On Analytic Functions of Complex Liu Process

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Abstract

A complex stochastic process is a fundamental connection between two dimensional stochastic processes. Similarly, a complex fuzzy process is proposed and the complex differential and complex Liu integral of complex Liu process are studied in this paper. Especially, some properties of analytic functions of complex Liu process are proved. Finally, complex differential equations are defined.

Keywords: Brownian motion, Fuzzy process, Stochastic integral

1 Introduction

Stochastic process has been widely applied in many disciplines to deal with dynamic random phenomena. In order to deal with dynamic fuzzy phenomena with time axis, Liu [3] gave a new concept of fuzzy process. Subsequently, Liu [3] presented a special fuzzy process, Liu process, which plays an important role in fuzzy process theory just like Brownian motion in stochastic process. Based on Liu process, the concept and properties of fuzzy differential and fuzzy integral were also studied by Liu. You [11] extended the work of Liu and studied the multi-dimensional liu process. In her paper, multi-dimensional fuzzy differential and integral were discussed.

Stochastic process may be complex. Complex stochastic process is widely used in stochastic signal analysis. It is different from the two dimensional stochastic process since it can reveal the relation of its real and imaginary parts. Similarly, we introduce the complex fuzzy process. Especially, complex Liu process is proposed to describe the spatial situation. It is a fundamental connection between two dimensional Liu process. Analytic function is a particular class of complex functions ant it possesses many better properties. An analytic function of standard complex Liu process have the same differential form as the real situation, however, the derivative is complex. Another extension is complex fuzzy integral. The complex fuzzy integral of a complex fuzzy process with standard complex Liu process is a complex fuzzy variable if the integral exists.

In this paper, some basic definitions and properties of credibility theory are recalled in Section 2. Complex fuzzy process is introduced in Section 3. Section 4 proposed the concept of Complex Liu process. In Section 4, the properties of complex fuzzy differential are studied. Complex fuzzy integral is discussed in Section 5. Finally, a brief conclusion is given.
2 Preliminaries

In this section, some basic definitions and properties of credibility theory are recalled.

Let $\Theta$ be a nonempty set, and $\mathcal{P}$ the power set of $\Theta$. For any $A \in \mathcal{P}$, Liu and Liu [5] presented a credibility measure $\text{Cr}\{A\}$ to describe the chance that fuzzy event $A$ occurs. In 2006, Li and Liu [2] proved that a set function $\text{Cr}$ is a credibility measure if and only if

(i) $\text{Cr}\{\Theta\} = 1$;
(ii) $\text{Cr}\{A\} \leq \text{Cr}\{B\}$, whenever $A \subset B$;
(iii) $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}$;
(iv) $\text{Cr}\{\bigcup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}$ with $\sup_i \text{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}, \text{Cr})$ is called a credibility space.

A fuzzy variable is defined as a function from a credibility space to the set of real numbers. As a extension, Yang [10] defined a complex fuzzy variable whose real and imaginary parts are both fuzzy variables.

3 Complex Fuzzy Process

Let $T$ be an index set and let $(\Theta, \mathcal{P}, \text{Cr})$ be a credibility space. A fuzzy process may be considered as a function from $T \times (\Theta, \mathcal{P}, \text{Cr})$ to the set of real numbers. In essential, a fuzzy process $X(t, \theta)$ is a function of two variables such that $X(t^*, \theta)$ is a fuzzy variable for each $t^*$. The symbol $X_t$ is used to replace $X(t, \theta)$ in the following section. Similarly, complex fuzzy process is defined as follows:

Definition 3.1 Let $T$ be an index set and let $(\Theta, \mathcal{P}, \text{Cr})$ be a credibility space. A complex fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \text{Cr})$ to the set of complex numbers.

Theorem 3.1 A process $X_t$ is a complex fuzzy process if and only if there exist two fuzzy process $X_{1t}$ and $X_{2t}$ such that $X_t = X_{1t} + iX_{2t}$.

Proof: Suppose that $X_t$ is a complex fuzzy process which is defined on $T \times (\Theta, \mathcal{P}, \text{Cr})$. By Definition 3.1, $X(t, \theta)$ is a complex number for every $t \in T$ and $\theta \in \Theta$. Write $X_1(t, \theta) = \text{Re}(X(t, \theta))$ and $X_2(t, \theta) = \text{Im}(X(t, \theta))$. Obviously, both $X_1$ and $X_2$ are functions from $T \times (\Theta, \mathcal{P}, \text{Cr})$ to the set of real numbers. Therefore, $X_1$ and $X_2$ are two fuzzy processes and $X_t = X_{1t} + iX_{2t}$.

Conversely, suppose that $X_{1t}$ and $X_{2t}$ are fuzzy processes defined on $T \times (\Theta, \mathcal{P}, \text{Cr})$. If $X_t = X_{1t} + iX_{2t}$, then $X(t, \theta) = X_{1t}(t, \theta) + iX_{2t}(t, \theta)$ is a complex number for $t \in T$ and $\theta \in \Theta$. By definition 3.1, $X_t$ is a complex fuzzy process.

Definition 3.2 (Liu [3][4]) A fuzzy process $X_t$ is said to have independent increments if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent fuzzy variables for any times $t_0 < t_1 < \cdots < t_k$. A fuzzy process $X_t$ is said to have stationary increments if, for any given $s > 0$, $X_{t+s} - X_t$ are identically distributed fuzzy variables for all $t$.

If the real and imaginary parts of a complex fuzzy process have independent and stationary increments, then the complex fuzzy process is called to have independent and stationary increments.
4 Complex Liu Process


**Definition 4.1 (Liu [3][4])** A fuzzy process \( C_t \) is said to be a Liu process if

(i) \( C_0 = 0 \),

(ii) \( C_t \) has stationary and independent increments,

(iii) every increment \( C_{t+s} - C_s \) is a normally distributed fuzzy variable with expected value \( \mu_t \) and variance \( \sigma_t^2 \) whose membership function is

\[
\mu(x) = 2 \left( 1 + \exp \left( \frac{\pi|x - \mu_t|}{\sqrt{6}\sigma_t} \right) \right)^{-1}, \quad -\infty < x < +\infty.
\]

The Liu process is said to be standard if \( \mu = 0 \) and \( \sigma = 1 \). Like a complex fuzzy process, we can give the definition of complex Liu process as follows.

**Definition 4.2** If \( C_{1t} \) and \( C_{2t} \) are independent Liu processes, then \( C_t = C_{1t} + iC_{2t} \) is called a complex Liu process. Here, \( i = \sqrt{-1} \) is an imaginary number.

Especially, a complex Liu process \( C_t = C_{1t} + iC_{2t} \) is said to be standard if \( C_{1t} \) and \( C_{2t} \) are both standard Liu processes.

5 Complex Fuzzy Differential

In this part, the chain rules for complex fuzzy differential will be discussed. Let \( C_t \) be a standard complex Liu process, and let \( dt \) be an infinitesimal time interval. Then \( dC_t = C_{t+dt} - C_t \) is a complex fuzzy process, where \( \text{Re}(dC_t) \) and \( \text{Im}(dC_t) \) are normally distributed fuzzy variables. This complex fuzzy process may be considered as white noise in spatial situation. To compute the complex fuzzy differential based on complex Liu process, a chain rule for differentiation in fuzzy process was given as follows:

**Theorem 5.1 (Liu Formula) (Liu [3])** Let \( C_t \) be a standard Liu process, and let \( g(t, x) \) be a continuously differentiable function. If fuzzy process \( X_t \) is given by \( dX_t = udt + v dC_t \), where \( u \) and \( v \) are absolutely integrable fuzzy processes. Define \( Y_t = g(t, C_t) \). Then we have the following chain rule

\[
dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial c}(t, X_t)dX_t.
\]

Assume that \( h \) is a function of two variables. Then \( h(t, z) \) is a complex function, where \( z = x + iy \).

**Theorem 5.2 (You [11])** Let \( h(t, x_1, x_2, \cdots, x_n) \) be a multivariate continuously differentiable function. If \( n \)-dimensional fuzzy process \( X_t \) is given by \( dX_t = udt + v dC_t \), and \( Y_t = h(t, x_1, x_2, \cdots, x_n) \). Then

\[
dY_t = \frac{\partial h}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(t, X_t)X_{it}dX_{it}.
\]
Theorem 5.3 Let $C_{1t}$ and $C_{2t}$ be two standard Liu processes, and let $h(t, z)$ be a differential complex function. If complex fuzzy process $X_t$ is given by $dX_t = dX_{1t} + idX_{2t}$, where $dX_{1t} = u_1 dt + v_{11} dC_{1t} + v_{12} dC_{2t}$ and $dX_{2t} = u_2 dt + v_{21} dC_{1t} + v_{22} dC_{2t}$, $u_1, u_2, v_{11}, v_{12}, v_{21}$ and $v_{22}$ are absolutely integrable real fuzzy processes. Define $Z_t = h(t, X_t)$, that is, $Z_t = u(t, X_{1t}, X_{2t}) + iv(t, X_{1t}, X_{2t})$. Then

$$dZ_t = \left(\frac{\partial u}{\partial t}(t, X_t)dt + \frac{\partial u}{\partial x}(t, X_t)dX_{1t} + \frac{\partial u}{\partial y}(t, X_t)dX_{2t}\right) + i\left(\frac{\partial v}{\partial t}(t, X_t)dt + \frac{\partial v}{\partial x}(t, X_t)dX_{1t} + \frac{\partial v}{\partial y}(t, X_t)dX_{2t}\right).$$

Equation (2) is right. The proof is complete.

Example 5.1 If $u_1 = u_2 = v_{12} = v_{21} = 0$ and $v_{11} = v_{22} = 1$ in the above theorem, then $X_t = C_{1t} + iC_{2t}$ is a standard complex Liu process. Let $h(t, z)$ be a differential complex function. If $Z_t = h(t, X_t)$, that is, $Z_t = u(t, C_{1t}, C_{2t}) + iv(t, C_{1t}, C_{2t})$. Then

$$dZ_t = \left(\frac{\partial u}{\partial t}(t, C_t)dt + \frac{\partial u}{\partial x}(t, C_t)dC_{1t} + \frac{\partial u}{\partial y}(t, C_t)dC_{2t}\right) + i\left(\frac{\partial v}{\partial t}(t, C_t)dt + \frac{\partial v}{\partial x}(t, C_t)dC_{1t} + \frac{\partial v}{\partial y}(t, C_t)dC_{2t}\right).$$

Example 5.2 Let $h(t, z) = z^2$. Thus, we have $Z_t = h(t, C_t) = C_t^2 = C_{1t}^2 - C_{2t}^2 + i \cdot (2C_{1t}C_{2t})$. Using Equation (2), we can get

$$dZ_t = 2(C_{1t}dC_{1t} - C_{2t}dC_{2t}) + i \cdot 2(C_{1t}dC_{2t} + C_{2t}dC_{1t}).$$

Especially, if $f$ is a continuously differential function and $C_t$ is a standard Brownian motion, then we have $dZ_t = f'(C_t) dC_t$ where $f'$ is the derivative of $f$. In complex case, if the function is an analytic function, we can get the same form as follows:

Theorem 5.4 Let be given the analytic function $F : C \rightarrow C, z \rightarrow F(z)$. That is, $F(z) = u(z) + iv(z)$ satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \quad z = x + iy. \quad (4)$$

Define $Z_t = F(C_t)$. Then we have the following expression

$$dZ_t = F'(C_t) dC_t \quad (5)$$

where $F'$ is the (complex) derivative of $F$. 

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Proof: According to Theorem 5.3, we have
\[
\begin{align*}
dZ_t &= \left( \frac{\partial u}{\partial x}(C_{1t}, C_{2t})dC_{1t} + \frac{\partial u}{\partial y}(C_{1t}, C_{2t})dC_{2t} \right) + i \left( \frac{\partial v}{\partial x}(C_{1t}, C_{2t})dC_{1t} + \frac{\partial v}{\partial y}(C_{1t}, C_{2t})dC_{2t} \right) \\
&= \left( \frac{\partial u}{\partial x}(C_{1t}, C_{2t}) + i \frac{\partial v}{\partial y}(C_{1t}, C_{2t}) \right) dC_{1t} + \left( \frac{\partial u}{\partial y}(C_{1t}, C_{2t}) + i \frac{\partial v}{\partial x}(C_{1t}, C_{2t}) \right) dC_{2t} \\
&= \left[ \frac{\partial u}{\partial x}(C_{1t}, C_{2t}) - i \frac{\partial v}{\partial y}(C_{1t}, C_{2t}) \right] dC_{1t} + i \left[ \frac{\partial u}{\partial y}(C_{1t}, C_{2t}) - i \frac{\partial v}{\partial x}(C_{1t}, C_{2t}) \right] dC_{2t} \\
&= F'(t) dC_{1t} + i F''(t) dC_{2t} \\
&= F'(t) dC_t.
\end{align*}
\]
The proof is complete.

6 Complex Liu Integral

In Liu [3][4], the concept of fuzzy integral of a fuzzy process with respect to the standard Brownian motion was defined. In You [11], this type of fuzzy integral was called Liu integral. In this paper, we continue to adopt the name and state the definition as follows.

Let \( X_t \) be a fuzzy process and let \( C_t \) be a standard Liu process. For any partition of closed interval \([a, b]\) with \( a = t_1 < t_2 < \cdots < t_{k+1} = b \), the mesh is written as \( \Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i| \). Then the fuzzy integral of \( X_t \) with respect to \( C_t \) is
\[
\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})
\]
provided that the limit exists and is a fuzzy variable.

Just like from the real calculus to complex calculus, we can define a complex Liu integral as a natural extension of ordinary Liu integral (6) in the following way.

**Definition 6.1** Let \( X_t = X_{1t} + iX_{2t} \) be a complex fuzzy process and let \( C_t = C_{1t} + iC_{2t} \) be a standard complex Liu process. Then the complex fuzzy integral of \( X_t \) with respect to \( C_t \) is defined by
\[
\int_a^b X_t dC_t = \int_a^b (X_{1t}dC_{1t} - X_{2t}dC_{2t}) + i \int_a^b (X_{1t}dC_{2t} + X_{2t}dC_{1t})
\]
provided that the limit exists and is a complex fuzzy variable.

Note that every integral in Equation (7) is ordinary Liu integral defined by (6).

**Example 6.1** Let \( C_t = C_{1t} + iC_{2t} \) be the standard complex Liu process. Then we have
\[
\begin{align*}
\int_0^t dC_s &= \int_0^t dC_{1s} + i \int_0^t dC_{2s} = C_{1t} + iC_{2t} = C_t, \\
\int_0^t dC_s &= t(C_{1t} + iC_{2t}) - \int_0^t (C_{1s} + iC_{2s})ds = tC_t - \int_0^t C_sds, \\
\int_0^t C_s dC_s &= \int_0^t (C_{1s}dC_{1s} - C_{2s}dC_{2s}) + i \int_0^t (C_{2s}dC_{1s} + C_{1s}dC_{2s}) \\
&= \frac{1}{2}(C_{1t}^2 - C_{2t}^2) - C_{1t}C_{2t} = \frac{1}{2}C_t^2.
\end{align*}
\]
**Theorem 6.1** *Integration by Parts* (Liu [3][4]) Suppose that \( C_t \) is a standard Liu process and \( F(t) \) is an absolutely continuous function. Then
\[
\int_0^t F(s)dC_s = F(t)C_t - \int_0^t C_s dF(s). \tag{8}
\]
Similarly, in complex cases, we have the integration by parts which has the same form as the real situation.

**Theorem 6.2** Suppose that \( C_t = C_{1t} + iC_{2t} \) is a standard complex Liu process and \( F_1(t) \) and \( F_2(t) \) are two absolutely continuous function. Define \( F(t) = F_1(t) + iF_2(t) \). Then
\[
\int_0^t F(s)dC_s = F(t)C_t - \int_0^t C_s dF(s). \tag{9}
\]

**Proof:** It follows from Definition 7 that
\[
\int_0^t F(s)dC_s = \int_0^t F_1(s)dC_{1s} - \int_0^t F_2(s)dC_{2s} + i \int_0^t F_1(s)dC_{2s} + i \int_0^t F_2(s)dC_{1s}
\]
\[
= \left( F_1(t)C_{1t} - \int_0^t C_{1s}dF_1(s) \right) - \left( F_2(t)C_{2t} - \int_0^t C_{2s}dF_2(s) \right) + i \left( F_1(t)C_{2t} - \int_0^t C_{2s}dF_1(s) \right) + i \left( F_2(t)C_{1t} - \int_0^t C_{1s}dF_2(s) \right)
\]
\[
= F_1(t)C_t + iF_2(t)C_t - \int_0^t (C_{1s} + iC_{2s})dF_1(s) - i \int_0^t (C_{1s} + iC_{2s})dF_2(s)
\]
\[
= F(t)C_t - \int_0^t C_s dF(s).
\]
The theorem is complete.

**7 Complex Fuzzy Differential Equation**

Suppose \( C_t = C_{1t} + iC_{2t} \) is a standard complex Liu process, and \( f \) and \( g \) are some given complex functions. We would like to find an unknown complex fuzzy process \( X_t \) such that
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{10}
\]
which is called a complex fuzzy differential equation.

**Example 7.1** Let \( C_t \) be a standard complex Liu process. Then the complex fuzzy differential equation
\[
dX_t = adt + bdC_t
\]
has a solution
\[
X_t = at + bC_t.
\]

**Example 7.2** Let \( C_t \) be a standard complex Liu process. Then the complex fuzzy differential equation
\[
dX_t = X_t dC_t
\]
has a solution
\[
X_t = \xi \cdot \exp(C_t)
\]
where \( \xi \) is a fuzzy variable.
8 Conclusions

In this paper, we first extended the concept of fuzzy process to the complex fuzzy process. Then the complex Liu process was proceeded. Finally complex fuzzy differential and complex Liu integral were studied.

References