Optimistic value based optimal control for uncertain linear singular systems and application to a dynamic input-output model

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Abstract

In this paper, optimal control problems for uncertain discrete-time singular systems and uncertain continuous-time singular systems are considered under optimistic value criterion. The above singular systems are assumed to be regular and impulse-free, and optimistic value method is employed to optimize uncertain objective functions. Firstly, based on Bellman’s principle of optimality, a recurrence equation is presented for settling optimal control problems subject to uncertain discrete-time singular systems. Then, by applying the principle of optimality and uncertainty theory, an equation of optimality for an optimal control model subject to an uncertain continuous-time singular system is derived. The optimal control problem can be settled through solving the equation of optimality. Two numerical examples and a dynamic input-output model are given to show the effectiveness of the results obtained.

1. Introduction

The study of optimal control greatly attracted the attention of many mathematicians for the necessity of strict expression form in optimal control theory. In past several decades, optimal control theory has achieved plenty of developments not only in theory but also in applications such as economics, production engineering and management. An optimal control problem for a given system is to choose the best decision such that an objective function is optimized. Kirk [1] investigated optimal control theory and obtained lots of meaningful results. Bryson, Ho and Siouris [2] studied optimal control problems in various aspects including optimization, estimation and control. Naidu [3] considered singular perturbations and time scales in control theory and its applications. Then Das and Mahanta [4] proposed a chattering free optimal second order sliding mode control to stabilize nonlinear systems.

From 1970s lots of researchers began to investigate stochastic optimal control problems, such as in Merton [5] for finance. In recent decades, the study of stochastic optimal control has been made considerable developments, for example, Fleming and Rishel [6], Harrison [7], Karatzas [8] and Cairns [9] studied optimal control problems of Brownian motion or stochastic differential equations and applications in finance and engineering. One of the main methods to handle optimal control problems is Bellman’s dynamic programming. The utilization of dynamic programming in optimization over Itô’s process was discussed in Dixit and Pindyck [10].

The complexity of the real world makes the events we face uncertain in various forms. Plenty of human uncertainty does not behave like randomness, such as the price of a new stock, oil filed reserves and bridge strength. In order to deal with these phenomena, an uncertainty theory was established in [11] and refined by Liu [12] as a branch of axiomatic mathematics for modeling human uncertainty. Furthermore, Liu [13] introduced uncertain process and canonical process as counterparts of stochastic process and Wiener process, respectively. Then the concept of uncertain differential equation was presented in [13]. Based on uncertain differential equation, Zhu [16] studied the excepted value model of uncertain optimal control problem. Employing Bellman’s principle of optimality, an equation of optimality was derived as a counterpart of HJB equation, and then he solved an uncertain portfolio selection problem. Moreover, by the equation of optimality, Yao and Qin [17] proposed an uncertain linear quadratic control model. Then Xu and Zhu [18], Kang and Zhu [19] studied uncertain bang-bang control problems for continuous-time system and multi-stage system, respectively. Gao and Yu [20] employed expected value criterion and optimistic value criterion to investigate a finite extensive game. Sheng and Zhu [21] discussed the optimistic value model of uncertain optimal control.
As we all know, expected value is the weighted average of uncertain variables in the sense of uncertain measure. However, sometimes we need to take other characters of uncertain variables into account. For example, if individual incomes in a city appear two-level differentiation phenomenon, and the difference between higher income and lower income is too large, in this case average income may not be discussed alone. Then optimistic value of individual incomes may be considered. We may study the problem such as that 90% of the individual incomes achieve how many dollars above.

Singular systems, also known as descriptor systems, implicit systems and generalized state-space systems, are described by differential-algebraic equations (DAEs). Singular systems [22,23] have been extensively studied during the past decades due to the fact that they are able to describe plenty of natural phenomena in physical systems such as economics, demography, microelectronic circuits and so on [24–26]. Liu, Lin and Chen [27] studied the admissibility problem for a class of linear singular systems with time-varying delays. However, because of the difficulty appearing in analysis, few results are concerned with the optimal control of singular systems.

One of the contributions of this paper is to transform an uncertain linear singular system to two uncertain sub-systems which are simpler for investigation. This result inspired by the work in [30] is presented in Section 3. Getting a recurrence equation for an optimal control problem subject to an uncertain discrete-time singular system is the second contribution. Compared with the work in [19], this recurrence equation is able to settle more discrete-time optimal control problems and has more applications in practice. The last one is an equation of optimality derived in Section 5 which turns an optimal control problem for an uncertain continuous-time singular system to become an easier question. This work based on the results in [16,21] enlarges the scope of uncertain optimal control.

Inspired by the previous works, we will investigate optimal control problems for uncertain discrete-time singular systems and uncertain continuous-time singular systems under optimistic value criterion. The organization of this paper is as follows: in the next section, we will review some concepts such as uncertain measure, uncertainty space, optimistic value of uncertain variable, uncertain process, canonical process, and uncertain differential equation. In Section 3, we will introduce a kind of equivalent restrict form of an uncertain discrete-time singular system provided that it is regular and impulse-free. Based on this equivalent restrict form and Bellman’s principle of optimality, a recurrence equation will be presented for setting an optimal control problem subject to an uncertain discrete-time singular system. In Section 4, we will obtain bang-bang optimal controls for two different types of uncertain discrete-time singular systems. In Section 5, we will consider an optimal control problem for an uncertain continuous-time singular system and get an equation of optimality for solving such problem. Two numerical examples and a dynamic input-output model are provided to illustrate the results obtained in Sections 4 and 5.

2. Preliminary

In convenience, we give some useful concepts at first. Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \in \mathcal{L} \) is called an event.

**Definition 2.1 ([11])**. A set function \( \mathcal{M} \) defined on the \( \sigma \)-algebra \( \mathcal{L} \) is called an uncertain measure if it satisfies the following four axioms:

Axiom 1. (Normality) \( \mathcal{M}(\Gamma) = 1 \);
Axiom 2. (Self-Duality) \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any event \( \Lambda \);
Axiom 3. (Countable Subadditivity) \( \mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i) \) for any events \( \Lambda_1, \Lambda_2, \ldots \).

**Definition 2.2 ([11]).** Let \( \Gamma \) be a nonempty set, \( \mathcal{L} \) the \( \sigma \)-algebra over \( \Gamma \), and \( \mathcal{M} \) an uncertain measure. Then the triplet \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is said to be an uncertainty space. An uncertain variable is a measurable function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers, i.e., for any Borel set of real numbers, the set \( \{ \xi \in B \} = \{ \tau \in \Gamma | \xi(\tau) \in B \} \) is an event.

**Definition 2.3 ([11]).** The uncertainty distribution \( \Phi : \mathbb{N} \rightarrow [0, 1] \) of an uncertain variable \( \xi \) is defined by \( \Phi(\xi) = \mathcal{M}\{\xi \leq x\} \).

**Definition 2.4 ([14]).** The uncertain variables \( \xi_1, \xi_2, \ldots, \xi_m \) are said to be independent if

\[
\mathcal{M}\left(\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right) = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}
\]

for any Borel sets \( B_1, B_2, \ldots, B_m \) of real numbers.

Independence of uncertain variables does not mean they are unrelated but that no “new” information about any uncertain variables can be obtained through observations of the others. For instance, uncertain variables are independent if they are defined on different uncertainty spaces.

**Definition 2.5 ([11]).** Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by

\[
E[\xi] = \int_{0}^{+\infty} \mathcal{M}(\xi \geq r)dr - \int_{-\infty}^{0} \mathcal{M}(\xi \leq r)dr
\]

provided that at least one of the two integrals is finite.

**Definition 2.6** Let \( \xi \) be an uncertain variable with finite expected value \( e \). Then the variance of \( \xi \) is defined by \( \mathbb{V}[\xi] = E[(\xi - e)^2] \).

**Definition 2.7 ([11]).** Let \( \xi \) be an uncertain variable, and \( a \in (0, 1) \). Then \( \xi_{\sup}(a) = \sup\{\mathcal{M}(\xi \geq r) \geq a\} \) is called the \( a \)-optimistic value of \( \xi \); and \( \xi_{\inf}(a) = \inf\{\mathcal{M}(\xi \leq r) \geq a\} \) is called the \( a \)-pessimistic value of \( \xi \).

“sup \( A \)” means the supremum of a given set \( A \), and “inf” the infimum of set \( A \).

**Theorem 2.1 ([11,12]).** Assume that \( \xi \) is an uncertain variable. Then we have

(a) if \( \lambda \geq 0 \), then \( \lambda \xi_{\sup}(a) = \lambda \xi_{\sup}(a) \), and \( \lambda \xi_{\inf}(a) = \lambda \xi_{\inf}(a) \);
(b) if \( \lambda < 0 \), then \( \lambda \xi_{\sup}(a) = \lambda \xi_{\inf}(a) \), and \( \lambda \xi_{\inf}(a) \); and \( \lambda \xi_{\inf}(a) \).
(c) \( \xi + \eta_{\sup}(a) = \xi_{\sup}(a) + \eta_{\sup}(a) \xi \) if \( \xi \) and \( \eta \) are independent.

Based on the uncertainty space, Liu introduced the concepts of uncertain process, canonical process, uncertain differential equation, and etc.

**Definition 2.8 ([14]).** Let \( T \) be an index set and let \( (\Gamma, \mathcal{L}, \mathcal{M}) \) be an uncertainty space. An uncertain process is a measurable function from \( T \times (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers, i.e., for each \( t \in T \)
and any Borel set $B$ of real numbers, the set
\[ \{ X_t \in B \} = \{ \gamma \in \Gamma | X_t(\gamma) \in B \} \]
is an event.

**Definition 2.9 ([14])**. An uncertain process $X_t$ is said to have independent increments if
\[ X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_l} - X_{t_{l-1}} \]
are independent uncertain variables for any times $t_0 < t_1 < \cdots < t_l$. An uncertain process $X_t$ is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

**Definition 2.10 ([14])**. An uncertain process $C_t$ is said to be canonical process if

(a) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(b) $C_t$ has stationary and independent increments,
(c) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is.

\[ \Phi(x) = \left(1 + \exp \left( -\frac{\pi}{\sqrt{3}} x \right) \right)^{-1}, \ x \in \mathfrak{N}. \]

Normal uncertain variable is an important type of uncertain variables. Its uncertainty distribution $\Phi(x)$ owns very special property as the following:
\[ 0 < \Phi(x) < 1, \forall x \in \mathbb{R}, \]
and
\[ \lim_{x \to -\infty} \Phi(x) = 0, \lim_{x \to +\infty} \Phi(x) = 1. \]

Liu [13] gave a proof of the existence of a canonical process. Based on canonical process, a new kind of uncertain differential was introduced by Liu [13]. The following concept of uncertain differential equation is important in theory and applications, and essential in the study of this paper.

**Definition 2.11 ([13])**. Suppose $C_t$ is a canonical process, and $g_t$ and $g_t$ are some given functions. Then
\[ dX_t = g_t(X_t, t)dt + g_t(X_t, t)dC_t \]
is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies this equation identically in $t$.


**Theorem 2.2 ([21])**. Let $\xi$ be a normal uncertain variable with expected value 0 and variance $\sigma^2 (\sigma > 0)$, whose uncertainty distribution is

\[ \Phi(x) = \left(1 + \exp \left( -\frac{\pi}{\sqrt{3}} x \right) \right)^{-1}, \ x \in \mathfrak{N}. \]

Then for any real number $a$, and any $\epsilon > 0$ small enough,
\[ a^2 + b^2 \text{sup}(\alpha) \leq \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \text{var} + \left( \frac{\sqrt{3}}{\pi} \ln \frac{2 - \epsilon}{\epsilon} \right)^2 \text{var}, \]
if $b > 0$; and
\[ a^2 + b^2 \text{sup}(\alpha) \geq \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \text{var} + \left( \frac{\sqrt{3}}{\pi} \ln \frac{2 - \epsilon}{\epsilon} \right)^2 \text{var}, \]
if $b < 0$; and also
\[ a^2 + b^2 \text{sup}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \text{var} + \left( \frac{\sqrt{3}}{\pi} \ln \frac{2 - \epsilon}{\epsilon} \right)^2 \text{var}, \]
if $b = 0$.

3. Optimal control for uncertain discrete-time singular system

3.1. Uncertain discrete-time singular system

In the first subsection, we introduce the following uncertain discrete-time singular system
\[ \begin{cases} \text{Ex}(j+1) = A\text{x}(j) + Bu(j) + D\xi_j, \\ j = 0, 1, 2, \ldots, N - 1, \end{cases} \]
where $x(j) \in \mathbb{R}^n$ is the state vector of the system at stage $j$, $u(j) \in \mathbb{U} \subset \mathbb{R}^m$ is the input variable at stage $j$ with the constraint domain $\mathbb{U}$, and $A, E, B \in \mathbb{R}^{n \times n}$ are known coefficient matrices associated with $x(j)$ and $u(j)$, respectively. $D \in \mathbb{R}^q$ is a given vector. The $E$ is a known (singular) matrix with $\text{rank}(E) = q \leq n$, and $\text{det}(\text{det}(E - A)) = r$ where $z$ is a complex variable. $\Delta_z, \Delta_{z^2}, \ldots, \Delta_{z^{N-1}}$ are given independent uncertain variables, and $\xi_0 = 0$.

**Definition 3.1 ([29])**. \( (E, A) \) is said to be regular if $\text{det}(E - A)$ is not identically zero; \( (E, A) \) is said to be impulse-free if $\text{deg}(\text{det}(E - A)) = \text{rank}(E)$.

**Lemma 3.1.** If $\ (E, A) \$ is regular and impulse-free, then system (1) is equivalent to system (2):
\[ \begin{cases} x(j + 1) = A_1 x(j) + B_1 u(j) + D_1 \xi_j, \\ 0 = x(j) + B_2 u(j) + D_2 \xi_j, \end{cases} \]
\[ j = 0, 1, 2, \ldots, N - 1, \]
\[ \text{where } x(j) = \begin{bmatrix} x_1(j) \\ x_2(j) \end{bmatrix}, \quad x_1(j) \in \mathbb{R}^r, \quad x_2(j) \in \mathbb{R}^{n-r}, \quad \text{and } A_1 \in \mathbb{R}^{r \times r}, \quad B_1 \in \mathbb{R}^{r \times m}, \quad B_2 \in \mathbb{R}^{(n-r) \times m}, \quad D_1 \in \mathbb{R}^{r}, \quad D_2 \in \mathbb{R}^{r-r}. \]

**Proof.** Because $\ (E, A) \$ is impulse-free, we get $\text{rank}(E) = \text{deg}(\text{det}(E - A)) = r$
according to **Definition 3.1**. Denote $\text{det}(E - A) = \sum_{i=0}^{r} d_i z^i$.

Obviously there exists a complex number $z_0 \neq 0$ such that $\text{det}(z_0 E - A) \neq 0$. Then denote $\hat{E} = (z_0 E - A)^{-1} E$, $\hat{A} = (z_0 E - A)^{-1} A$.

From the following equality:
\[ \text{det}(zE - A) = \text{det}(z_0 E - A)^{-1} E = \text{det}(z_0 E - A)^{-1} \text{det}(z(z_0 E - A) - E) = \text{det}(z_0 E - A)^{-1} \text{det}(z(z_0 E - A) - E z A) = \text{det}(z_0 E - A)^{-1} \text{det}(\frac{z}{z_0 E - A} - E) = \text{det}(z_0 E - A)^{-1} \sum_{i=0}^{r} d_i \left(\frac{z}{z_0 E - A} - E\right)^i = \text{det}(z_0 E - A)^{-1} z^{-i} \sum_{i=0}^{r} d_i z^{i} (z_0 E - A)^{i}, \]

we know that the total multiplicity of non-zero eigenvalues concerning $E$ is $r$, and the multiplicity of zero eigenvalue is $n - r$. Hence there exists a nonsingular matrix $T$ such that
\[ T^{-1} ET = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}. \]
where $E_j \in \mathbb{R}^{r \times r}$ is nonsingular and made up of several Jordan blocks. For
\[
\bar{z}_0 E - \bar{A} = z_0(z_0 E - A)^{-1}E - (z_0 E - A)^{-1}A
\]
we obtain $\bar{A} = z_0 \bar{E} - l$. Therefore
\[
T^{-1} \bar{A} T = T^{-1}(z_0 E - l) T
\]
\[
= \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix} (z_0 E - l_1) \begin{bmatrix} I \\ 0 \end{bmatrix}
\]
where $l_1, l_2 \in \mathbb{R}^{r \times r}$ and $l_2 \in \mathbb{R}^{r \times (n-1)}$ are both identity matrices. Now let
\[
P = \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix}, Q = T^{-1}(z_0 E - A)^{-1}, \quad \text{and then}
\]
\[
PEQ = \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix}^{-1} z_0 E - A \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix}
\]
\[
= \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix}
\]
\[
PAQ = \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix}^{-1} z_0 E - A \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix}
\]
\[
= \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & -l_2 \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 \\ A_2 \end{bmatrix},
\]
where $A_2 = E_1(z_0 E_1 - l_2)$. Let
\[
\begin{bmatrix} x_1(j) \\ x_2(j) \end{bmatrix} = Q^{-1} x(j),
\]
where $x_1(j), x_2(j) \in \mathbb{R}^{r \times t}$, and denote
\[
P B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad PD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.
\]
Then pre-multiplying both sides of the equation $Ex(j + 1) = Ax(j) + Bu(j) + Dz_j$ by matrix $P$, through deduction and calculation, it is not hard to verify the equivalence between system $\{1\}$ and system $\{2\}$. This completes the proof. □

**Remark 3.1.** Inspired by the investigation concerning linear singular systems in [30], we present system (2) called a restrained equal system for uncertain singular system (1) in Lemma 3.1. Such restrained equal system is composed of two sub-systems which are described as uncertain difference equation and uncertain algebraic equation. It may provide a solid foundation to settle the main problems in this paper, and therefore Lemma 3.1 will play an important role from Sections 3 to 5.

**Remark 3.2.** In Section 3 and 4, it is always assumed that $(E, A)$ is regular and impulse-free. Under such assumption, we study the corresponding optimal control problem subject to a class of uncertain singular systems in the following subsection.

### 3.2. Optimal control problem of uncertain discrete-time singular system

In this subsection, let us consider the following optimal control problem subject to an uncertain discrete-time singular system:
\[
\left\{ \begin{array}{l}
J(0, x_0) = \sup_{u \in (\mathbb{R}^{n \times 1})^N} \sum_{j=0}^{N-1} f(x(j), u(j), j) \\
\text{subject to} \\
Ex(j + 1) = Ax(j) + Bu(j) + Dz_j,
\end{array} \right. \quad \alpha \in (0, 1), \quad \text{and} \quad \left[ \sum_{j=0}^{N-1} f(x(j), u(j), j) \right]_{\sup} \text{ (3)}
\]
where $\alpha \in (0, 1)$, and $\left[ \sum_{j=0}^{N-1} f(x(j), u(j), j) \right]_{\sup} \text{ (3)}$ stands for the $\alpha$-optimistic value to the uncertain variable $\sum_{j=0}^{N-1} f(x(j), u(j), j)$ according to Definition 2.7. Now we assume the initial state $x(0)$ is an allowed completeness condition [30] which means that $x(0) = -Bz(0)$ is satisfied, where $x(0) = Q^{-1} x(0)$. By Lemma 3.1, problem (3) is equivalent to problem (4):
\[
\left\{ \begin{array}{l}
J(0, x_0, x_{2,N}) = \sup_{u \in (\mathbb{R}^{n \times 1})^N} \sum_{j=0}^{N-1} g_j(x(j), \zeta_j, u(j), j) + g_{\bar{N}}(x_N, x_{2,N}, u_N, N) \\
\text{subject to} \\
x_1(j + 1) = A_1 x_1(j) + B_2 u(j) + D_1 z_j, \\
0 = x_2(j) + (D_2 z_j), \\
\text{for } j = 0, 1, 2, \ldots, N - 1, \\
x(0) = x_0, x_N = x_{2,N},
\end{array} \right. \quad \text{where} \quad \alpha \in (0, 1), \quad \zeta_j = u(0), \quad u_j = (u^0, u^1, \ldots, u^N), \quad \text{and} \quad Q = (Q_1, Q_2), \quad Q_1 \in \mathbb{R}^{r \times r},
\]

\[
\left\{ \begin{array}{l}
J(k, x_k, x_{2,N}) = \sup_{u \in (\mathbb{R}^{n \times 1})^N} \sum_{j=k+1}^{N-1} g_j(x(j), \zeta_j, u(k), k) + g_{\bar{N}}(x_N, x_{2,N}, u(N), N) \\
\text{subject to} \\
x_1(j + 1) = A_1 x_1(j) + B_2 u(j) + D_1 z_j, \\
0 = x_2(j) + (D_2 z_j), \\
\text{for } j = k, k + 1, \ldots, N - 1, \\
x(0) = x_0, x_N = x_{2,N}.
\end{array} \right. \quad \text{where} \quad Q = (Q_1, Q_2), \quad Q_1 \in \mathbb{R}^{r \times r},
\]

\[
\left\{ \begin{array}{l}
J(k, x_k, x_{2,N}) = \sup_{u \in (\mathbb{R}^{n \times 1})^N} \sum_{j=k+1}^{N-1} g_j(x(j), \zeta_j, u(k), k) + g_{\bar{N}}(x_N, x_{2,N}, u(N), N) \\
\text{subject to} \\
x_1(j + 1) = A_1 x_1(j) + B_2 u(j) + D_1 z_j, \\
0 = x_2(j) + (D_2 z_j), \\
\text{for } j = k, k + 1, \ldots, N - 1, \\
x(0) = x_0, x_N = x_{2,N}.
\end{array} \right. \quad \text{where} \quad Q = (Q_1, Q_2), Q_1 \in \mathbb{R}^{r \times r},
\]

\[
J(k, x_k, x_{2,N}) = \sup_{u \in (\mathbb{R}^{n \times 1})^N} \sum_{j=k+1}^{N-1} g_j(x(j), \zeta_j, u(k), k) + g_{\bar{N}}(x_N, x_{2,N}, u(N), N).
\]
Taking the supremum with respect to $u(j)$, $j = k + 1, k + 2, \ldots, N$

$$j(k, x_{\alpha,k}, x_{\beta,N}) \geq \left[ \sum_{j=k}^{N-1} g_j(x(j), \zeta_j, u(j), j) + g_0(x(N), x_{\beta,N}, u(N), N) \right]_{\sup} \quad (\alpha)$$

in (6), and then $u(k)$ in (6), we get $j(k, x_{\alpha,k}, x_{\beta,N}) \geq \tilde{j}(k, x_{\alpha,k}, x_{\beta,N})$.

On the other hand, for every $u(j)$, $j = k, k + 1, \ldots, N$, we have

$$\sup \left[ \sum_{j=k}^{N-1} g_j(x(j), \zeta_j, u(j), j) + g_0(x(N), x_{\beta,N}, u(N), N) \right] \leq \left[ g_0(x_{\beta,N}, u(k), k) + \sum_{j=k+1}^{N-1} g_j(x(j), \zeta_j, u(j), j) + g_0(x(N), x_{\beta,N}, u(N), N) \right]_{\sup} \quad (\alpha)$$

$$\sup \left[ \max_{u(k)} \left[ \sum_{j=k}^{N-1} g_j(x(j), \zeta_j, u(j), j) + g_0(x(N), x_{\beta,N}, u(N), N) \right] \right]_{\sup} \geq \tilde{j}(k, x_{\alpha,k}, x_{\beta,N})$$

Hence, $j(k, x_{\alpha,k}, x_{\beta,N}) \leq \tilde{j}(k, x_{\alpha,k}, x_{\beta,N})$, and then $j(k, x_{\alpha,k}, x_{\beta,N}) = \tilde{j}(k, x_{\alpha,k}, x_{\beta,N})$. The theorem is proved.

**Remark 3.3.** When we look for $u^*(k)$, $x_k(k)$ can be treated as a constant quantity, but $x_k(k)$ cannot be regarded constant because $x_k(k)$ is related to $u(k)$ and $\xi_k$. Through the equality $x_k(k) = -B_2u(k) - D_2\xi_k$, we are able to transform $f(x(k), u(k), k)$ into $g_0(x_k(k), \xi_k, u(k), k)$. In a word, $j(k, x_k)$ has nothing to do with $x_k(k)$, so it can be denoted as $j(k, x_{\alpha,k}, x_{\beta,N})$ which is ruled by the state $x_{\alpha,k}$ and the terminal state $x_{\beta,N}$. Such fact is an important difference between the linear singular system and the linear normal system.

**Remark 3.4.** Note that when $E$ is invertible, the uncertain singular system becomes uncertain normal system and the optimal control problem of the uncertain normal system [16] has been tackled in recent years. In this section, the optimal control problem in our investigation is extended from the uncertain normal system to the uncertain singular system. As is well known, the uncertain singular system has more extensive applications than the uncertain normal system in science and industry.

**Theorem 3.1** tells us that to solve problem (4) turns to solve the simpler problems (5) step by step from the last stage to the initial stage in reverse order.

### 4. Bang-bang optimal control for uncertain discrete-time singular system

In this section, we will investigate optimal control problems subject to two kinds of uncertain discrete-time singular systems. Before discussing these problems, we first present the following lemma.

**Lemma 4.1.** Let $\xi$ be a linear uncertain variable $\mathcal{L}(a, b)(b > a)$ with the distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a, \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\ 1, & \text{if } x > b. \end{cases}$$

Then its $\alpha$-optimistic and $\alpha$-pessimistic values are $\xi_{\sup}(\alpha) = (b - a)(1 - \alpha) + a$ and $\xi_{\inf}(\alpha) = (b - a)\alpha + a$, respectively.

**Proof.** Based on Axiom 2 in **Definition 2.1**, for $\alpha \in (0, 1]$, we solve an inequality $M(\xi) \geq r \geq \alpha$ with respect to $r$, settling process is not complex:

$$M(\xi) \geq r \geq \alpha$$

$$1 - \Phi(r) \leq \alpha$$

$$1 - \frac{r - a}{b - a} \leq \alpha$$

$$r \leq (b - a)(1 - \alpha) + a.$$

Then we easily obtain

$$\xi_{\inf}(\alpha) = \sup\{r | M(\xi) \geq \alpha\} = \sup\{r | r \leq (b - a)(1 - \alpha) + a\} = (b - a)(1 - \alpha) + a,$$

and using similar method we get $\xi_{\sup}(\alpha) = (b - a)\alpha + a$. This completes the proof. \hfill \Box

#### 4.1. Bang-bang optimal control problem of uncertain linear singular system

Now let us consider the following optimal control problem with a linear objective function subject to an uncertain linear discrete-time singular system:
\[ J(0, x_0) = \sup_{u \in \mathcal{U}_{ad}} \left\{ \sum_{j=0}^{N-1} \left[ r_j^x X(j) + \beta_j^x u(j) \right] + r_N^x X(N) \right\} \] 

subject to

\[ \begin{align*}
&J(x(j+1)) = A X(j) + B u(j) + D_i^P r_j^x, \\
&0 = x_j(j) + B u(j) + D_i^P r_j^x, \\
&x_0 = x_{0,0}, x_N = x_{N,N},
\end{align*} \]

where \( r_j \in \mathbb{R}, j = 0, 1, \ldots, N - 1 \) are known coefficient vectors, \( \beta_j \in \mathbb{R}, j = 0, 1, \ldots, N - 1 \) are independent linear uncertain variables, especially \( \beta_0 = 0 \), and \( \mathcal{U}_{ad} = [-1, 1]^m \).

Also we assume the initial state \( x(0) \) is an allowed completeness condition. By Lemma 3.1, problem (7) equals to problem (8):

\[ \begin{align*}
J(0, x_0, x_{N,N}) &= \sup_{u \in \mathcal{U}_{ad}} \left\{ \sum_{j=0}^{N-1} \left[ r_j^Q X(j) + \beta_j^Q u(j) - r_j^Q D_j^P (\rho + \beta_j^Q) \right] + r_N^Q X(N) \right\} \\
&\text{subject to} \quad x_j(j + 1) = A X(j) + B u(j) + D_i^Q r_j^Q, \\
&0 = x_j(j) + B u(j) + D_i^Q r_j^Q, \\
&x_0 = x_{0,0}, x_N = x_{N,N},
\end{align*} \]

where \( Q = \{Q, Q_1, Q_2\}, Q_1 \in \mathbb{R}^{m \times r}, Q_2 \in \mathbb{R}^{(m-r) \times r} \).

Denote the optimal control for problem (8) by \( u^*(N) \). By employing the recurrence Eq. (5), the exact solution for problem (8) will be obtained.

**Theorem 4.1.** The optimal controls \( u^*(k) \) of problem (8) are provided by

\[ u^*_i(N) = \begin{cases} 
\text{sign}(h^k_i), & \text{if } h^k_i \neq 0 \; \text{undetermined, otherwise}, \\
\end{cases} \]

and the optimal values are

\[ J(N, x_{N,N}, x_{N,N}) = r_N^Q X(N) + r_N^Q X(N), \]

\[ j(k, x_{N,N}, x_{N,N}) = \sum_{j=k}^{N} r_j^Q A_{j-k}^T x_k + \sum_{j=k}^{N-1} e_j, \]

where

\[ e_k = \begin{cases} 
\left( \sum_{j=k+1}^{N} r_j^Q A_{j-k}^T D_1 - r_j^Q A_{j-k}^T D_2 \right) & \text{if } \sum_{j=k+1}^{N} r_j^Q A_{j-k}^T D_1 - r_j^Q A_{j-k}^T D_2 \geq 0 \\
\left[ c_k - c_{k-1} X(j) + \alpha \right] & \text{if } \sum_{j=k+1}^{N} r_j^Q A_{j-k}^T D_1 - r_j^Q A_{j-k}^T D_2 < 0,
\end{cases} \]

and especially \( e_0 = 0 \) and

\[ h^k = \sum_{i=k+1}^{N} r_i^Q A_{i-k}^T B_1 + \beta_i - r_i^Q A_{i-k}^T B_2, \]

and \( h^k \) is the \( s \)-th component of the vector \( h^k \) and \( u^*_i(k) \) is the \( s \)-th component of the input vector \( u^*(k) \) for \( k = N - 1, N - 2, \ldots, 0 \) and \( s = 1, 2, \ldots, m \).

**Proof.** Denote the optimal control for problem (8) by \( u^*(0), u^*(1), \ldots, u^*(N) \). By applying the recurrence Eq. (5), we have

\[ J(N, x_{N,N}, x_{N,N}) = \sup_{u_N \in \mathcal{U}_{ad}} \left\{ r_N^Q X(N) + r_N^Q X(N) \right\} \]

where \( u^*_i(N) \leq 1, s = 1, 2, \ldots, m \).

Therefore we obtain

\[ u^*_i(N - 1) = \begin{cases} 
\text{sign}(h^k_i), & \text{if } h^k_i \neq 0 \; \text{undetermined, otherwise}, \\
\end{cases} \]

for \( s = 1, 2, \ldots, m \), and by Lemma 4.1 and Theorem 2.1 we have

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

Denote \( h^{N-1} = \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \).

Therefore we obtain

\[ u^*_i(N - 1) = \begin{cases} 
\text{sign}(h^k_i), & \text{if } h^k_i \neq 0 \; \text{undetermined, otherwise}, \\
\end{cases} \]

for \( s = 1, 2, \ldots, m \), and by Lemma 4.1 and Theorem 2.1 we have

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

Denote \( e_{N-1} = \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \).

Therefore we get

\[ J(N - 1, x_{N-1,N}, x_{N-1,N}) = \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( h^{N-1} \right)^T u^*(N - 1) + \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

\[ \left( r_N^Q D_1 - r_N^Q D_2 \right) \text{sup} (a) \]

For \( k = N - 2 \), by using the recurrence Eq. (5), we have
\[f(N-2, x_{N-2}, x_N) = \sup_{\alpha \in \mathbb{R}^{N-2}} \left\{ (t_{N-2}Q_{N-2} + r_{N-2}Q_{N-2}B_2u(N-2) - r_{N-2}Q_{N-2}D_{N-2}) \right\} \]

\[= (\sup_{\alpha \in \mathbb{R}^{N-2}} \left\{ (t_{N-2}Q_{N-2} + r_{N-2}Q_{N-2}B_2u(N-2) - r_{N-2}Q_{N-2}D_{N-2}) \right\} \]

\[= (h^{N-2}) = (h^{N-2}) \]
and

\[ l_k = \begin{cases} \frac{b_k^2}{4a_k} & \text{if } a_k < 0 \text{ and } 2a_k \leq b_k \leq -2a_k \\ a_k - b_k, & \text{if } a_k = 0 \text{ and } b_k \leq 0, \text{ or } a_k < 0 \text{ and } b_k < 2a_k, \\ a_k + b_k, & \text{if } a_k = 0 \text{ and } b_k > 0, \text{ or } a_k < 0 \text{ and } b_k > -2a_k, \\ b_k, & \text{if } a_k > 0 \text{ and } b_k > 0. \\ \end{cases} \]

for \( k = N - 1, N - 2, \ldots, 1, 0 \).

**Proof.** Denote the optimal control for problem (10) by \( u^*(0), u^*(1), \ldots, u^*(N) \). By applying the recurrence Eq. (5), we have

\[ J(N, x_{1,N}, x_{2,N}) = \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N}^Q \phi_{x_1,N} + r_{1N}^Q \phi_{x_2,N} \right\} \]

where \( u^*(N) \leq 1 \). For \( k = N - 1 \), by using the recurrence equation (5), we have

\[ J(N - 1, x_{1,N-1}, x_{2,N}) = \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + (d_{N-1} - r_{1N-1}^Q \mathcal{D} B) u(N - 1) - r_{1N-1}^Q \mathcal{B} \delta^2(N - 1) \right\} \]

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + (d_{N-1} - r_{1N-1}^Q \mathcal{D} B) u(N - 1) - r_{1N-1}^Q \mathcal{B} \delta^2(N - 1) \right\} \]

Hence, we obtain

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

Denote

\[ a_{N-1} = r_{1N}^Q \mathcal{H} \]

\[ b_{N-1} = r_{1N}^Q \mathcal{B} \]

Hence we know

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

where \( L(u(N - 1)) = (a_{N-1} \mu^2(N - 1) + b_{N-1} \mu(N - 1)) \). If \( a_{N-1} = 0 \) and \( b_{N-1} \leq 0 \), then \( u^*(N - 1) = -1 \) is the maximum point of \( L(u(N - 1)) \), and

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

Denote

\[ b_{N-1} = \sup_{u_{N-1} \in U_{N-1}} L(u(N - 1)). \]

Then by Lemma 4.1 and Theorem 2.1 we know

\[ \sup_{u_{N-1} \in U_{N-1}} \left\{ r_{1N-1}^Q \phi_{x_1,N-1} + r_{1N}^Q \phi_{x_2,N} \right\} \]

Denote \( e_{N-1} = \left( r_{1N}^Q \mathcal{D} - r_{1N}^Q \mathcal{B} \right) \mathcal{K} \). Then we have
\[ j(N - 1, X_{iN-1}, X_{iN}) \]
\[ = \left( r_{iN}^T Q_{1} + r_{iN}^T Q_{1} A_{1} X_{iN-1} + r_{iN}^T Q_{2} X_{iN} + \sup_{u(N-1) \in U_{df}} I(u(N - 1)) \right) \]
\[ + \left( r_{iN}^T Q_{D1} - r_{iN}^T Q_{D2} \zeta_{N-1} \right) \]
\[ = \left( r_{iN}^T Q_{1} + r_{iN}^T Q_{1} A_{1} X_{iN-1} + \delta_{N-1} + r_{iN}^T Q_{2} X_{iN} \right). \]

By induction, we are able to get the conclusion of the theorem. This completes the proof. □

5. Optimal control for uncertain continuous-time singular system

In the beginning of this section, we consider the following uncertain continuous-time singular system
\[
\begin{align*}
\dot{X}_t &= A X_t dt + B X_t dC_t, \\
X_{t0} &= X_0, \quad t \geq 0,
\end{align*}
\]
(11)
where \( X_t \in \mathbb{R}^n \) is the state vector of the system, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n} \) are known coefficient matrices associated with \( X_t \), respectively. \( E \) is a known (singular) matrix with rank \( (E) = q \leq n \), and \( deg(det E A) = r \) where \( z \) is a complex variable. \( C_t \) is a canonical process defined on uncertainty space, representing the noise of the system.

**Lemma 5.1.** System (11) has a unique solution if \((E, A)\) is regular and impulse-free, and \( \text{rank}(E, B) = \text{rank}E \). Moreover, the solution is sample-continuous.

**Proof.** Let \[
\begin{align*}
X_{t1} &= Q^{-1} X_t, \quad \text{where} \quad X_{t1} \in \mathbb{R}^r \quad \text{and} \quad X_{t2} \in \mathbb{R}^{n-r}.
\end{align*}
\]
Then system (11) is equivalent to
\[
\begin{align*}
\dot{X}_{t1} &= A_{1} X_{t1} dt + B_{1} X_{t1} dC_t, \\
0 &= X_{t2} dt,
\end{align*}
\] or
\[
\begin{align*}
\dot{X}_{t1} &= A_{1} X_{t1} dt + B_{1} X_{t1} dC_t, \\
0 &= X_{t2} dt,
\end{align*}
\] (12)
for all \( t \geq 0 \). Applying Theorem 4 in [28], the equation
\[\dot{X}_{t1} = A_{1} X_{t1} dt + B_{1} X_{t1} dC_t\]
has a unique solution \( X_{t1} \) on interval \([0, +\infty)\). Obviously, \( X_t = Q \begin{bmatrix} X_{t1} \\ X_{t2} \end{bmatrix} \) for all \( t \geq 0 \), which is the unique solution to (11) on \([0, +\infty)\). Finally, for each \( t \in \Gamma \), according to the result in [28], we have
\[
\| X(t) - X(t) \| = \| Q \int_0^t A_{1} X_{t1}(s) ds + Q \int_0^t B_{1} X_{t1}(s) dC_s \| \rightarrow 0
\]
as \( r \rightarrow t \). Thus \( X_t \) is sample-continuous, and this completes the proof. □

Then we start to investigate the corresponding optimal control problem subject to an uncertain continuous-time singular system. An uncertain optimal control problem is to select the best decision such that an objective function related to an uncertain process provided by an uncertain differential equation is optimized. Now we use the optimistic value-based method to optimize the uncertain objective function. That is, we assume that an uncertain variable is larger than the other one if the optimistic value of it is larger.

Unless stated otherwise, it is always assumed that the control system is regular and impulse-free. In this section, under this assumption, the following optimal control problem for an uncertain continuous-time singular system is introduced:
\[
\begin{align*}
J(0, X_0) &= \sup_{\pi \in \Pi(t)} \left[ \int_0^T f(X_t, u(s), s) ds + G(X_T, T) \right] \sup_{\alpha} (\alpha), \\
\text{subject to} & \quad \dot{X}_t = \left[ A X_t + B u(s) \right] ds + D u(s) dC_t, \quad X_{t0} = X_0.
\end{align*}
\]
where \( X_t \in \mathbb{R}^n \) is the state vector, \( u(s) \in \mathbb{R}^m \) is the input variable, \( f \) is the objective function, and \( G \) is the function of terminal reward. For a given \( u(s) \), \( X_t \) is defined by the uncertain differential equations. The function \( J(0, X_0) \) is the expected optimal value obtainable in \([0, T]\) with the initial state that at time 0 we are in state \( X_0 \).

For any \( 0 < t < T \), \( J(t, X_t) \) is the expected optimal reward obtainable in \([t, T]\) with the condition that at time \( t \) we are in state \( X_t = X \). That is, we have
\[
\begin{align*}
J(t, X_t) &= \sup_{\alpha \in \Pi(t)} \left[ \int_t^T f(X_t, u(s), s) ds + G(X_T, T) \right] \sup_{\alpha} (\alpha), \\
\text{subject to} & \quad \dot{X}_t = \left[ A X_t + B u(s) \right] ds + D u(s) dC_t, \quad X_{t0} = X_0.
\end{align*}
\]
(13)

Now we present the following principle of optimality for uncertain optimal control problem.

**Theorem 5.1.** [16] (Principle of Optimality) For any \((t, X(t)) \in [0, T] \times \mathbb{R}^n, \) and \( \delta t > 0 \) with \( t + \delta t < T \), we have
\[
J(t, X(t)) = \sup_{\pi(t)} \left[ J(t, X(t)) + J(t + \delta t, X + \delta X_t) + O(\delta t) \right] \sup_{\alpha}(\alpha),
\]
(14)
where \( X + \delta X_t = X_{t+\delta t} \).

Also, we assume that \( C_t \) is a canonical process. Consider uncertain optimal control problem (13) and the following result called the equation of optimality is derived.

**Theorem 5.2.** (Equation of Optimality) \((E, A)\) is assumed to be regular and impulse-free, and \( P D = 0 \). Let \( J(t, X(t)) \) be twice differentiable on \([0, T] \times \mathbb{R}^n \) and \( u(s) \) derivable on \([0, T] \). Then we get
\[
\begin{align*}
-\dot{J}(t, X(t)) &= \sup_{u(t)} \left[ f(X(t), u(t), t) + \nabla_X J(t, X(t)) + P \int_0^t \nabla_X J(t, X(t)) + P \right], \\
&= P \left[ A_{1} X + B_{1} u(t) \right] - B_{1} \dot{u}(t),
\end{align*}
\]
(15)
where \( P = Q \left[ A_{2} X + B_{2} u(t) \right] - B_{2} \dot{u}(t), \) and \( Q = Q \left[ Q_{2} \right], \) \( Q \in \mathbb{R}^{n \times r}, \) \( Q_{2} \in \mathbb{R}^{n \times (n-r)}, \) and \( Q = Q \left[ Q_{2} \right]. \) \( P \in \mathbb{R}^{n \times n}, \) \( P_{2} \in \mathbb{R}^{n \times (n-r)}. \)

**Proof.** Because \((E, A)\) is regular and impulse-free, by Lemma 3.1 there exist invertible matrices \( P \) and \( Q \) such that
\[
PEQ = \left[ \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right], \quad \text{and} \quad \text{PAEQ} = \left[ A_{1} \right] \left[ I_{n-r} \right], \quad PB = \left[ B_{1} \right],
\]
and from \( P D = 0 \) we get
\[
P D = \left[ \begin{array}{cc} P & D_1 \\ P & D_2 \end{array} \right] = \left[ \begin{array}{cc} 0 & D_1 \\ 0 & D_2 \end{array} \right],
\]
where \( D_1 = RD_1 \). Let \( X_s = Q \begin{bmatrix} X_{s1} \\ X_{s2} \end{bmatrix} \) for any \( s \in [t, T] \) and especially at time \( t \) we have
\[
\begin{align*}
\dot{X}_{s1} &= \left[ A_{1} X_{s1} + B_{1} u(s) \right] ds + D_{1} u(s) dC_s, \\
0 &= \left[ X_{s2} + B_{2} u(s) \right] ds,
\end{align*}
\]
where \( s \in [t, T] \). Since at any time \( s \in [t, T] \) we have
\[ X_{2+} = -B_2 u(s) \]

Let \( s = t \) and \( s = t + \delta t \), respectively. We get the following two equations:

\[
\begin{align*}
X_2 &= -B_2 u(t) \\
X_{2+} = -B_2 u(t + \delta t) 
\end{align*}
\]

Using the latter equation minus the former one, we obtain

\[
\delta X_2 = -B_2 u(t + \delta t) + O(\delta t),
\]

where \( u(t + \delta t) = u(t) + \delta t\delta t + O(\delta t) \), because \( u(s) \) is derivable on \([t, T]\). Obviously we know

\[
\delta X_i = \left[ A_i X_i + B_i u(t) \right] \delta t + B_i u(t) \delta C_i + O(\delta t),
\]

where \( \delta C_i \sim N(0, \delta t^2) \) which means \( \delta C_i \) is a normally distributed uncertain variable with expected value 0 and variance \( \delta t^2 \). Because \( X_i = \left[ X_{i,s} \right] \) is correct for any \( s \in [t, T] \), we obtain

\[
\delta X_i = \left[ A_i X_i + B_i u(t) \right] \delta t + Q_i \delta t \delta C_i + O(\delta t).
\]

where \( Q = [Q_1, Q_2] \) and \( Q_1 \in \mathbb{R}^{n_r \times n_r} \). \( Q_2 \in \mathbb{R}^{n_r \times (n_r - 1)} \). Now denote

\[
p = \left[ A_i X_i + B_i u(t) \right] - B_i u(t).
\]

Then we have

\[
\delta X_i = p \delta t + q \delta C_i + O(\delta t).
\]

By employing Taylor series expansion, we obtain

\[
J(t + \delta t, X + \delta X_i) = J(t, X) + J_i(t, X) \delta t + \nabla J(t, X)^T \delta X_i + \frac{1}{2} J_{ii}(t, X) \delta X_i^2 + \frac{1}{2} \delta X_i^T \nabla^2 J(t, X) \delta X_i + O(\delta t).
\]

Substituting Eq. (16) into Eq. (14) yields

\[
0 = \sup_{u(t)} \left\{ f(X, u(t), t) \delta t + J_i(t, X) \delta t + \left[ \nabla J(t, X)^T \delta X_i + \nabla J_{ix}(t, X) \delta X_i \right] \sup (\alpha) + O(\delta t) \right\}.
\]

Then we know

\[
\left[ \nabla J(t, X)^T \delta X_i + \nabla J_{ix}(t, X) \delta X_i \delta t + \frac{1}{2} \delta X_i^T \nabla^2 J(t, X) \delta X_i \right] \sup (\alpha) = \left[ \nabla J(t, X)^T (p \delta t + q \delta C_i + O(\delta t)) + \frac{1}{2} \nabla J_{ix}(t, X) (p \delta t + q \delta C_i + O(\delta t)) \delta t + \frac{1}{2} (p \delta t + q \delta C_i + O(\delta t)) \delta X_i \right] \sup (\alpha) = \nabla J(t, X)^T p \delta t + \left[ \nabla J(t, X)^T q \delta t + \nabla J_{ix}(t, X) q \delta t \right] \sup (\alpha) + O(\delta t),
\]

where \( a = \frac{1}{2} \nabla^2 J(t, X) \delta X_i^2 \) and \( b = \frac{1}{2} \nabla^2 J(t, X) \delta X_i^2 \).

Substituting Equation (18) into (17) results in

\[
0 = \sup_{u(t)} \left\{ f(X, u(t), t) \delta t + J_i(t, X) \delta t + \left[ a \delta C_i + b \delta C_i^2 \right] \sup (\alpha) + O(\delta t) \right\}.
\]

Obviously we have

\[
a \delta C_i - b \delta C_i^2 \leq a \delta C_i + b \delta C_i^2 \leq a \delta C_i + b \delta C_i^2.
\]

Applying Theorem 2.2 that for any \( \epsilon > 0 \) small enough, we get

\[
\left[ a \delta C_i + b \delta C_i^2 \right] \sup (\alpha) \leq \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \right]^2 b \delta t^2, \quad \left[ a \delta C_i - b \delta C_i^2 \right] \sup (\alpha) \geq \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha - \epsilon}{\alpha + \epsilon} \right]^2 b \delta t^2.
\]

Combining Inequalities (20) and (21), then we obtain

\[
a \delta C_i + b \delta C_i^2 \leq \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \right]^2 b \delta t^2, \quad a \delta C_i + b \delta C_i^2 \geq \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha - \epsilon}{\alpha + \epsilon} \right]^2 b \delta t^2.
\]

According to Eq. (19) and Inequality (22), for \( \delta t > 0 \), there exists a control \( u(t) \) such that

\[
-\epsilon \delta t \leq \left\{ f(X, u(t), t) \delta t + J_i(t, X) \delta t + \left[ a \delta C_i + b \delta C_i^2 \right] \sup (\alpha) + O(\delta t) \right\}.
\]

Dividing both sides of this inequality by \( \delta t \), we have

\[
-\frac{\epsilon}{\delta t} \leq f(X, u(t), t) + J_i(t, X) + \nabla J(t, X)^T \delta C_i + O(\delta t) + \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \right]^2 b \delta t^2 + O(\delta t) \delta t.
\]

Since \( \|a\| \leq \nabla J(t, X)^T q \delta t \) as \( \delta t \to 0 \). Letting \( \delta t \to 0 \) and then \( \epsilon \to 0 \), it is easy to know

\[
0 \leq J_i(t, X) + \sup_{u(t)} \left\{ f(X, u(t), t) + \nabla J(t, X)^T \delta t + \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha + \epsilon}{\alpha - \epsilon} \right]^2 b \delta t^2 \right\}.
\]

On the other hand, according Eq. (19) and Inequality (22), using the similar approach, we are able to obtain

\[
0 \geq J_i(t, X) + \sup_{u(t)} \left\{ f(X, u(t), t) + \nabla J(t, X)^T \delta t + \left[ \frac{\sqrt{2}}{\pi} \ln \frac{1 - \alpha - \epsilon}{\alpha + \epsilon} \right]^2 b \delta t^2 \right\}.
\]
By Inequalities (24) and (25), we get the Eq. (15). This completes the proof.

**Remark 5.1.** The solutions of the presented model (13) may be obtained from settling the equation of optimality (15). The vector \( \mathbf{p} = \left[ g(t)(A\mathbf{x}_t + B\mu(t)) - B\tilde{u}(t) \right] \) is related to the function \( \tilde{u}(t) \) which is totally different from the optimal control problem of the uncertain normal system, and it will bring lots of matters in solving Eq. (15). In some special cases, this equation of optimality may be settled to get analytical solution such as the following example. Otherwise we have to employ numerical methods to obtain the solution approximately.

### 6. Examples

In this section, two numerical examples will be presented to show the effectiveness of results obtained in SubSection 4.1 and Section 5.

**Example 6.1.** Firstly, based on Theorem 4.1, we consider the following optimal control problem

\[
j(0, x_0) = \sup_{u \in U_{ad}} \left\{ \sum_{j=0}^{T} \left[ r_j^x(x(j)) + \beta_j^x u(j) \right] + r_T^x(x(T)) \right\}
\]

subject to

\[
E\mathbf{x}(j + 1) = A\mathbf{x}(j) + B\mathbf{u}(j) + D\mathbf{d}_j,
\]

where \( U_{ad} = \{-1, 1\}^3 \),

\[
E = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix},
B = \begin{bmatrix}
-1 & 1 & 0 \\
0 & 1 & -1
\end{bmatrix},
D = \begin{bmatrix}
-3 \\
-1
\end{bmatrix},
\]

and

\[
\eta_1 = -L(2, 4), \quad \eta_2 = -L(-1, 3), \quad \eta_3 = -L(0, 5), \quad \eta_4 = -L(2, 6)
\]

are linear uncertain variables, and especially \( \eta_0 = 0 \); and

\[
\eta_0 = (9, -4, -19, 9)^T, \quad \eta_1 = (11, -20, 8, 18)^T, \quad \eta_2 = (6, 11, -8, 7)^T, \quad \eta_3 = (19, 6, 3, -7)^T, \quad \eta_4 = (-20, 10, -11, 25)^T, \quad \eta_5 = (8, 5, -9, 8)^T, \quad \eta_6 = (-8, 9, 11, -5)^T, \quad \eta_7 = (8, 17, -8, 3)^T, \quad \eta_8 = (-3, -14, 8, 5)^T;
\]

and

\[
\beta_0 = (4, -7, 5)^T, \quad \beta_1 = (12, 3, -5)^T, \quad \beta_2 = (-8, -4, 9)^T, \quad \beta_3 = (4, 1, 9)^T, \quad \beta_4 = (2, 8, -4)^T, \quad \beta_5 = (-7, 11, 5)^T, \quad \beta_6 = (-5, 6, -2)^T, \quad \beta_7 = (3, -7, -1)^T.
\]

Through calculating, we know

\[
det(2E - A) = \det \begin{bmatrix}
z & -1 & 0 & 0 \\
-1 & z & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 1
\end{bmatrix} = z^2 + z + 1.
\]

Obviously, \( \det(2E - A) \) is not identically zero and \( \deg(\det(2E - A)) = \text{rank}(E) \), namely, \( (E, A) \) is regular and impulse-free. By using Lemma 3.1, through deduction we obtain two invertible matrices \( P \) and \( Q \):

\[
P = \begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix},
Q = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

\[
P E Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
P A Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
P B = \begin{bmatrix}
1 & -1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix},
P D = \begin{bmatrix}
2 & -3 \\
1 & 0
\end{bmatrix}.
\]

\[
E = \begin{bmatrix}
-1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix},
Q_1 = \begin{bmatrix}
1 & 0 \\
-1 & 1 \\
0 & 0
\end{bmatrix},
Q_2 = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

And it is not difficult to obtain that
e_0 = 0, \quad e_1 = -290.4, \quad e_2 = 17.6, \quad e_3 = 1168, \quad e_4 = -518, \quad e_5 = 16, \quad e_6 = 309.4, \quad e_7 = 120.4.

Because \( x_0 = Q x_0, Q x_0 = (1, 1)^T \) is easily derived. Based on Lemma 3.1, problem (26) is equivalent to problem (27):

\[
j(0, x_0, x_2, y) = \sup_{u \in U_{ad}} \left\{ \sum_{j=0}^{T} \left[ r_j^x(x(j)) + \beta_j^x u(j) \right] + r_T^x(x(T)) \right\}
\]

subject to

\[
E \mathbf{x}(j + 1) = A \mathbf{x}(j) + B \mathbf{u}(j) + D \mathbf{d}_j,
\]

\[
0 = x_0(j), \quad B_0 = 0
\]

\[
1, 2, ..., 7.
\]

\[
x_0 = (1, 1)^T, \quad x_0, x_8 = (-1, 1)^T.
\]

The optimal controls and optimal values are obtained by Theorem 4.1 and listed in Table 1.

**Remark 6.1.** In Column 3 of Table 1, the corresponding states \( x_k(k + 1) \) which are derived from \( x_k(k + 1) = A_k x_k(k) + B_k \mu_k(k) + D_k \mathbf{d}_k \) for initial state \( x_k(0) = (1, 1)^T \), where \( \mu_k \) is the realization of uncertain variable \( \xi_k \), and may be generated by \( \mu_k = (1 - a_k) x_k + a_k x_k^c \) for a random number \( a_k \in [0, 1] \) for \( k = 1, 2, ..., 7 \).

Next we investigate an optimal control model subject to a discrete-time singular system without uncertainty compared with problem (26):

\[
j(0, x_0) = \sup_{u \in U_{ad}} \left\{ \sum_{j=0}^{T} \left[ r_j^x(x(j)) + \beta_j^x u(j) \right] + r_T^x(x(T)) \right\}
\]

subject to

\[
E \mathbf{x}(j + 1) = A \mathbf{x}(j) + B \mathbf{u}(j),
\]

\[
x_0 = (1, 1, 1)^T, \quad j = 0, 1, 2, ..., 7.
\]

(28)
Table 1
The optimal results of problem (26).

<table>
<thead>
<tr>
<th>Stage</th>
<th>$\mu$</th>
<th>$x_i^0(k)$</th>
<th>$x_i^t(k)$</th>
<th>$y_i^0$, $y_i^t$</th>
<th>$u_i^0(k)$</th>
<th>$J(k, x_i^0, x_i^t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1, 1)</td>
<td>(1, 1, 1, 3)</td>
<td>(1, 1, 1, 3)</td>
<td>(56, -44, 8)</td>
<td>(1, -1, 1)</td>
<td>2187</td>
</tr>
<tr>
<td>1</td>
<td>2.764</td>
<td>( -5, -1)</td>
<td>( -5, 2.236, -1, -9.764)</td>
<td>( -9, 128, 3)</td>
<td>( -1, 1, 1)</td>
<td>2126</td>
</tr>
<tr>
<td>2</td>
<td>0.356</td>
<td>( -7, -11.292)</td>
<td>( -7, 6.936, -11.292, 6.664)</td>
<td>( -47, -67, 40)</td>
<td>( -1, -1, 1)</td>
<td>2367.4</td>
</tr>
<tr>
<td>3</td>
<td>3.17</td>
<td>(1.292, 5.932)</td>
<td>(1.292, -7.394, 5.932, 0.122)</td>
<td>(95, -139, 21)</td>
<td>(1, -1, 1)</td>
<td>2215.4</td>
</tr>
<tr>
<td>4</td>
<td>2.152</td>
<td>( -10.224, -10.218)</td>
<td>( -10.224, 15.29, -10.218, -14.376)</td>
<td>( -58, 205, -75)</td>
<td>(1, -1, -1)</td>
<td>2011.9</td>
</tr>
<tr>
<td>5</td>
<td>-0.419</td>
<td>(23.442, -14.68)</td>
<td>(23.442, -5.343, -14.68, 23.861)</td>
<td>(13, -114, 67)</td>
<td>(1, -1, 1)</td>
<td>1931.4</td>
</tr>
<tr>
<td>6</td>
<td>1.053</td>
<td>( -11.762, 24.699)</td>
<td>( -11.762, 28.61, -16.921, -10.815)</td>
<td>(27, -10, -26)</td>
<td>(1, -1, 1)</td>
<td>1451.3</td>
</tr>
<tr>
<td>8</td>
<td>(33.858, -20.367)</td>
<td>(33.858, -14.491, -20.367, 34.858)</td>
<td>u(8) e [-1, 1]^3</td>
<td>112.654</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The optimal controls and optimal values are derived for $x_{0,0} = (1, 1)^T$ with $x_{0,2} = (-1, 1)^T$, and listed in Table 2.

From Table 1 and 2, the optimal controls for problem (26) and problem (28) have no difference, so they can be illustrated in Fig. 1. But the state vectors of two problems above are distinctive, we draw Figs. 2 and 3 of state vectors about problem (26) and problem (28), respectively.

Remark 6.2. Comparing uncertain optimal control problem (26) with optimal control problem (28) without uncertainty, obviously we see that problem (28) is just a special case of problem (26) which tells us the latter problem is more important than the former problem in theory and has more applications in practice. From Fig. 2, we notice that every component of state vector is integer which seems too idealistic, while uncertain factors have been considered in problem (26) which makes the state vectors in Fig. 3 more accurate.

Example 6.2. In Section 5, we got Theorem 5.2 named equation of optimality for the uncertain optimal control problem subject to an uncertain continuous-time singular system. For this problem, we can sometimes use analytical method to obtain the optimal controls and the optimal values with the initial state of the system at all time. As an numerical example, we consider the following problem:

$$
J(t, X_t) = \sup_{u(t) \in \mathcal{U}(t)} \left[ \int_0^t r(s) X_t(s, u(s)) ds \right]_{sup}$$

subject to

$$EdX_t = [AX_t + Bu(s)] ds + Du(s) dC, \text{ and } X_t = X.$$  (30)

where $X_t \in \mathbb{R}^3$ is the state vector, $r(s) \in \mathbb{R}^3$ is the coefficient of $X_t$, $U_{ad} = \{-1, 1\}$, and

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, D = \begin{bmatrix} 2/3 \\ 1/2 \\ 0 \end{bmatrix}.$$  

and

$$r^T(s) = [1, 0, -2] e^{-s}.$$  

Through calculating, we know

$$\det(E) - A) = \det(2z - 1) = (z - 1)^2.$$  

Obviously, $\det(E - A)$ is not identically zero and $\text{deg}(\det(E - A)) = \text{rank}(E)$, namely, $(E, A)$ is regular and impulse-free. By Using Lemma 3.1, through deduction we obtain two invertible matrices $P$ and $Q$:

$$P = \begin{bmatrix} 0 & 1/2 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$  

such that

$$PEQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, PAQ = \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, PB = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, PD = \begin{bmatrix} -1/12 \\ -1/6 \\ -1/3 \end{bmatrix}.$$  

Easily, we can see

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} -1/12 \\ -1/6 \\ -1/3 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}.$$  

where $P_2 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$. Denote $X = [x_i, x_j, x_k]^T$, and we assume that $x_i + 2x_j = 0$. Because

$$Q^{-1} = \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \\ 1 & -1 \end{bmatrix},$$

and

$$X_i^T = Q^{-1} X_i,$$  

we obtain $X_i = [x_i, x_j]^T$. Combining these results and Eq. (15), we know
Table 2
The optimal results of problem (28).

<table>
<thead>
<tr>
<th>Stage</th>
<th>$x_1^T(k)$</th>
<th>$x_2^T(k)$</th>
<th>$u(k)^T$</th>
<th>$u^T(k)$</th>
<th>$J(k, x_k, x_{2k})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1, 1)</td>
<td>(1, 1, 1, 1)</td>
<td>(56, -44, 8)</td>
<td>(1, -1, 1)</td>
<td>1378</td>
</tr>
<tr>
<td>1</td>
<td>(-5, -1)</td>
<td>(-5, 1, -1, -7)</td>
<td>(-9, 128, 3)</td>
<td>(-1, -1, 1)</td>
<td>1387</td>
</tr>
<tr>
<td>2</td>
<td>(7, -3)</td>
<td>(7, -1, -3, 7)</td>
<td>(-47, -67, 40)</td>
<td>(-1, -1, 1)</td>
<td>1652</td>
</tr>
<tr>
<td>3</td>
<td>(-7, 7)</td>
<td>(-7, 3, 7, -5)</td>
<td>(95, -139, 21)</td>
<td>(-1, -1, 1)</td>
<td>1647</td>
</tr>
<tr>
<td>4</td>
<td>(-3, -9)</td>
<td>(-3, 9, -9, -5)</td>
<td>(-58, 205, -75)</td>
<td>(-1, -1, 1)</td>
<td>1742</td>
</tr>
<tr>
<td>5</td>
<td>(15, -1)</td>
<td>(15, -11, -1, 15)</td>
<td>(-13, -114, 67)</td>
<td>(-1, -1, 1)</td>
<td>1528</td>
</tr>
<tr>
<td>6</td>
<td>(-17, 15)</td>
<td>(-17, 3, 15, -15)</td>
<td>(27, -10, -26)</td>
<td>(-1, -1, 1)</td>
<td>1347</td>
</tr>
<tr>
<td>7</td>
<td>(1, -19)</td>
<td>(1, 15, -19, -1)</td>
<td>(-36, 78, -48)</td>
<td>(-1, -1, 1)</td>
<td>961</td>
</tr>
<tr>
<td>8</td>
<td>(21, 3)</td>
<td>(21, -25, 3, 22)</td>
<td>(-8) ∈ [ -1, 1]^2</td>
<td>(1, 1, 1)</td>
<td>421</td>
</tr>
</tbody>
</table>

We conjecture that $f(t, X) = k_f^T(tX)$, and let $\alpha = 0.2$. Then

$$J(t, X) = -k_f^T(tX, \nabla_x J(t, X)) = k_f(t),$$

and

$$r(t)Xu(t) + \nabla_x J(t, X)^T p + \sqrt{\frac{3}{\pi}} \ln \frac{1 - \alpha}{\alpha} (\nabla_x J(t, X)^T q)$$

$$= (x_1 - 2x_2)e^{-t}u(t) + \frac{1}{2}(x_1 + 2u(t) - 2(x_1 + x_2 + 2u(t)))e^{-t}$$

$$+ \sqrt{\frac{3}{\pi}} \ln \frac{1 - \alpha}{\alpha} ku(t)e^{-t},$$

$$= (x_1 - 2x_2 - 6k)e^{-t}u(t) + \sqrt{\frac{3}{\pi}} \ln 4ku(t)e^{-t}.$$  

Applying Eq. (15), we get

$$k(x_1 - 2x_2)e^{-t} = \sup_{u(t) \in [-1, 1]} \left( x_1 - 2x_2 - 2k e^{-t}u(t) + \frac{\sqrt{3}}{\pi} \ln 4ku(t)e^{-t} \right)$$

$$= e^{-t} \sup_{u(t) \in [-1, 1]} \left( x_1 - 2x_2 - 2k u(t) + \frac{\sqrt{3}}{\pi} \ln 4ku(t) \right).$$  

(31)

Dividing Eq. (31) by $e^{-t}$, we obtain

$$k(x_1 - 2x_2) = \sup_{u(t) \in [-1, 1]} \left( x_1 + 2x_2 - 2k u(t) + \frac{\sqrt{3}}{\pi} \ln 4ku(t) \right).$$  

(32)

If $ku(t) \geq 0$, Eq. (32) turns to be

$$k(x_1 - 2x_2) = \sup_{u(t) \in [-1, 1]} \left( x_1 + 2x_2 - 2ku(t) + \frac{\sqrt{3}}{\pi} \ln 4ku(t) \right)$$

$$= x_1 - 2x_2 - \frac{2 - \frac{\sqrt{3}}{\pi} \ln 4k}{k},$$  

(33)

and then

$$k^2(x_1 - 2x_2)^2 = \left( x_1 - 2x_2 - \frac{2 - \frac{\sqrt{3}}{\pi} \ln 4k}{k} \right)^2.$$  

namely

$$a^2 - b_1^2 k^2 + 2ab_1 k - a^2 = 0,$$
where $a = x_1 - 2x_3$, and $b_1 = 2 - \sqrt{3}/\pi > 0$. Because $ku(t) \geq 0$, and by Eq. (33) the symbols of $k$ and $a$ must keep coincident, we know

\[
k = \begin{cases} 
\frac{a}{2b_1}, & \text{if } a = \pm b_1 \\
0, & \text{if } a = 0 \\
\frac{-a}{a - b_1}, & \text{if } a < -b_1 \\
\frac{a}{a + b_1}, & \text{if } a > b_1.
\end{cases}
\]

And the optimal control

\[
u^*(t) = \text{sign}(a - bk).
\]

If $ku(t) < 0$, Eq. (32) turns to be

\[
k(x_1 - 2x_3) = \sup_{u \in \{-1, 1\}} \left[ (x_1 + 2x_3 - 2k)u(t) - \frac{\sqrt{3}}{\pi}\ln4ku(t) \right]
\]

\[
= \left| x_1 - 2x_3 - \left(2 + \frac{\sqrt{3}}{\pi}\ln4\right)k \right|.
\]

Using the similar method, we are able to obtain

\[
k = \begin{cases} 
\frac{a}{2b_2}, & \text{if } a = \pm b_2 \\
\frac{a}{a + b_2}, & \text{if } -b_2 < a < 0 \\
\frac{-a}{a - b_2}, & \text{if } 0 < a < b_2,
\end{cases}
\]

where $a = x_1 - 2x_3$, and $b_2 = 2 + \frac{\sqrt{3}}{\pi}\ln4 > 0$. And the optimal control

\[
u^*(t) = \text{sign}(a - bk).
\]

When $b_1 < a < b_2$, we know

\[
b_1 - b_2 + 2a > 3b_2 - b_2 = 4\left(1 - \frac{\sqrt{3}}{\pi}\ln4\right) > 0,
\]

and obviously

\[
-\frac{a}{a - b_2} > 0, \quad \frac{a}{a + b_2} > 0.
\]

When $-b_2 < a < -b_1$, similarly we get that

\[
-\frac{a}{a - b_1} > 0, \quad \frac{a}{a + b_1} > 0.
\]

To optimize the objective function $j(t, X)$, we combine the above results, and then obtain the accurate optimal control

\[
u^*(t) = \begin{cases} 
\text{sign}(a - bk), & \text{if } a = \pm b_1, 0, 0, \text{or } a > b_2, \\
\text{sign}(a - bk), & \text{if } 0 < a < b_1, \text{or } b_1 < a < b_2.
\end{cases}
\]

**Remark 6.3.** Problem (30) is a very special case of problem (13) considered in Section 5 usually being impossible to get analytical solution. In such case, we can discretize the problem in Example 6.2 to be a discrete-time problem in Example 6.1, and then employ Theorem 4.1 to get optimal controls approximative, finally obtaining its optimal controls accurate by passing to the limit.

### 7. Application

In order to display the power of Theorem 4.2, we now consider a dynamic input-output system concerning managing a company of auto parts, in which $A = (a_p)_{3 \times 3}$ is the matrix of consuming coefficient and $T = (t_p)_{3 \times 3}$ the matrix of consumption coefficient obtained by the wage paying workers and salesmen. These coefficients are listed in Table 3.

According to Table 3 we know that getting the outputs $x_1, x_2, x_3$ will consume the amounts of products, equipments and funds as follows:

\[
\begin{bmatrix}
q_1 x_1 + q_2 x_2 + q_3 x_3 \\
q_2 x_1 + q_2 x_2 + q_3 x_3 \\
q_3 x_1 + q_2 x_2 + q_3 x_3
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0.2 \\
0 & 0 & 0 \\
0.5 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= Ax.
\]

All wage, which can be treated as the input of funds, for workers and salesmen is:

\[
\begin{bmatrix}
0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= T x.
\]

In this input-output model, it is assumed that the input $Ax(j)$ and the consumption $Tx(j)$ happen in the same year with the output $x(j)$. Next, we give matrix $B = (b_p)_{3 \times 4}$ standing for the matrix of fixed-asset investing coefficient in Table 4. From Table 4, obtaining the outputs $x_1, x_2, x_3$ may cost the amount of fixed-asset investment as the following:

\[
\begin{bmatrix}
b_{11} x_1 + b_{12} x_2 + b_{13} x_3 \\
b_{21} x_1 + b_{22} x_2 + b_{23} x_3 \\
b_{31} x_1 + b_{32} x_2 + b_{33} x_3
\end{bmatrix}
= \begin{bmatrix}
b_{11} \\
b_{21} \\
b_{31}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= B x.
\]

In order to get the output $x(j)$ in $j$-th year, fixed asset $Bx(j)$ should be invested. Similarly, in $j + 1$-th year getting the output $x(j + 1)$ needs fixed asset $Bx(j + 1)$. That is to say, from $j$-th year to $j + 1$-th year, the increment of fixed asset is $B(x(j + 1) - x(j))$. For gaining more profits, advertisement expenses $u(j)$ in $j$-th year will bring an extra income

\[
\Delta x(j) = \hat{A} \hat{u}^2(j) + \hat{B}u(j), \quad u(j) \in [0, 50].
\]

We employ an uncertain variable $\xi_j$ to illustrate the unexpected expenditures in $j$-th year, such as repairing the factories, loss caused by defective products and so on. To keep balance, the following equation is derived:

\[
\begin{bmatrix}
0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x(j) + \Delta x(j) \\
x(j + 1) \\
x(j + 2) \\
x(j + 3)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{A} x(j) + \hat{B} u(j) + \hat{H} \hat{u}^2(j) + \hat{D} \xi_j \\
\hat{B} x(j + 1) - \hat{A} x(j) \\
\hat{B} x(j + 2) - \hat{A} x(j + 1) \\
\hat{B} x(j + 3) - \hat{A} x(j + 2)
\end{bmatrix}
\]

\[
\Delta x(j) = \hat{A} \hat{u}^2(j) + \hat{B}u(j), \quad u(j) \in [0, 50].
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x(j) + \Delta x(j) \\
x(j + 1) \\
x(j + 2) \\
x(j + 3)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{A} x(j) + \hat{B} u(j) + \hat{H} \hat{u}^2(j) + \hat{D} \xi_j \\
\hat{B} x(j + 1) - \hat{A} x(j) \\
\hat{B} x(j + 2) - \hat{A} x(j + 1) \\
\hat{B} x(j + 3) - \hat{A} x(j + 2)
\end{bmatrix}
\]

\[
\Delta x(j) = \hat{A} \hat{u}^2(j) + \hat{B}u(j), \quad u(j) \in [0, 50].
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x(j) + \Delta x(j) \\
x(j + 1) \\
x(j + 2) \\
x(j + 3)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{A} x(j) + \hat{B} u(j) + \hat{H} \hat{u}^2(j) + \hat{D} \xi_j \\
\hat{B} x(j + 1) - \hat{A} x(j) \\
\hat{B} x(j + 2) - \hat{A} x(j + 1) \\
\hat{B} x(j + 3) - \hat{A} x(j + 2)
\end{bmatrix}
\]

\[
\Delta x(j) = \hat{A} \hat{u}^2(j) + \hat{B}u(j), \quad u(j) \in [0, 50].
\]
Whole wage: \( \omega_1 = 0 \), \( \omega_2 = 0 \), \( \omega_3 = 0.02 \), \( \omega_4 = 0.7735 \).

From matrix \( Q \), we get

\[
Q = \begin{bmatrix}
0.9055 & -0.1055 & 0.2 \\
0.684 & 19.316 & -20 \\
4.5274 & -0.5274 & -4
\end{bmatrix}
\]
The optimal results of problem (36).

<table>
<thead>
<tr>
<th>Stage</th>
<th>( \mu_k )</th>
<th>( X^*_k(k) )</th>
<th>( a_k )</th>
<th>( b_k )</th>
<th>( u(k) )</th>
<th>( J(k, X_k(0), X_5(5)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(11.072, 0.2292)</td>
<td>0.5906</td>
<td>-22.1488</td>
<td>50</td>
<td>1233.1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(11.1899, 4.4921)</td>
<td>0.4669</td>
<td>-31.1258</td>
<td>0</td>
<td>1206.8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(11.6513, 12.5466)</td>
<td>0.1270</td>
<td>-6.3497</td>
<td>50</td>
<td>1176.7</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(12.4291, 31.3263)</td>
<td>0.0574</td>
<td>-3.3458</td>
<td>0</td>
<td>1112.1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(14.3470, 70.8007)</td>
<td>0.0225</td>
<td>-1.5471</td>
<td>0</td>
<td>927.9998</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(18.4197, 156.4109)</td>
<td>0.0225</td>
<td>-1.5471</td>
<td>0</td>
<td>607.1983</td>
<td></td>
</tr>
</tbody>
</table>

\[ Q^{-1} = \begin{bmatrix} 1 & 0.006 & 0.02 \\ 0.0516 & -0.2078 \\ 0 & -0.2 \end{bmatrix} \]

and thus

\[ X(0) = Q^{-1} x(0) = \begin{bmatrix} 11.072 \\ 0.2292 \\ 0 \end{bmatrix} \]

which implies that

\[ X(0) = (11.072, 0.2292)^T, X(5) = 0. \]

Based on Lemma 3.1, problem (35) is equivalent to problem (36):

\[ J(0, X(0), X(5)) \]

subject to

\[ X(j + 1) = \hat{A} X(j) + \hat{B} X(j) + \hat{H} X(j) + \hat{D} \xi \]

\[ 0 = X(j), \quad j = 0, 1, 2, 3, 4, \]

\[ X(0) = (11.072, 0.2292)^T, X(5) = 0. \] (36)

The optimal controls and optimal values are obtained by Theorem 4.2 and listed in Table 5. The data in the table are derived for the initial state \( X(0) = (11.072, 0.2292)^T \) and the terminal state \( X(5) = 0 \).

**Remark 7.1.** In Column 3 of Table 5, the corresponding states \( X_k(k) \) which are derived from \( X_k(k + 1) = A_k X_k(k) + B_k u(k) + H_k u^2(k) + D_k \xi_k \) for initial state \( X_{0,0} = (1, 1)^T \), where \( \xi_k \) is the realization of uncertain variable \( x_k \), and may be generated by \( \xi_k = (1 - \alpha_k) x_k + \alpha_k c_k \) for a random number \( \alpha_k \in [0, 1], k = 1, 2, ..., 6 \).

8. Conclusions

Applying optimistic value criterion, we studied optimal control problems for uncertain discrete-time singular systems and uncertain continuous-time singular systems in this paper. In order to settle these two types of optimal control problems, we presented the recurrence equation and the equation of optimality based on Bellman’s principle of optimality in dynamic programming and some results in uncertainty theory. When the objective function is linear in the optimal control problem subject to uncertain discrete-time singular systems, the corresponding optimal controls are bang-bang. For the optimal control problem subject to an uncertain continuous-time singular system, the equation of optimality was derived to solve such problem provided that the input vector is derivable. In addition, we gave two numerical examples and a dynamic input-output model as applications of the results obtained in Section 4 and 5. Throughout this paper, we assumed that the uncertain singular system is regular and impulse-free. To the best of our knowledge, under such assumption we obtained a recurrence equation and an equation of optimality for uncertain discrete-time optimal problem and uncertain continuous-time optimal problem, respectively. If the singular system is regular while not impulse-free, we could not transform it into two sub-systems as Lemma 3.1. Such singular system called noncausal system which is more difficult to handle in analysis. We may consider optimal control problems subject to uncertain noncausal systems in the future.

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**References**


