfolio Selection Problem

Let $X_t$ be the wealth of an investor at time $t$. The investor allocates a fraction $w$ of the wealth in a risky asset and remainder in a sure asset. The sure asset produces a rate of return $b$. The risky asset yields a mean rate of return $\mu$ along with a variance of $\sigma^2$ per unit time. That is to say, the risky asset earns a return $dr_t$ in time interval $(t, t + dt)$, where $dr_t = \mu dt + \sigma dC_t$, and $C_t$ is a standard Liu process. Thus

$$X_{t+dt} = X_t + b(1-w)dt + dr_t(wX_t)$$

$$= X_t + b(1-w)X_t dt + (\mu dt + \sigma dC_t)(wX_t)$$

$$= X_t + [\mu w + b(1-w)]X_t dt + \sigma w X_t dC_t.$$

If we consider the consumption rate by an amount $p$, we obtain

$$dX_t = [\mu w X_t + b(1-w)X_t - p]dt + \sigma w X_t dC_t.$$ 

A portfolio selection problem is provided by

$$J(t, x) \equiv \max_{p, w} E \left[ \int_0^\infty e^{-\beta t} \frac{p^\lambda + [(\mu - \sigma^2)x]_+}{\lambda} dt \right]$$

subject to

$$dX_t = [\mu w X_t + b(1-w)X_t - p]dt + \sigma w X_t dC_t,$$

where $\beta > 0$, $0 < \lambda < 1$. By the equation of optimality (4.2), we have that

$$-J_t = \max_{p, w} \left\{ e^{-\beta t} \frac{p^\lambda + [(\mu - \sigma^2)x]_+}{\lambda} + (\mu - b)x \right\} = \max_{p, w} L(p, w),$$

where $L(p, w)$ represents the term in the braces. The optimal $(p, w)$ satisfies

$$\frac{\partial L(p, w)}{\partial p} = e^{-\beta t} p^{\lambda-1} - J_x,$$

$$\frac{\partial L(p, w)}{\partial w} = e^{-\beta t} (\mu - \sigma^2)x \lambda^{-1} [(\mu - \sigma^2)x]_+ + J_x (\mu - b)x = 0,$$

or

$$p = (J_x e^{-\beta t}) \frac{\lambda+1}{\lambda}, \quad w = \left[ \frac{(\mu - b)J_x e^{\beta t}}{\mu - \sigma^2} \right] \frac{1}{(\mu - \sigma^2)x}.$$

Hence

$$-J_t = \frac{1}{\lambda} e^{-\beta t} \left\{ (J_x e^{-\beta t}) \frac{\lambda+1}{\lambda} + \left[ \frac{(\mu - b)J_x e^{\beta t}}{\mu - \sigma^2} \right] \frac{\lambda}{\lambda+1} \right\} + \frac{\mu - b}{\mu - \sigma^2} J_x \left[ \frac{(b - \mu) J_x e^{\beta t}}{\mu - \sigma^2} \right] \frac{1}{\lambda+1}$$

$$+ bx J_x - J_x (J_x e^{-\beta t}) \frac{\lambda+1}{\lambda},$$

or

$$-J_t e^{\beta t} = \left[ \frac{1}{\lambda} - 1 \right] \left[ 1 + \left( \frac{b - \mu}{\mu - \sigma^2} \right) \frac{1}{\lambda+1} \right] (J_x e^{\beta t}) \frac{\lambda+1}{\lambda} + bx J_x e^{\beta t}.$$ 

(5.1)

We conjecture that $J(t, x) = k x^\lambda e^{-\beta t}$. Then

$$J_t = -k \beta x^\lambda e^{-\beta t}, \quad J_x = k \lambda x^{\lambda-1} e^{-\beta t}.$$
Substituting them into Equation (5.1) yields

\[ k\beta x^\lambda = \left( \frac{1}{\lambda} - 1 \right) \left[ 1 + \left( \frac{b - \mu}{\mu - \sigma^2} \right)^{\frac{1}{\lambda}} \right] (k\lambda)^{\frac{1}{\lambda}} x^\lambda + ka\lambda x^\lambda, \]

or

\[ (k\lambda)^{\frac{1}{\lambda}} = \frac{\beta - b\lambda}{(1 - \lambda) \left[ 1 + \left( \frac{b - \mu}{\mu - \sigma^2} \right)^{\frac{1}{\lambda}} \right]}. \]

So we get

\[ k\lambda = \left\{ \frac{\beta - b\lambda}{(1 - \lambda) \left[ 1 + \left( \frac{b - \mu}{\mu - \sigma^2} \right)^{\frac{1}{\lambda}} \right]} \right\}^{\lambda - 1}. \]

Therefore the optimal consumption rate and the optimal fraction of investment on risky asset is determined, respectively, by

\[ p = x(k\lambda)^{\frac{1}{\lambda}}, \quad w = \left( \frac{b - \mu}{\mu - \sigma^2} \right)^{\frac{1}{\lambda}} \frac{(k\lambda)^{\frac{1}{\lambda}}}{\mu - \sigma^2}. \]

**Remark 5.1.** Note that the optimal consumption rate calls for the investor to consume a constant fraction of wealth at each moment, and optimal fraction of investment on risky asset is independent of total wealth. These conclusions are similar to that in the case of randomness [2].

### 6 Conclusion

Based on the concept of Liu process, we studied a fuzzy optimal control problem: optimizing the expected value of an objective function subject to a fuzzy differential equation. By using the Bellman’s principle of optimality in dynamic programming, we presented the principle of optimality and a fundamental result called equation of optimality for fuzzy optimal control. As application of the equation of optimality, we solved a portfolio selection problem.

### References


Appendix

Let us give a formula for computing the expected value of $a\xi + b\xi^2$ if $\xi$ is a fuzzy variable.

**Theorem 6.1.** Let $\xi$ be a fuzzy variable with an even and integrable membership function $\mu(x)$ which is decreasing on $[0, +\infty)$ and $\mu(0) = 1$. Then

$$E[a\xi + b\xi^2] = \frac{1}{2} \int_0^{+\infty} \mu(x)(a + 2bx)dx - \frac{1}{2} \int_0^{-\infty} \mu(x)(a - 2bx)dx$$  \hspace{1cm} (6.1)

if $a \geq 0, b > 0$, or $a \leq 0, b < 0$, and

$$E[a\xi + b\xi^2] = \frac{1}{2} \int_0^{x_0} \mu(x)(a + 2bx)dx - \frac{1}{2} \int_0^{-x_0} \mu(x)(a - 2bx)dx$$  \hspace{1cm} (6.2)

if $a \geq 0, b < 0$, or $a \leq 0, b > 0$, where $x_0 = |a|/(2|b|)$.

**Proof:** (1) If $b > 0$, then $a\xi + b\xi^2 \geq -\frac{a^2}{4b} \equiv y_0$. For any $y \geq y_0$, let

$$x_1 = -a + \sqrt{a^2 + 4by} \quad \text{and} \quad x_2 = -a - \sqrt{a^2 + 4by}.\quad \text{Then}$$

$$\{a\xi + b\xi^2 = y\} = \{\xi = x_1\} \cup \{\xi = x_2\}.$$  

Thus, the membership function of fuzzy variable $a\xi + b\xi^2$ is that if $y < y_0$, $\mu_{a\xi + b\xi^2}(y) = 0$, if $y \geq y_0$,

$$\mu_{a\xi + b\xi^2}(y) = 2\mathbb{C}r\{a\xi + b\xi^2 = y\} \land 1$$

$$= 2(\mathbb{C}r\{\xi = x_1\} \lor \mathbb{C}r\{\xi = x_2\}) \land 1$$

$$= (2\mathbb{C}r\{\xi = x_1\} \land 1) \lor (2\mathbb{C}r\{\xi = x_2\} \land 1)$$

$$= \mu(x_1) \lor \mu(x_2).$$
If $a \geq 0$, then $\mu(x_2) \leq \mu(x_1)$. So, $\mu_\alpha + \beta \xi^2(y) = \mu(x_1)$ for $y \geq y_0$. Thus

$$E[\alpha \xi + b \xi^2] = \int_0^{+\infty} \operatorname{Cr}(\alpha \xi + b \xi^2 \geq y) \, dy - \int_{y_0}^0 \operatorname{Cr}(\alpha \xi + b \xi^2 \leq y) \, dy$$

$$= \int_0^{+\infty} \frac{1}{2} \mu(x_1) \, dy - \int_{y_0}^0 \frac{1}{2} \mu(x_1) \, dy \quad \text{(let } x_1 = x, \text{ then } ax + bx^2 = y)$$

$$= \frac{1}{2} \int_0^{+\infty} \mu(x)(a + 2bx) \, dx - \frac{1}{2} \int_{x_0}^x \mu(x)(a + 2bx) \, dx$$

$$= \frac{1}{2} \int_0^{+\infty} \mu(x)(a + 2bx) \, dx - \frac{1}{2} \int_{x_0}^x \mu(x)(a - 2bx) \, dx.$$ 

where $x_0 = a/(2b)$. We obtain the formula (6.1). If $a \leq 0$, then $\mu(x_2) \geq \mu(x_1)$. So, $\mu_\alpha + \beta \xi^2(y) = \mu(x_2)$ for $y \geq y_0$. Thus

$$E[\alpha \xi + b \xi^2] = \int_0^{+\infty} \operatorname{Cr}(\alpha \xi + b \xi^2 \geq y) \, dy - \int_{y_0}^0 \operatorname{Cr}(\alpha \xi + b \xi^2 \leq y) \, dy$$

$$= \int_0^{+\infty} \frac{1}{2} \mu(x_2) \, dy - \int_{y_0}^0 \frac{1}{2} \mu(x_2) \, dy \quad \text{(let } x_2 = x, \text{ then } ax + bx^2 = y)$$

$$= \frac{1}{2} \int_0^{+\infty} \mu(x)(a + 2bx) \, dx - \frac{1}{2} \int_{x_0}^x \mu(x)(a + 2bx) \, dx$$

$$= \frac{1}{2} \int_0^{+\infty} \mu(x)(a + 2bx) \, dx - \frac{1}{2} \int_{x_0}^x \mu(x)(a - 2bx) \, dx.$$ 

where $x_0 = -a/(2b)$. The formula (6.2) is proved.

(2) If $b < 0$, then $\alpha \xi + b \xi^2 \leq -\frac{a^2}{2b}$. The formula (6.1) and (6.2) can be proved same as in (1).

**Remark 6.1.** When $b = 0$, define $x_0 = +\infty$. Letting $a = 1$, the formula (6.1) or (6.2) reduces the formula of computing the expected value $E[\xi]$ provided by Definition 2.1. If $a = 0$, the formula (6.1) or (6.2) reduces the formula of computing the expected value $E[b \xi^2]$ as

$$E[b \xi^2] = b \int_0^{+\infty} x \mu(x) \, dx.$$