Stability of high-order uncertain differential equations

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Abstract. High-order uncertain differential equations are applied to model differentiable uncertain systems with high-order differentials. In order to describe the influence of the initial value on the solution, this paper proposes two concepts of stability for high-order uncertain differential equation, including stability in measure and stability in mean. Most important of all, some stability theorems are given for a high-order uncertain differential equation. In addition, this paper shows that the given condition is not necessary for a high-order uncertain differential equation being stable via a counterexample. Lastly, this paper will discuss the relationship between stability in measure and stability in mean.

Keywords: High-order uncertain differential equation, uncertain process, Liu process, stability

1. Introduction

For a long time, indeterministic phenomenon was mainly described by probability and fuzzy set theory. However, except for randomness and fuzziness, human uncertainty is another source of indeterminate information, and lots of surveys show that human uncertainty behaves neither like randomness nor fuzziness. In order to deal with the information associated with human uncertainty, an uncertainty theory was founded by Liu [7] using uncertain measure to deal with the belief degree in 2007 and Liu [9] perfected it with presenting product uncertain measure. Uncertain measure, as a set function satisfying the normality, duality, subadditivity and production axioms, is used to indicate the belief degree that an event occurs. Meanwhile, uncertain variable, a basic concept, was proposed by Liu [7]. Uncertain variable, as a measurable function on an uncertainty space, is used to model a quantity with human uncertainty. For describing the uncertain variable, Liu [7] introduced a concept of uncertainty distribution. Besides, Peng and Iwamura [13] gave a sufficient and necessary condition for the uncertainty distribution of an uncertain variable. Liu [9] gave a concept of independence with respect to uncertain variables, based on which the operational law was established by Liu [10] and concepts of expected value, variance and entropy are also proposed to describe an uncertain variable. In addition, the independence of uncertain vectors was discussed by Liu [11].

In order to model the dynamic system involving random factors, stochastic differential equation is proposed that it is a type of differential equations driven by Wiener process. In fuzzy set theory, there exists fuzzy differential equation deal with the dynamic system involving fuzzy factors. Similarly, in uncertainty theory, uncertain process is a sequence of uncertain variables driven by the time or the space. In order to model a dynamic system with human uncertainty, in 2008, Liu [8] gave a concept of uncertain process for modeling the evolution of uncertain phenomena. Then Liu [9] designed a Liu process as a counterpart of standard Wiener process. It is a stationary independent increment uncertain process with normal increments, and its almost all sample
paths are Lipschitz continuous. After that, Liu [9] founded an uncertain calculus theory to deal with the integral and differential of an uncertain process with respect to Liu process. Uncertain differential equation is a type of differential equations driven by Liu process, it first proposed by Liu [8], and it aims to describe the evolution of dynation uncertain systems. As the stochastic differential equation [14] and the fuzzy differential equation [1] existence and uniqueness are also the fundamental problems in the uncertain differential equation. Recently, Allahviranloo et al. [2, 3] provided some good ideal to further study the high-order fuzzy differential equation. Chen and Liu [4] provided a sufficient condition for an uncertain differential equation being stable. After a while, Gao [5] gave an existence and uniqueness theorem under weaker conditions. The concept of stability for an uncertain differential equation having a unique solution was proposed by Liu [9] in the sense of uncertain measure. Then Yao et al. [20] gave a sufficient condition for stability. After that, Yao et al. [21] proposed a concept of stability in mean, and gave a sufficient condition for stability. Sheng and Wang [16] studied another type of stability in mean. The rest of this paper is organized as follows: Section 2 will introduce some concepts about uncertain variables, uncertain processes, and uncertain differential equations.

2. Preliminaries

In this section, we will introduce some fundamental concepts and properties concerning uncertain variables, uncertain processes, and uncertain differential equations.

Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \) in \( \mathcal{L} \) is called an event and assigned a number \( \mathcal{M}(\Lambda) \) to indicate the belief degree with which we believe \( \Lambda \) will happen. In order to deal with belief degrees rationally, Liu [7] suggested the following three axioms:

1. (Normality Axiom) \( \mathcal{M}(\Gamma) = 1 \) for the universal set \( \Gamma \);
2. (Duality Axiom) \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any event \( \Lambda \);
3. (Subadditivity Axiom) For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have \( \mathcal{M}\left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i) \).

Definition 2.1. (Liu [7]) The set function \( \mathcal{M} \) is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.

The triplet \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is called an uncertainty space. Furthermore, the product uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L} \) was defined by Liu [9] as follows:

- (Product Axiom) Let \( (\Gamma_k, \mathcal{L}_k, \mathcal{M}_k) \) be uncertainty spaces for \( k = 1, 2, \ldots \). The product uncertain measure \( \mathcal{M} \) is an uncertain measure satisfying \( \mathcal{M}\{ \prod_{k=1}^{\infty} \Lambda_k \} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Lambda_k) \) where \( \Lambda_k \) are arbitrary events chosen from \( \mathcal{L}_k \) for \( k = 1, 2, \ldots \), respectively.

Definition 2.2. (Liu [7]) An uncertain variable is a measurable function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set \( \{ \xi \in B \} = \{ \gamma \in \Gamma | \xi(\gamma) \in B \} \) is an event.

Definition 2.3. (Liu [7]) Suppose \( \xi \) is an uncertain variable. Then the uncertainty distribution of
\( \xi \) is defined by \( \Phi(x) = \mathcal{M}\{\xi \leq x\} \) for any real number \( x \).

An uncertainty distribution \( \Phi(x) \) is said to be regular if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1) \). Inverse uncertainty distribution plays an important role in the operations of independent uncertain variables. In the following, the concept of inverse uncertainty distribution will be presented.

The operational law of independent uncertain variables was given by Liu [10] in order to calculate the uncertainty distribution of a strictly increasing or decreasing function of uncertain variables. Before introducing the operational law, the concept of independence of uncertain variables is presented as follows:

**Definition 2.4.** (Liu [9]) The uncertain variables \( \xi_1, \xi_2, \ldots, \xi_n \) are said to be independent if
\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}
\]
for any Borel sets \( B_1, B_2, \ldots, B_n \).

For ranking uncertain variables, the concept of expected value was proposed by Liu [7] as follows:

**Definition 2.5.** (Liu [7]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by
\[
E[\xi] = \int_{-\infty}^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}dx
\]
provided that at least one of the two integrals is finite.

**Theorem 2.1.** (Liu [7]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then \( E[\xi] = \int_{-\infty}^{+\infty} x d\Phi(x) \). If the uncertainty distribution \( \Phi \) is regular, then we also have \( E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha \).

### 3. High-order uncertain differential equations

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The study of uncertain process was started by Liu [8] in 2008.

**Definition 3.1.** (Liu [8]) Let \( T \) be an index set and let \( (\Gamma, \mathcal{L}, \mathcal{M}) \) be an uncertainty space. An uncertain process is a measurable function from \( T \times (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers such that \( \{X_t \in B\} \) is an event for any Borel set \( B \) for each \( t \).

After that, Liu [9] designed a process which is one of the most important uncertain processes, it is named as Liu process thereafter.

**Definition 3.2.** (Liu [9]) An uncertain process \( C_t \) is said to be a Liu process if (i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous, (ii) \( C_t \) has stationary and independent increments, (iii) every increment \( C_{t+s} - C_s \) is a normal uncertain variable with expected value 0 and variance \( t^2 \), whose uncertainty distribution is
\[
\Phi(x) = \left( 1 + \exp\left( -\frac{\pi x}{\sqrt{3}} \right) \right)^{-1}, \quad x \in \mathbb{R}.
\]

**Theorem 3.1.** (Yao et al. [20]) Let \( C_t \) be a Liu process on an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \). Then there exists an uncertain variable \( K \) such that \( K(\gamma) \) is a Lipschitz constant of the sample path \( C_t(\gamma) \) for each \( \gamma \), we have
\[
\lim_{x \to +\infty} \mathcal{M}\{K \leq x\} = 1 \quad \text{and} \quad \mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} \geq 2 \left( 1 + \exp\left( -\frac{\pi x}{\sqrt{3}} \right) \right)^{-1} - 1.
\]

**Definition 3.3.** (Yao [19]) Let \( C_t \) be a Liu process, and \( f \) and \( g \) are two given functions. Then
\[
\frac{d^n X_t}{dr^n} = f\left(t, X_t, \frac{dX_t}{dr}, \ldots, \frac{d^{n-1}X_t}{dr^{n-1}}\right) + g\left(t, X_t, \frac{dX_t}{dr}, \ldots, \frac{d^{n-1}X_t}{dr^{n-1}}\right) \frac{dC_t}{dr}
\]
is called an \( n \)-order uncertain differential equation. 

An uncertain process that satisfies (9.1) identically at each time \( t \) is called a solution of the high-order uncertain differential equation.

High-order uncertain differential equations are essentially high-order differential equation driven by the Liu process. High-order uncertain differential equations are used to model differentiable uncertain systems with high-order differentials.

Yao showed the high-order uncertain differential equation with an initial value \( X_0 \) has a unique solution if the coefficients satisfy the following conditions.

**Theorem 3.2.** (Yao [19]) The \( n \)-order uncertain differential equation (2) has a unique solution if for any \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n \) and \( t \geq 0 \), the coefficients \( f(t, x_1, x_2, \ldots, x_n) \) and \( g(t, x_1, x_2, \ldots, x_n) \) satisfy the linear growth condition.
\[ |f(t, x_1, \cdots, x_n)| + |g(t, x_1, \cdots, x_n)| \leq L \left( 1 + \sum_{i=1}^{n} |x_i| \right) \]  

and the Lipschitz condition

\[ |f(t, x_1, \cdots, x_n) - f(t, y_1, \cdots, y_n)| + |g(t, x_1, \cdots, x_n) - g(t, y_1, \cdots, y_n)| \leq L \cdot \sum_{i=1}^{n} |x_i - y_i| \]

for some constant \( L \).

For the \( n \)-order uncertain differential Equation (2), we denote

\[ X_{1t} = X_t, X_{2t} = \frac{dX_t}{dt}, \cdots, X_{nt} = \frac{d^{n-1}X_t}{dt^{n-1}}. \]

Then we have

\[
\begin{align*}
  \frac{dX_{1t}}{dt} &= X_{2t} \\
  \frac{dX_{2t}}{dt} &= X_{3t} \\
  & \vdots \\
  \frac{dX_{nt-1t}}{dt} &= X_{nt} \\
  \frac{dX_{nt}}{dt} &= f(t, X_{1t}, X_{2t}, \cdots, X_{nt}) \\
  & + g(t, X_{1t}, X_{2t}, \cdots, X_{nt}) \frac{dC_t}{dt}.
\end{align*}
\]

Denote the vectors

\[ \mathbf{X}_t = [X_{1t}, X_{2t}, \cdots, X_{nt}]^T \]

\[ = \left[ X_t, \frac{dX_t}{dt}, \cdots, \frac{d^{n-1}X_t}{dt^{n-1}} \right]^T, \]

\[ f(t, \mathbf{X}_t) = [X_{2t}, X_{3t}, \cdots, X_{nt}, f(t, X_{1t}, X_{2t}, \cdots, X_{nt})]^T \]

and

\[ g(t, \mathbf{X}_t) = [0, 0, \cdots, g(t, X_{1t}, X_{2t}, \cdots, X_{nt})]^T. \]

Then the \( n \)-order uncertain differential equation (2) can be transformed into an \( n \)-dimensional uncertain differential equation driven by a Liu process, that is

\[ d\mathbf{X}_t = f(t, \mathbf{X}_t)dt + g(t, \mathbf{X}_t)dC_t, \]

where \( f(t, \mathbf{X}_t), g(t, \mathbf{X}_t) \) are a vector-valued function form \( T \times \mathbb{R}^n \) to \( \mathbb{R}^n \).

In other words, the equation

\[
\frac{d^n X_t}{dt^n} = f(t, X_t, \frac{dX_t}{dt}, \cdots, \frac{d^{n-1}X_t}{dt^{n-1}}) + g(t, X_t, \frac{dX_t}{dt}, \cdots, \frac{d^{n-1}X_t}{dt^{n-1}}) \frac{dC_t}{dt}
\]

and the equation

\[ d\mathbf{X}_t = f(t, \mathbf{X}_t)dt + g(t, \mathbf{X}_t)dC_t, \]

are equivalent.

The uncertain differential equation

\[ d\mathbf{X}_t = f(t, \mathbf{X}_t)dt + g(t, \mathbf{X}_t)dC_t \]

is equivalent to the uncertain integral equation

\[ \mathbf{X}_s = \mathbf{X}_0 + \int_0^s f(t, \mathbf{X}_t)dt + \int_0^s g(t, \mathbf{X}_t)dC_t. \]

The \( n \)-order uncertain differential equation in the form of

\[
\frac{d^n X_t}{dt^n} = \left( u_{0t}, u_{1t}X_t, u_{2t}\frac{dX_t}{dt}, \cdots, u_{nt}\frac{d^{n-1}X_t}{dt^{n-1}} \right) + \left( v_{0t}, v_{1t}X_t, v_{2t}\frac{dX_t}{dt}, \cdots, v_{nt}\frac{d^{n-1}X_t}{dt^{n-1}} \right) \frac{dC_t}{dt}
\]

is called a linear \( n \)-order uncertain differential equation and it can be transformed into a linear \( n \)-dimensional uncertain differential equation

\[ d\mathbf{X}_t = (U_{1t}\mathbf{X}_t + U_{2t})dt + (V_{1t}\mathbf{X}_t + V_{2t})dC_t, \]

driven by the Liu process, where

\[ U_{1t} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \]

\[ U_{2t} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \]

\[ V_{1t} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \]

\[ V_{2t} = [0, \cdots, 0, u_{0t}]^T, \quad V_{2t} = [0, \cdots, 0, v_{0t}]^T. \]
4. Stability of high-order uncertain differential equations

In this section, we will discuss the stability in measure and in mean for high-order uncertain differential equation. For an $n$-dimensional vector $x = [x_1, x_2, \ldots, x_m]^T$ and an $m \times n$ matrix $A = [a_{ij}]$, we use the infinite normal $|x| = \sqrt{\sum_{i=1}^{m} |x_i|}$, $|A| = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$.

4.1. Stability in measure

In this subsection, we investigate the stability in measure for high-order uncertain differential equation and give a stability theorem.

Definition 4.1. The $n$-order uncertain differential equation

$$\frac{d^n X_t}{dt^n} = f\left(t, X_t, \frac{dX_t}{dt}, \ldots, \frac{d^{n-1}X_t}{dt^{n-1}}\right) + g\left(t, X_t, \frac{dX_t}{dt}, \ldots, \frac{d^{n-1}X_t}{dt^{n-1}}\right) \frac{dC_t}{dt}$$

is said to be stable in measure if for any given $\varepsilon > 0$, we have

$$\lim_{|X_0 - Y_0| \to 0} M\{\sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon\} = 1, \quad (8)$$

where $X_t$ and $Y_t$ are any two solutions of its equivalent equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

with different initial values $X_0$ and $Y_0$.

Example 4.1. The following uncertain differential equation

$$\frac{d^2 X_t}{dt^2} = -\frac{1}{t} \exp\left(-\frac{dX_t}{dt}\right) + \exp\left(-\frac{dX_t}{dt}\right) \frac{dC_t}{dt},$$

and its equivalent equation is

$$dX_t = \left(\begin{array}{c} \frac{dX_t}{dt} \\ -\frac{1}{t} \exp\left(-\frac{dX_t}{dt}\right) \\ 0 \end{array}\right) dt + \left(\begin{array}{c} 0 \\ \exp\left(-\frac{dX_t}{dt}\right) \\ 0 \end{array}\right) dC_t.$$

It two solutions with different initial values $X_0$ and $Y_0$ are

$$X_t = \ln(1 + 1/t^2 + C_t) \cdot X_0,$$

$$Y_t = \ln(1 + 1/t^2 + C_t) \cdot Y_0,$$

respectively. Then we have

$$|X_t - Y_t| = \left|\ln(1 + 1/t^2 + C_t) \cdot (X_0 - Y_0)\right| \leq \ln(1 + 1/t^2 + C_t) \cdot |X_0 - Y_0|.$$

As a result, for any given $\varepsilon > 0$, we have

$$\lim_{|X_0 - Y_0| \to 0} M\{\sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon\} \leq \lim_{|X_0 - Y_0| \to 0} M\{\sup_{t \geq 0} |\ln(1 + 1/t^2 + C_t)| \cdot |X_0 - Y_0| \leq \varepsilon\} = 1.$$

Thus the 2-order uncertain differential equation is stable in measure.

Example 4.2. The following second order uncertain differential equation

$$\frac{d^2 X_t}{dt^2} = \frac{dX_t}{dt} + \frac{dX_t}{dt} \frac{dC_t}{dt}, \quad (10)$$

and its equivalent equation is

$$dX_t = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right) X_t dt + \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) X_t dC_t.$$

It two solutions with different initial values $X_0$ and $Y_0$ are

$$X_t = \left(\begin{array}{c} 0 \\ \exp(t) \\ 0 \end{array}\right) \cdot X_0,$$

$$Y_t = \left(\begin{array}{c} 0 \\ \exp(t + C_t) \\ 0 \end{array}\right) \cdot Y_0,$$

respectively. Then we have

$$|X_t - Y_t| = \left|\begin{array}{c} 0 \\ \exp(t) \\ 0 \end{array}\right| \cdot (X_0 - Y_0) \leq \left|\begin{array}{c} 0 \\ \exp(t + C_t) \\ 0 \end{array}\right| \cdot |X_0 - Y_0| = \exp(t + C_t) \cdot |X_0 - Y_0|.$$
As a result, for any given $\varepsilon > 0$, we have
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{ \sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon \}
\leq \lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{ \sup_{t \geq 0} \exp(t + C_t) \cdot |X_0 - Y_0| \leq \varepsilon \}
\leq \mathcal{M}\{C_t \geq 0, \forall t \geq 0\} \wedge \lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{ \sup_{t \geq 0} t \cdot |X_0 - Y_0| \leq \varepsilon \}
= 1/2.
\]

Thus the 2-order uncertain differential equation is not stable in measure.

**Theorem 4.1.** Suppose the high-order uncertain differential equation
\[
\frac{d^n X_t}{dt^n} = f(t, X_t, \frac{dX_t}{dt}, \ldots, \frac{d^{n-1} X_t}{dt^{n-1}}) + g(t, X_t, \frac{dX_t}{dt}, \ldots, \frac{d^{n-1} X_t}{dt^{n-1}}) dC_t
\]
is stable in measure if the coefficient functions $f(t, X_t)$ and $g(t, X_t)$ satisfy the strong Lipschitz condition
\[
|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)|
\leq L_t |x - y|, \quad \forall x, y \in \mathbb{R}^n, t \geq 0
\]
where $|x - y| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|}$ and $L_t$ are functions satisfying
\[
\int_{0}^{+\infty} L_t dt < +\infty.
\]

**Proof.** Assume that $X_t$ and $Y_t$ are two solutions of the $n$-order uncertain differential Equation (13) with two different initial values $X_0$ and $Y_0$, respectively, i.e.,
\[
X_t = X_0 + \int_{0}^{t} f(s, X_s)ds + \int_{0}^{t} g(s, X_s)dC_s,
\]
\[
Y_t = Y_0 + \int_{0}^{t} f(s, Y_s)ds + \int_{0}^{t} g(s, Y_s)dC_s.
\]

Then we have
\[
X_t - Y_t
= (X_0 - Y_0) + \int_{0}^{t} (f(s, X_s) - f(s, Y_s)) ds + \int_{0}^{t} (g(s, X_s) - g(s, Y_s)) dC_s.
\]

According to the strong Lipschitz condition, we can obtain
\[
|X_t(\gamma) - Y_t(\gamma)|
\leq |X_0 - Y_0| + \left| \int_{0}^{t} f(s, X_s(\gamma)) - f(s, Y_s(\gamma)) ds \right|
+ \left| \int_{0}^{t} g(s, X_s(\gamma)) - g(s, Y_s(\gamma)) dC_s(\gamma) \right|
\leq |X_0 - Y_0| + \int_{0}^{t} |f(s, X_s(\gamma)) - f(s, Y_s(\gamma))| ds
+ \int_{0}^{t} |g(s, X_s(\gamma)) - g(s, Y_s(\gamma))| K(\gamma) ds
\]
where $K(\gamma)$ is the Lipschitz constants of $C_t(\gamma)$. By the Gronwall’s inequality, for any $t \geq 0$, we can also obtain
\[
|X_t(\gamma) - Y_t(\gamma)| 
\leq |X_0 - Y_0| \exp \left( \int_{0}^{t} (1 + K(\gamma)) L_s \, ds \right) 
\leq |X_0 - Y_0| \exp \left( (1 + K(\gamma)) \int_{0}^{+\infty} L_s \, ds \right).
\]
So we have
\[
\sup_{t \geq 0} |X_t - Y_t| 
\leq |X_0 - Y_0| \exp \left( \int_{0}^{+\infty} L_s \, ds \right) 
\cdot \exp \left( K \int_{0}^{+\infty} L_s \, ds \right) \quad (16)
\]
almost surely, where $K$ is a nonnegative uncertain variable such that
\[
\mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} 
\geq 2 \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3} \sigma} \right) \right)^{-1} - 1.
\]
by Theorem 3.1.

Then for any given $\epsilon \geq 0$, there exists a real number $T = T(\epsilon)$ such that
\[
\mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq T\} \geq 1 - \epsilon.
\]
We take
\[
\delta = \exp \left( -(1 + T) \int_{0}^{+\infty} L_s \, ds \right) \epsilon.
\]
Then we have $|X_t(\gamma) - Y_t(\gamma)| \leq \delta$, $\forall t \geq 0$ provided that $|X_0 - Y_0| \leq \delta$ and $K(\gamma) \leq T$. So if $|X_0 - Y_0| \leq \delta$ we have
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\left\{ \sup_{t \geq 0} |X_t - Y_t| \leq \epsilon \right\} = 1.
\]
Hence, it follows from the definition of stability in measure that the high-order uncertain differential equation is stable in measure under the strong Lipschitz condition. The theorem is proved.

**Example 4.3.** The following 2-order uncertain differential equation
\[
\frac{d^2 X_t}{dr^2} = -\exp(-r^2)\frac{dX_t}{dr} + \exp(-t)\frac{dX_t}{dr} \cdot \frac{dC_t}{dt} \quad (17)
\]
Its equivalent equation is
\[
\frac{dX_t}{dr} = \begin{pmatrix} 0 & \exp(-r^2) \\ 0 & - \exp(-t^2) \end{pmatrix} X_t \, dt 
+ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \, X_t \, dC_t.
\]
Note that the function $f(t, x) = \exp(-r^2) \cdot x$ satisfies
\[
|f(t, x) - f(t, y)| \leq \exp(-r^2) \cdot |x - y|
\]
and the function $g(t, x) = \exp(-t) \cdot x$ satisfies
\[
|g(t, x) - g(t, y)| \leq \exp(-t) \cdot |x - y|.
\]
Since the function $L_t = \exp(-r^2) + \exp(-t)$ satisfies
\[
\int_{0}^{+\infty} L_t \, dt < +\infty,
\]
the 2-order uncertain differential equation is stable in measure.

**Remark 4.1.** Theorems 4.1, 4.2 give the sufficient condition but not the necessary condition for high-order uncertain differential equation being stable in measure.

**4.2. Stability in mean**

In this subsection, we investigate the stability in mean for high-order uncertain differential equation and give some stability theorems.

**Definition 4.2.** The high-order uncertain differential equation
\[
\frac{d^n X_t}{dr^n} = f \left( t, X_t, \frac{dX_t}{dr}, \cdots, \frac{d^{n-1} X_t}{dr^{n-1}} \right) 
+ g \left( t, X_t, \frac{dX_t}{dr}, \cdots, \frac{d^{n-1} X_t}{dr^{n-1}} \right) \frac{dC_t}{dt}
\]
is said to be stable in mean if for any given \( \varepsilon > 0 \), we have

\[
\lim_{|X_0 - Y_0| \to 0} E \left[ \sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon \right] = 1, \tag{18}
\]

where \( X_t \) and \( Y_t \) are any two solutions of its equivalent equation

\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t
\]

with different initial values \( X_0 \) and \( Y_0 \).

**Example 4.4.** The following uncertain differential equation

\[
d^2X_t = \frac{dX_t}{dt} = -\frac{dX_t}{dt} - \frac{dX_t}{dt} \cdot \frac{dC_t}{dt}, \tag{19}
\]

It two solutions with different initial values \( X_0 \) and \( Y_0 \) are

\[
X_t = X_0 \int_0^t \exp(-s + C_s)ds,
\]

\[
Y_0 = Y_0 \int_0^t \exp(-s + C_s)ds,
\]

respectively. Then we have

\[
|X_t - Y_t| = \left| \int_0^t \exp(-s + C_s)ds \cdot (X_0 - Y_0) \right|
\]

\[
\leq \int_0^t \exp(-s + C_s)ds \cdot |X_0 - Y_0|. \tag{20}
\]

As a result, for any given \( \varepsilon > 0 \), we have

\[
\lim_{|X_0 - Y_0| \to 0} E \left[ \sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon \right]
\]

\[
\leq E \left[ \sup_{t \geq 0} \int_0^t \exp(-s + C_s)ds \cdot |X_0 - Y_0| \leq \varepsilon \right]
\]

\[
= 1.
\]

Thus the 2-order uncertain differential equation is stable in mean.

**Theorem 4.3.** Suppose the \( n \)-order uncertain differential equation

\[
\frac{d^n X_t}{dt^n} = f(t, X_t, \frac{dX_t}{dt}, \cdots, \frac{d^{n-1} X_t}{dt^{n-1}}) + g(t, X_t, \frac{dX_t}{dt}, \cdots, \frac{d^{n-1} X_t}{dt^{n-1}}) \frac{dC_t}{dt}, \tag{21}
\]

and its equivalent equation

\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t
\]

is stable in mean if the coefficient functions \( f(t, X_t) \) and \( g(t, X_t) \) satisfy the strong Lipschitz condition

\[
|f(t, x) - f(t, y)| \leq L_1|x - y|,
\]

\[
|g(t, x) - g(t, y)| \leq L_2|x - y|,
\]

\[
\forall x, y \in \mathbb{R}^m, t \geq 0 \tag{22}
\]

where \( L_1 \) and \( L_2 \) are two functions satisfying

\[
\int_0^{+\infty} L_1 dt < +\infty, \int_0^{+\infty} L_2 dt < \frac{\pi}{\sqrt{3}} \tag{23}
\]

**Proof.** Assume that \( X_t \) and \( Y_t \) are two solutions of the \( n \)-order uncertain differential Equation (27) with two different initial values \( X_0 \) and \( Y_0 \), respectively, i.e.,

\[
X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dC_s,
\]

\[
Y_t = Y_0 + \int_0^t f(s, Y_s)ds + \int_0^t g(s, Y_s)dC_s.
\]

Then we have

\[
X_t - Y_t
\]

\[
= (X_0 - Y_0) + \int_0^t (f(s, X_s) - f(s, Y_s)) ds
\]

\[
+ \int_0^t (g(s, X_s) - g(s, Y_s)) dC_s
\]

\[
= (X_0 - Y_0) + \int_0^t (f(s, X_s) - f(s, Y_s)) ds
\]

\[
+ \int_0^t (g(s, X_s) - g(s, Y_s)) dC_s.
\]

According to the strong Lipschitz condition, we can obtain

\[
|X_t(\gamma) - Y_t(\gamma)|
\]

\[
\leq |X_0 - Y_0| + \int_0^t f(s, X_s(\gamma)) - f(s, Y_s(\gamma))ds
\]

\[
+ \left\| \int_0^t g(s, X_s(\gamma)) - g(s, Y_s(\gamma))dC_s(\gamma) \right\|
\]

\[
\leq |X_0 - Y_0| + \int_0^t |f(s, X_s(\gamma)) - f(s, Y_s(\gamma))| ds
\]

\[
+ \int_0^t |g(s, X_s(\gamma)) - g(s, Y_s(\gamma))| K(\gamma) ds
\]
where $K(\gamma)$ is the Lipschitz constants of $C_\gamma$. By the Gronwall’s inequality, for any $t \geq 0$, we can also obtain

$$|X_t(\gamma) - Y_t(\gamma)|$$

$$\leq |X_0 - Y_0| \exp \left( \int_0^t (1 + K(\gamma)) L_s ds \right)$$

$$\leq |X_0 - Y_0| \exp \left( (1 + K(\gamma)) \int_0^+ L_s ds \right).$$

So we have

$$\sup_{t \geq 0} |X_t - Y_t|$$

$$\leq |X_0 - Y_0| \exp \left( \int_0^+ L_s ds \right)$$

$$\cdot \exp \left( K \int_0^+ L_s ds \right)$$

(24)

almost surely, where $K$ is a nonnegative uncertain variable, we have

$$\mathbb{M}\{\gamma \in \Gamma | K(\gamma) \leq x\}$$

$$\geq 2 \left( 1 + \exp \left( \frac{-\pi x}{2\sqrt{3}\sigma} \right) \right)^{-1} - 1.$$  

by Theorem 3.1.

Taking expected value on both sides of (24), we have

$$E \left[ \sup_{t \geq 0} |X_t - Y_t| \right]$$

$$\leq |X_0 - Y_0| \exp \left( \int_0^+ L_{1s} ds \right)$$

$$\cdot E \left[ \exp \left( K \int_0^+ L_{2s} ds \right) \right].$$

Since

$$\int_0^+ L_{1s} ds < +\infty,$$

we have

$$E \left[ \exp \left( \int_0^+ L_{1s} ds \right) \right] < +\infty,$$

and if

$$\int_0^+ L_{2s} ds < \frac{\pi}{\sqrt{3}},$$

then we have

$$E \left[ \exp \left( K \int_0^+ L_{2s} ds \right) \right]$$

$$= \int_0^+ \mathbb{M}\left\{ \exp \left( K \int_0^+ L_{2s} ds \right) \geq x \right\} dx$$

$$= \int_0^1 \mathbb{M}\left\{ \exp \left( K \int_0^+ L_{2s} ds \right) \geq x \right\} dx$$

$$+ \int_1^+ \mathbb{M}\left\{ \exp \left( K \int_0^+ L_{2s} ds \right) \geq x \right\} dx$$

$$\leq 1 + \int_1^+ \mathbb{M}\left\{ \exp \left( \int_0^+ L_{2s} ds \right) \geq x \right\} dx$$

$$= 1 + \int_0^+ L_{2s} ds$$

$$\cdot \int_0^+ \exp \left( y \int_0^+ L_{2s} ds \right) \mathbb{M}\{K \geq y\} dy$$

$$= 1 + \int_0^+ L_{2s} ds \int_0^+ \exp \left( y \int_0^+ L_{2s} ds \right)$$

$$\cdot \left\{ 1 - \left( 2 \left( 1 + \exp \left( \frac{-\pi x}{2\sqrt{3}\sigma} \right) \right)^{-1} - 1 \right) \right\} dy$$

$$= 1 + 2 \left( \int_0^+ L_{2s} ds \right) \cdot \int_0^+ \exp \left( y \int_0^+ L_{2s} ds \right)$$

$$\cdot \left( 1 + \exp \left( \frac{-\pi x}{2\sqrt{3}\sigma} \right)^{-1} \right) dy$$

$$= 1 + 2 \int_1^+ \left\{ 1 + \frac{x}{\sqrt{3}} \int_0^+ L_{2s} ds \right\}^{-1} dx$$

$$< +\infty.$$
Hence, it follows from the definition of stability in mean that the high-order uncertain differential equation is stable in mean under the strong Lipschitz condition. The theorem is proved.

**Example 4.5.** The following 2-order uncertain differential equation

\[
\frac{d^2 X_t}{dt^2} = -\exp(-t^2) \frac{dX_t}{dt} + \exp(-t) \frac{dX_t}{dt} \cdot \frac{dC_t}{dt},
\]

Its equivalent equation is

\[
dX_t = \begin{pmatrix} 0 & \exp(-t^2) \\ 0 & -\exp(-t^2) \end{pmatrix} X_t dt + \begin{pmatrix} 0 & 0 \\ 0 & \exp(-t) \end{pmatrix} X_t dC_t.
\]

Note that the function \( f(t, x) = \exp(-t^2) \cdot x \) satisfies

\[
|f(t, x) - f(t, y)| \leq \exp(-t^2) \cdot |x - y|
\]

and the function \( g(t, x) = \exp(-t) \cdot x \) satisfies

\[
|g(t, x) - g(t, y)| \leq \exp(-t) \cdot |x - y|.
\]

Since the function \( L_{1t} = \exp(-t^2) \) satisfies

\[
\int_0^{+\infty} L_{1t} dt < +\infty,
\]

and \( L_{2t} = \exp(-t) \) satisfies

\[
\int_0^{+\infty} L_{2t} dt = 1 < \frac{\pi}{\sqrt{3}},
\]

the 2-order uncertain differential equation is stable in mean.

**Remark 4.2** Theorem 4.3 gives the sufficient condition but not the necessary condition for high-order uncertain differential equation being stable in mean.

### 4.3. Relationship of stability in measure and stability in mean

In this section, the relationship between stability in measure and stability in mean for a high-order uncertain differential equation is discussed.

**Theorem 4.4.** If an uncertain differential equation is stable in mean, then it is stable in measure.

**Proof.** It follows from the definition of stability in mean that for two solutions \( X_t \) and \( Y_t \) with different initial values \( X_0 \) and \( Y_0 \), we have

\[
\lim_{|X_0 - Y_0| \to 0} \sup_{t \geq 0} |X_t - Y_t| = 0, \forall t \geq 0.
\]

Then for any given real number \( \varepsilon \geq 0 \), we have

\[
\lim_{|X_0 - Y_0| \to 0} \frac{\mathbb{M}\left\{\sup_{t \geq 0} |X_t - Y_t| \geq \varepsilon\right\}}{\varepsilon} = 0,
\]

\( \forall t \geq 0 \) by Markov inequality. Thus, for a high-order uncertain differential equation is stable in mean implies stable in measure.

**Remark 4.3.** For a high-order uncertain differential equation, generally, stability in measure does not imply stability in mean.

**Example 4.6.** The following 2-order uncertain differential equation

\[
\frac{d^2 X_t}{dt^2} = -\exp(-t^2) \frac{dX_t}{dt} + \exp(-2t) \frac{dX_t}{dt} \cdot \frac{dC_t}{dt}.
\]

Its equivalent equation is

\[
dX_t = \begin{pmatrix} 0 & \exp(-t^2) \\ 0 & -\exp(-2t) \end{pmatrix} X_t dt + \begin{pmatrix} 0 & 0 \\ 0 & \exp(-2t) \end{pmatrix} X_t dC_t.
\]

Note that the function \( f(t, x) = \exp(-t^2) \cdot x \) satisfies

\[
|f(t, x) - f(t, y)| \leq \exp(-t^2) \cdot |x - y|
\]

and the function \( g(t, x) = \exp(-2t) \cdot x \) satisfies

\[
|g(t, x) - g(t, y)| \leq \exp(-2t) \cdot |x - y|.
\]

Since the function \( L_t = \exp(-t^2) + \exp(-2t) \) satisfies

\[
\int_0^{+\infty} L_t dt < +\infty,
\]
the 2-order uncertain differential equation is stable in measure. But the function $L_{1t} = \exp(-t^2)$ satisfies
\[ \int_{0}^{+\infty} L_{1t} \, dt < +\infty, \]
and $L_{2t} = \exp(-2t)$ satisfies
\[ \int_{0}^{+\infty} L_{2t} \, dt = 2 > \frac{\pi}{\sqrt{3}}, \]
the 2-order uncertain differential equation is not stable in mean.

5. Conclusions

High-order uncertain differential equations describe the dynamic uncertain systems involving high-order differentials. It is essentially a system of uncertain differential equations. This paper proposed some concepts of stability for a high-order uncertain differential equation. Some theorems on stability in measure and in mean were proved, in which the sufficient conditions the high-order uncertain differential equation being stable in measure and in mean were provided. In addition, the relationship between stability in measure and stability in mean were discussed about the high-order uncertain differential equation.

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References