Existence and Uniqueness Theorem on Uncertain Delay Differential Equation with Local Lipschitz Condition

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Abstract

Uncertain delay differential equation (UDDE) is a type of differential equations driven by Liu process. It has been proved that uncertain delay differential equation has a unique solution in the finite domain, under the conditions that the coefficients are global Lipschitz continuous. This paper will extend this existence and uniqueness theorem from finite domain to infinite domain under the local Lipschitz condition and linear growth condition.

Keywords: uncertainty theory, uncertain delay differential equation, local Lipschitz condition

1 Introduction

Probability theory, which was founded by Kolmogorov in 1933, has been used to model indeterminacy phenomena for a long time. A premise of applying probability theory is that the obtained probability distribution is close enough to the real frequency. However, due to some privacy or technological reasons, we usually have little or no sample data to estimate the probability distribution of the variable. In this case, we have to invite some experts to give their belief degree that each event will occur. A lot of surveys showed that human beings usually estimate a much wider range of values than the object actually takes (Liu \cite{16}). This conservatism of human beings makes the belief degree deviate far from the frequency. Hence, it is inappropriate to treat the belief degree as a random variable and to model indeterminacy phenomena in this case by probability theory.

In order to deal with belief degree mathematically, Liu proposed uncertainty theory in 2007 \cite{7} by uncertain measure and refined it in 2010 \cite{11}. So far, it has been applied to many fields, and has brought many branches such as uncertain programming \cite{9}, uncertain risk analysis \cite{12}, uncertain inference \cite{13} and uncertain logic \cite{15}. To learn more, the readers may refer to Liu’s works \cite{7, 16}.

In order to describe the evolution of an uncertain phenomenon, Liu \cite{8} proposed the concept of uncertain process, and designed a canonical Liu process which is an uncertain process with stationary and independent normal uncertain increments \cite{10}. Meanwhile, Liu \cite{10} founded uncertain calculus to deal with the integral and differential of an uncertain process with respect to Liu process, and Chen and Ralescu \cite{3} proposed an uncertain integral with respect to general Liu process. Following that, Liu and Yao \cite{14} extended uncertain integral...


In many system models, people usually assume that the future state of the system is independent with the past states and is determined solely by the present. However, some real phenomena depend not only on the state of the system at a given instant but depend upon the history of the trajectory until this instant. In this case, it is inappropriate to stick with uncertain differential equation. Here, we had better employ uncertain delay differential equation to establish a mathematical formulation for such system. For uncertain delay differential equation, scholars began to explore existence and uniqueness theorem of solutions since 2010. Barbacioru [1] first proved a local existence and uniqueness result for a special type of uncertain delay differential equation (UDDE). Ge and Zhu [5] presented an existence and uniqueness theorem of solution for UDDE in the finite domain under the global Lipschitz condition and linear growth condition by Banach fixed point theorem in 2012.

However, the global global Lipschitz condition of the existing theorem is too strict. As is known, there are few functions satisfying global Lipschitz continuity. The vast majority of functions are only local Lipschitz continuous. In this paper, we aim at a general existence and uniqueness theorem on any UDDE under local Lipschitz condition. The remainder of this paper is organized as follows. The next section is intended to introduce some basic concepts and theorems of uncertainty theory used throughout this paper. Section 3 introduces the concept of UDDE and an existence and uniqueness theorem under the global Lipschitz condition. After that, the existence and uniqueness theorem on UDDE under the local Lipschitz condition is proved in Section 4. Finally, we make a brief conclusion in Section 5.

2 Preliminary

In this section, we will introduce some basic concepts and theorems about uncertainty theory which are used throughout this paper.

In order to provide a quantitative measurement that an uncertain phenomenon will occur, an axiomatic
The definition of uncertain measure is defined as follows.

**Definition 2.1** (Liu [7]) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( M \) is called an **uncertain measure** if it satisfies the following axioms:

Axiom 1. (**Normality Axiom**) \( M\{\Gamma\} = 1 \);

Axiom 2. (**Duality Axiom**) \( M\{\Lambda\} + M\{\Lambda^c\} = 1 \) for any \( \Lambda \in \mathcal{L} \);

Axiom 3. (**Subadditivity Axiom**) For every countable sequence of \( \{\Lambda_i\} \in \mathcal{L} \), we have

\[
M\left\{\bigcup_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.
\]

The triplet \((\Gamma, \mathcal{L}, M)\) is called an **uncertainty space**, and each element \( \Lambda \) in \( \mathcal{L} \) is called an **event**. In addition, in order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [10] by the following product axiom:

Axiom 4. (**Product Axiom**) Let \((\Gamma_k, \mathcal{L}_k, M_k)\) be uncertainty spaces for \( k = 1, 2, \cdots \). The product uncertain measure \( M \) is an uncertain measure satisfying

\[
M\left\{\prod_{k=1}^{\infty} \Lambda_k \right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \cdots \), respectively.

**Definition 2.2** (Liu [7]) An **uncertain variable** \( \xi \) is a measurable function from an uncertainty space \((\Gamma, \mathcal{L}, M)\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set

\[
\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}
\]

is an event.

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The uncertain process is defined as follows.

**Definition 2.3** (Liu [8]) Let \((\Gamma, \mathcal{L}, M)\) be an uncertainty space and let \( T \) be a totally ordered set (e.g., time). An **uncertain process** is a function \( X_t(\gamma) \) from \( T \times (\Gamma, \mathcal{L}, M) \) to the set of real numbers such that

\[
\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}
\]

is an event for any Borel set \( B \) at each time \( t \). For each \( \gamma \in \Gamma \), \( X_t(\gamma) \) is called a **sample path** of \( X_t \).

**Definition 2.4** (Liu [10]) An uncertain process \( C_t \) \( (t \geq 0) \) is called a **canonical Liu process** if

(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,

(ii) \( C_t \) is a stationary independent increment process,

(iii) every increment \( C_{t+t} - C_t \) is a normal uncertain variable with expected value 0 and variance \( t^2 \), whose uncertainty distribution is

\[
\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3t}}\right)\right)^{-1}.
\]
Based on the canonical Liu process, Liu [10] proposed the concept of uncertain integral, which is regarded as an uncertain counterpart of the Ito integral.

**Definition 2.5** (Liu [10]) Let $X_t$ be an uncertain process and let $C_t$ be a canonical Liu process. For any partition of the closed interval $[a,b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as
\[
\Delta = \sup_{1 \leq i \leq k} |t_{i+1} - t_i|.
\]
Then the uncertain integral of $X_t$ with respect to $C_t$ is defined by
\[
\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})
\]
provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be uncertain integrable.

**Definition 2.6** (Liu [8]) Suppose $C_t$ is a canonical Liu process, and $f$ and $g$ are two functions. Given an initial value $X_0$,
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t
\]
is called an uncertain differential equation with an initial value $X_0$. A solution is an uncertain process $X_t$ that satisfies Equation (1) identically in $t$.

**Theorem 2.1** (Chen and Liu [2]) Suppose $C_t$ is a canonical Liu process, and $X_t$ is an integrable uncertain process on $[a,b]$ with respect to $t$. Then the inequality
\[
\left| \int_a^b X_t dC_t \right| \leq K(\gamma) \int_a^b |X_t(\gamma)| d\gamma
\]
holds, where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$.

### 3 Uncertain delay differential equation

Let $X_t$ be an uncertain process and let $C_t$ be a canonical Liu process. Consider the uncertain delay differential equation [1]:
\[
\begin{aligned}
&dX_t = f(t, X_t, X_{t-\tau})dt + g(t, X_t, X_{t-\tau})dC_t, \quad t \in [0, +\infty) \\
&X_t = \varphi(t), \quad t \in [-\tau, 0]
\end{aligned}
\]
where $\tau > 0$ is called a time delay, and $f$, $g$ and $\varphi$ are three given continuous functions.

In 2012, Ge and Zhu [5] gave the following existence and uniqueness theorem of solutions for UDDEs in the finite domain $[0, T]$.

**Theorem 3.1** (Ge and Zhu [5]) The UDDE
\[
\begin{aligned}
&dX_t = f(t, X_t, X_{t-\tau})dt + g(t, X_t, X_{t-\tau})dC_t, \quad t \in [0, T] \\
&X_t = \varphi(t), \quad t \in [-\tau, 0]
\end{aligned}
\]
has a unique solution if the coefficients \( f(t, x, y) \) and \( g(t, x, y) \) satisfy the global Lipschitz condition

\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| + |g(t, x_1, y_1) - g(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|), \quad \forall x_1, x_2, y_1, y_2 \in \mathbb{R}, t \in [0, T]
\]

and the linear growth condition

\[
|f(t, x, y)| + |g(t, x, y)| \leq L(1 + |x| + |y|), \quad \forall x, y \in \mathbb{R}, t \in [0, T]
\]

for some constant \( L \). Moreover, the solution is sample-continuous.

In fact, the global Lipschitz condition in Theorem 3.1 is too strict, and few equations can satisfy this condition. In this paper, we will extend this theorem by replacing global Lipschitz condition by local Lipschitz condition, and extend the finite domain \([0, T]\) into infinite domain \([0, +\infty]\).

For the sake of simplicity, our discussion is not based on the form of UDDE (3) but on its equivalent integral form, or say, uncertain integral equation

\[
\begin{align*}
X_t &= X_0 + \int_0^t f(s, X_s, X_{s-\tau})ds + \int_0^t g(s, X_s, X_{s-\tau})dC_s, \quad t \in [0, +\infty) \\
X_t &= \varphi(t), \quad t \in [-\tau, 0].
\end{align*}
\]

### 4 Existence and uniqueness theorem with local Lipschitz condition

According to the definition of canonical Liu Process \( C_t \), almost all sample paths of \( C_t \) are Lipschitz continuous functions. That is, there exists a set \( \Gamma_0 \) in \( \Gamma \) with \( \mathcal{M}\{\Gamma_0\} = 1 \) such that for any \( \gamma \in \Gamma_0 \), \( C_t(\gamma) \) is Lipschitz continuous. For the sake of simplicity, in this paper we set \( \Gamma_0 = \Gamma \). Thus, for each \( \gamma \), there exists a positive number \( K(\gamma) \) such that

\[
|C_s(\gamma) - C_t(\gamma)| \leq K(\gamma)|s - t|, \quad \forall s, t \geq 0.
\]

Besides, the uncertain integral of \( C_t \) is equivalent to Riemann-Stieltjes integral from the point of each sample path. Hence, we can focus on the following integral equation

\[
\begin{align*}
X_t(\gamma) &= X_0(\gamma) + \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma))ds + \int_0^t g(s, X_s(\gamma), X_{s-\tau}(\gamma))dC_s(\gamma), \quad t \in [0, +\infty) \\
X_t(\gamma) &= \varphi(t), \quad t \in [-\tau, 0].
\end{align*}
\]

Our aim is to prove that under some conditions, for each sample path \( \gamma \), the integral equation (5) has a unique solution on \([0, +\infty)\).

At first, we prove the existence and uniqueness theorem on UDDE in a local interval \([t_0, t_0 + \alpha]\) for some positive \( \alpha \). The integral equation (5) becomes

\[
\begin{align*}
X_t(\gamma) &= X_{t_0}(\gamma) + \int_{t_0}^t f(s, X_s(\gamma), X_{s-\tau}(\gamma))ds + \int_{t_0}^t g(s, X_s(\gamma), X_{s-\tau}(\gamma))dC_s(\gamma), \quad t \in [t_0, t_0 + \alpha] \\
X_t(\gamma) &= \varphi(t), \quad t \in [t_0 - \tau, t_0].
\end{align*}
\]
Theorem 4.1 Fixing $\gamma \in \Gamma$, the uncertain integral equation (6) has a unique solution in the interval $[t_0, t_0+\alpha]$ if both the coefficients $f$ and $g$ are locally Lipschitz continuous in $x$.

That is, for each $D = \{(t,x,y)| t \in [t_0, t_0 + a], \ x \in [X_{t_0}(\gamma) - b, X_{t_0}(\gamma) + b], \ y \in \mathbb{R}\}$, there exists a positive constant $L_D$

$$|f(t,x_1,y) - f(t,x_2,y)| \vee |g(t,x_1,y) - g(t,x_2,y)| \leq L_D |x_1 - x_2|$$

where $(t,x_1,y),(t,x_2,y) \in D$, $H = \max_D \{|f(t,x,y)| + K(\gamma)|g(t,x,y)|\}$, $K(\gamma)$ is the Lipschitz constant to $C_t(\gamma)$, and $\alpha = \min\{a,b/H,\tau\}$.

**Proof:** We will prove this theorem in three steps by using the following successive approximations

$$
\begin{align*}
X_t^{(0)}(\gamma) &= X_{t_0}(\gamma) \\
X_{t_0-\gamma}(\gamma) &= X_{t_0-\gamma}(\gamma) \\
X_t^{(n+1)}(\gamma) &= X_t(\gamma) + \int_{t_0}^t f(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))ds + \int_{t_0}^t g(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))dC_s(\gamma), \ t \in [t_0, t_0 + \alpha] \\
X_t(\gamma) &= \varphi(t), \ t \in [t_0 - \tau, t_0].
\end{align*}
$$

(7)

Obviously, for each $n \geq 0, \{X_t^{(n)}(\gamma)\}$ is continuous in $t$.

**Step 1.** In this step, we will prove that $(t,X_t^{(n)}(\gamma),X_{t-\gamma}^{(n)}(\gamma)) \in D, n \geq 0$ when $t \in [t_0, t_0 + \alpha]$.

Here, we employ mathematical induction.

When $n = 0$,

$$
\begin{align*}
\begin{cases}
 t \in [t_0, t_0 + a] \\
 X_t^{(0)}(\gamma) = X_{t_0}(\gamma) \in [X_{t_0}(\gamma) - b, X_{t_0}(\gamma) + b] \\
 X_{t_0-\gamma}(\gamma) = X_{t_0-\gamma}(\gamma) \in \mathbb{R}.
\end{cases}
\end{align*}
$$

Hence the conclusion is obvious established.

Assume that when $t \in [t_0, t_0 + \alpha]$, $(t,X_t^{(n)}(\gamma),X_{t-\gamma}^{(n)}(\gamma)) \in D$, we have

$$
\begin{align*}
|X_t^{(n+1)}(\gamma) - X_t(\gamma)| &= \left| \int_{t_0}^t f(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))ds + \int_{t_0}^t g(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))dC_s(\gamma) \right| \\
&\leq \left| \int_{t_0}^t f(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))ds \right| + K(\gamma) \left| \int_{t_0}^t g(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))ds \right| \\
&\leq \left| \int_{t_0}^t f(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma))ds \right| + K(\gamma) \left| g(s,X_s^{(n)}(\gamma),X_{s-\gamma}^{(n)}(\gamma)) \right| \ ds \\
&\leq H \cdot |t - t_0| \leq H \cdot \alpha \leq b,
\end{align*}
$$

and

$$
X_t^{(n+1)}(\gamma) \in \mathbb{R}.
$$

This means that $(t,X_t^{(n)}(\gamma),X_{t-\gamma}^{(n)}(\gamma)) \in D, \ n = 0, 1, 2, \cdots$, when $t \in [t_0, t_0 + \alpha]$. 

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Step 2. In this step, we will prove that the sequence \( \{X_t^{(n)}(\gamma)\}_{n=0}^{+\infty} \) given by (7) converges uniformly to the solution of integral equation (6) on \([t_0, t_0 + \alpha]\) as \( n \to +\infty \).

First, we will prove

\[
|X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma)| \leq \frac{H(L_D + K(\gamma)L_D)^n}{(n+1)!} |t - t_0|^{n+1}.
\]

Similar to Step 1, we also employ mathematical induction in this step.

When \( n = 0 \),

\[
|X_t^{(1)}(\gamma) - X_t^{(0)}(\gamma)| = \left| \int_{t_0}^t f(s, X_s^{(0)}(\gamma), X_s^{(0)}(\gamma))ds + \int_{t_0}^t g(s, X_s^{(0)}(\gamma), X_s^{(0)}(\gamma))dC_s(\gamma) \right|
\]

\[
\leq \left| \int_{t_0}^t f(s, X_s^{(0)}(\gamma), X_s^{(0)}(\gamma))ds \right| + K(\gamma) \left| \int_{t_0}^t g(s, X_s^{(0)}(\gamma), X_s^{(0)}(\gamma))ds \right|
\]

\[
\leq \int_{t_0}^t \left| f(s, X_s^{(0)}(\gamma), X_s^{(0)}(\gamma)) \right| ds + K(\gamma) \left| g(s, X_s^{(0)}(\gamma), X_s^{(0)}(\gamma)) \right| ds
\]

\[
\leq H \cdot |t - t_0|.
\]

Assume that when \( t \in [t_0, t_0 + \alpha] \), \( |X_t^{(n)}(\gamma) - X_t^{(n-1)}(\gamma)| \leq \frac{H(L_D + K(\gamma)L_D)^{n-1}}{n!} |t - t_0|^n \), we have

\[
|X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma)| = \left| \int_{t_0}^t f \left( s, X_s^{(n)}(\gamma), X_s^{(n)}(\gamma) \right) ds + \int_{t_0}^t g \left( s, X_s^{(n)}(\gamma), X_s^{(n)}(\gamma) \right) dC_s(\gamma) \right|
\]

\[
- \int_{t_0}^t f \left( s, X_s^{(n-1)}(\gamma), X_s^{(n-1)}(\gamma) \right) ds - \int_{t_0}^t g \left( s, X_s^{(n-1)}(\gamma), X_s^{(n-1)}(\gamma) \right) dC_s(\gamma)
\]

\[
\leq \int_{t_0}^t \left| f \left( s, X_s^{(n)}(\gamma), X_s^{(n)}(\gamma) \right) - f \left( s, X_s^{(n-1)}(\gamma), X_s^{(n-1)}(\gamma) \right) \right| ds
\]

\[
+ \int_{t_0}^t \left| g \left( s, X_s^{(n)}(\gamma), X_s^{(n)}(\gamma) \right) - g \left( s, X_s^{(n-1)}(\gamma), X_s^{(n-1)}(\gamma) \right) \right| dC_s(\gamma)
\]

\[
\leq \int_{t_0}^t L_D \left| X_s^{(n)}(\gamma) - X_s^{(n-1)}(\gamma) \right| ds + K(\gamma) \int_{t_0}^t L_D \left| X_s^{(n)}(\gamma) - X_s^{(n-1)}(\gamma) \right| ds
\]

\[
\leq L_D (1 + K(\gamma)) \int_{t_0}^t \left| X_s^{(n)}(\gamma) - X_s^{(n-1)}(\gamma) \right| ds
\]

\[
\leq L_D (1 + K(\gamma)) \frac{H(L_D + K(\gamma)L_D)^{n-1}}{n!} |t - t_0|^n ds
\]

\[
= \frac{H(L_D + K(\gamma)L_D)^n}{n!} \int_{t_0}^t |s - t_0|^n ds
\]

\[
= \frac{H(L_D + K(\gamma)L_D)^n}{(n+1)!} |t - t_0|^{n+1}.
\]

The above inequality gives an upper bound of \( |X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma)| \) on \([t_0, t_0 + \alpha]\), for \( n = 0, 1, 2, \cdots \).
Obviously, \( \forall \varepsilon > 0 \), there exists a integer \( N \) such that

\[
\sum_{n \geq N} \left| X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma) \right| \leq \sum_{n \geq N} \frac{H(L_D + K(\gamma)L_D)^n}{(n+1)!} |t-t_0|^{n+1}
\]

\[
= \frac{H}{L_D + K(\gamma)L_D} \sum_{n \geq N} \frac{(L_D + K(\gamma)L_D)^n}{(n+1)!} |t-t_0|^{n+1}
\]

\[
\leq \frac{H}{L_D + K(\gamma)L_D} \sum_{n \geq N} \frac{(L_D + K(\gamma)L_D)^n}{(n+1)!} \alpha^{n+1}
\]

\[
\leq \frac{H}{L_D + K(\gamma)L_D} \sum_{n \geq N} \frac{(\alpha L_D(1 + K(\gamma)))^{n+1}}{(n+1)!} \left( \lim_{n \to +\infty} \frac{\alpha^{n+1}}{(n+1)!} = 0 \right)
\]

\[
< \varepsilon.
\]

Since \( X_t^{(n)}(\gamma) = X_t^{(n)}(\gamma) + \sum_{i=1}^{n} (X_t^{(i)}(\gamma) - X_t^{(i-1)}(\gamma)) \), the above inequality indicates that \( X_t^{(n)}(\gamma) \) converges uniformly on \([t_0, t_0 + \alpha] \) as \( n \to +\infty \).

Denote

\[
X_t(\gamma) = \lim_{n \to +\infty} X_t^{(n)}(\gamma).
\]

Taking the limit on both sides of the equation

\[
X_t^{(n+1)}(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s^{(n)}(\gamma), X_{s-\tau}^{(n-1)}(\gamma)) ds + \int_{t_0}^{t} g(s, X_s^{(n)}(\gamma), X_{s-\tau}^{(n-1)}(\gamma)) dC_s(\gamma),
\]

we have

\[
X_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s(\gamma), X_{s-\tau}(\gamma)) ds + \int_{t_0}^{t} g(s, X_s(\gamma), X_{s-\tau}(\gamma)) dC_s(\gamma).
\]

That is to say, the sequence \( \{X_t^{(n)}(\gamma)\} \) given by (7) converges uniformly to the solution of integral equation (6) on \([t_0, t_0 + \alpha]\) as \( n \to +\infty \). Since each \( X_t^{(n)}(\gamma) \) is continuous, \( X_t(\gamma) \) is also continuous on \([t_0, t_0 + \alpha]\). This completes the proof of existence.

**Step 3.** In this step, we will prove that \( X_t(\gamma) \) obtained in Step 2 is the unique solution of integral equation (6) on \([t_0, t_0 + \alpha]\).

Suppose that \( \tilde{X}_t(\gamma) \) is another solution of of integral equation (6), i.e.,

\[
\begin{cases}
\tilde{X}_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, \tilde{X}_s(\gamma), \tilde{X}_{s-\tau}(\gamma)) ds + \int_{t_0}^{t} g(s, \tilde{X}_s(\gamma), \tilde{X}_{s-\tau}(\gamma)) dC_s(\gamma), & t \in [t_0, t_0 + \beta] \\
\tilde{X}_t(\gamma) = \varphi(t), & t \in [t_0 - \tau, t_0]
\end{cases}
\]

where \( 0 < \beta \leq \alpha \).

Following the local Lipchitz condition, we have

\[
\left| X_t(\gamma) - \tilde{X}_t(\gamma) \right| = \left| \int_{t_0}^{t} \left( f(s, X_s(\gamma), X_{s-\tau}(\gamma)) - f(s, \tilde{X}_s(\gamma), \tilde{X}_{s-\tau}(\gamma)) \right) ds 
\right.

\[
\left. + \int_{t_0}^{t} \left( g(s, X_s(\gamma), X_{s-\tau}(\gamma)) - g(s, \tilde{X}_s(\gamma), \tilde{X}_{s-\tau}(\gamma)) \right) dC_s(\gamma) \right|
\]

\[
\leq L_D \int_{t_0}^{t} \left| X_s(\gamma) - \tilde{X}_s(\gamma) \right| ds + K(\gamma) L_D \int_{t_0}^{t} \left| X_s(\gamma) - \tilde{X}_s(\gamma) \right| ds
\]

\[
= (L_D + K(\gamma)L_D) \int_{t_0}^{t} \left| X_s(\gamma) - \tilde{X}_s(\gamma) \right| ds.
\]
By Gronwall inequality [6], the above expression can lead to

$$|X_t(\gamma) - X_t(\gamma)| \leq 0 \cdot \exp (t(L_D + K(\gamma)L_D)) = 0,$$

that is, $X_t(\gamma) = \bar{X}_t(\gamma)$, $\forall t \in [t_0, t_0 + \alpha]$. This completes the proof of uniqueness. Up to now, the proof is complete.

According to Theorem 4.1, the integral differential equation (6) has a unique solution on some local interval $[0, t_0 + \alpha]$. Next, Theorem 4.2 will show that the solution of integral differential equation (6) can be extended to the infinite domain $[0, +\infty)$.

**Theorem 4.2** Fixing $\gamma \in \Gamma$, the uncertain integral equation (5) has a unique solution on $[0, +\infty)$ if the coefficients $f$ and $g$ satisfy both local Lipschitz condition appeared in Theorem 4.1 and local linear growth condition, i.e., for each $T > 0$, there exists a constant $C_T$ such that

$$|f(t,x,y)| \vee |g(t,x,y)| \leq C_T(1 + |x| + |y|), \forall x, y \in \mathbb{R}, t \in [0, T].$$

**Proof:** Define $\zeta = \{ t : \text{integral equation (5) has a unique continuous solution on } [0,t]\}$, and $\zeta = \sup \zeta$. According to Theorem 4.1, the set $\zeta$ is nonempty. We will prove that $\zeta = +\infty$.

Assume that $\zeta < +\infty$, and a contradiction will be derived. As the definition, $X_t(\gamma)$ is the unique solution of integral equation (5) on $[0, \zeta)$. Then,

$$|X_t(\gamma)| + |X_{t-\tau}(\gamma)| \leq |X_0(\gamma)| + \left| \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right| + K(\gamma) \left| \int_0^t g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right|$$

$$+ |X_{t-\tau}(\gamma)| + \left| \int_0^{t-\tau} f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right| + K(\gamma) \left| \int_0^{t-\tau} g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right|$$

$$= |X_0(\gamma)| + |X_{t-\tau}(\gamma)| + \left| \int_{-\tau}^0 f(s, X_s(\gamma), X_{s-\tau}(\gamma)) + K(\gamma) g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right|$$

$$+ \left| \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma)) + K(\gamma) g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right|$$

$$\leq |X_0(\gamma)| + |X_{t-\tau}(\gamma)| + M + C\zeta (1 + K(\gamma)) \int_{-\tau}^0 1 + |X_s(\gamma)| + |X_{s-\tau}(\gamma)| \, ds$$

$$+ C\zeta + K(\gamma) \int_0^t 1 + |X_s(\gamma)| + |X_{s-\tau}(\gamma)| \, ds$$

$$\leq |X_0(\gamma)| + |X_{t-\tau}(\gamma)| + M + 2\zeta C\zeta (1 + K(\gamma))$$

$$+ 2C\zeta (1 + K(\gamma)) \int_0^t |X_s(\gamma)| + |X_{s-\tau}(\gamma)| \, ds, \forall t \in [0, \zeta).$$

where $M = \left| \int_{-\tau}^0 f(s, X_s(\gamma), X_{s-\tau}(\gamma)) + K(\gamma) g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right|$.

Setting $A = |X_0(\gamma)| + |X_{t-\tau}(\gamma)| + M + 2\zeta C\zeta (1 + K(\gamma))$, by Gronwall inequality [6], we have

$$|X_t(\gamma)| + |X_{t-\tau}(\gamma)| \leq A \exp (2C\zeta (1 + K(\gamma))) \zeta = H_0 < +\infty, \forall t \in [0, \zeta),$$

and this completes the proof of Theorem 4.2.
that is, $|X_t(\gamma)| + |X_{t-\tau}(\gamma)|$ is bounded on $[0, \varsigma)$.

Hence,

$$
|X_{t_1}(\gamma)| + |X_{t_2}(\gamma)| \leq \left| \int_{t_1}^{t_2} f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds \right| + \left| \int_{t_1}^{t_2} g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, dC_s(\gamma) \right|
$$

$$
\leq \int_{t_1}^{t_2} |f(s, X_s(\gamma), X_{s-\tau}(\gamma))| \, ds + K(\gamma) \int_{t_1}^{t_2} |g(s, X_s(\gamma), X_{s-\tau}(\gamma))| \, ds
$$

$$
\leq C_{\varsigma} \int_{t_1}^{t_2} 1 + |X_s(\gamma)| + |X_{s-\tau}(\gamma)| \, ds + C_{\varsigma} K(\gamma) \int_{t_1}^{t_2} 1 + |X_s(\gamma)| + |X_{s-\tau}(\gamma)| \, ds
$$

$$
= C_{\varsigma} (1 + K(\gamma)) \int_{t_1}^{t_2} 1 + |X_s(\gamma)| + |X_{s-\tau}(\gamma)| \, ds
$$

$$
\leq C_{\varsigma} (1 + K(\gamma)) \int_{t_1}^{t_2} (1 + H_0) \, ds
$$

$$
\leq C_{\varsigma} (1 + K(\gamma)) (1 + H_0) |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, \varsigma).
$$

It follows that

$$
\lim_{x \to \varsigma-} X_t(\gamma)
$$

exists. Set

$$
X_{\varsigma}(\gamma) = \lim_{x \to \varsigma-} X_t(\gamma).
$$

Then $X_{\varsigma}(\gamma)$ is continuous on $[0, \varsigma]$, and

$$
\begin{align*}
X_t(\gamma) &= X_0(\gamma) + \int_0^t f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds + \int_0^t g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, dC_s(\gamma), \quad t \in [0, \varsigma] \\
X_{\varsigma}(\gamma) &= \varphi(t), \quad t \in [-\tau, 0].
\end{align*}
$$

Consider the following integral equation

$$
\begin{align*}
X_t(\gamma) &= X_{\varsigma}(\gamma) + \int_{\varsigma}^{t} f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds + \int_{\varsigma}^{t} g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, dC_s(\gamma), \quad t \in (\varsigma, +\infty) \\
X_t(\gamma) &= X_0(\gamma) + \int_0^{t} f(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, ds + \int_0^{t} g(s, X_s(\gamma), X_{s-\tau}(\gamma)) \, dC_s(\gamma), \quad t \in [0, \varsigma] \\
X_{\varsigma}(\gamma) &= \varphi(t), \quad t \in [-\tau, 0].
\end{align*}
$$

(8)

Theorem 4.1 indicates that there exists a positive number $\alpha$ such that integral equation (8) has a unique continuous solution $\tilde{X}_t(\gamma)$ on $[\varsigma, \varsigma + \alpha]$.

Hence, setting function

$$
Y_t(\gamma) = \begin{cases} 
X_t(\gamma), & \text{if } t \in [0, \varsigma] \\
\tilde{X}_t(\gamma), & \text{if } t \in (\varsigma, \varsigma + \alpha]
\end{cases}
$$

$Y_t(\gamma)$ is the unique continuous solution of integral equation (4) on $[0, \varsigma + \alpha]$.

It is a contradiction from $\varsigma = \sup \varsigma < +\infty$. Thus, $\varsigma = +\infty$, and the solution of integral equation (6) can be extended uniquely to $[0, +\infty)$. Thus the proof is complete. \qed
Remark 4.1 When functions $f$ and $g$ in equation (3) are independent with the present state $X_t$, the equation (3) can be written as

$$
\begin{align*}
\begin{cases}
\d X_t = f(t, X_{t-\tau})dt + g(t, X_{t-\tau})dC_t, & t \in [0, +\infty) \\
X_t = \varphi(t), & t \in [-\tau, 0].
\end{cases}
\end{align*}
$$

(9)

For the uncertain delay differential equation (9), we have explicitly that for

$$
X_t = X_0 + \int_0^t f(s, X_{s-\tau}) \, ds + \int_0^t g(s, X_{s-\tau}) \, dC_s
$$

$$
= \Phi(0) + \int_0^t f(s, \Phi(s - \tau)) \, ds + \int_0^t g(s, \Phi(s - \tau)) \, dC_s,
$$

for $0 \leq t \leq \tau$. Then, for $\tau \leq t \leq 2\tau$,

$$
X_t = X_\tau + \int_\tau^t f(s, X_{s-\tau}) \, ds + \int_\tau^t g(s, X_{s-\tau}) \, dC_s.
$$

Repeat this procedure over the intervals $[2\tau, 3\tau]$, etc. Finally, we can obtain the explicit solution of uncertain delay differential equation (9).

5 Conclusions

Uncertain delay differential equation is an important tool to deal with dynamic systems including the past states in uncertain environments. This paper mainly gave an existence and uniqueness theorem for uncertain delay differential equations under the local Lipschitz condition and the linear growth condition. Furthermore, we extended this existence and uniqueness theorem from finite domain to infinite domain.

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References


