Uncertain Integral with respect to Multiple Canonical Processes

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Abstract

Uncertain calculus is a branch of mathematics that deals with the integral of functions of uncertain process. This paper will extend uncertain integral from single canonical process to multiple ones. Some mathematical properties of uncertain integral with respect to multiple canonical processes are proved, including the fundamental theorem of uncertain calculus.

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1 Introduction

In our daily life, due to economical or technical difficulties, very often we are lack of observed data about the unknown state of nature. Then we have to invite some domain experts to evaluate their belief degree that each event will occur. Because human being tends to overweight unlikely events [3], the belief degree usually has a much larger range than the real frequency. In 2012, Liu [9] declared that probability theory fails to model the belief degree under this situation via a counterexample about the strength of a bridge.

In order to deal with the belief degree, an uncertainty theory was founded by Liu [5] in 2007, and refined by Liu [8] in 2010 based on normality, duality, subadditivity and production axioms. Sometimes, the uncertain phenomena evolve with time. For modeling such phenomena, a concept of uncertain process was proposed by Liu [1] as a sequence of uncertain variables indexed by time. After that, Liu [7] designed a canonical process which is an independent and stationary increment uncertain process with normal uncertain variables as the increments. Meanwhile, Liu [7] founded uncertain calculus to deal with the integral and differential of a function of uncertain process with respect to canonical process. Canonical process is a type of continuous uncertain process, and can only model continuous uncertain systems. In order to model the sudden jumps in an uncertain system, Liu [7] proposed uncertain renewal process. Inspired by Liu [7], Yao [13] founded uncertain calculus with respect to uncertain renewal process. After that, Chen [2] generalized the work by Liu [7] and Yao [13], and proposed uncertain calculus with respect to finite variation process.


In this paper, we will present uncertain calculus with multiple canonical processes. The rest of this paper is structured as follows. The next section is intended to introduce some concepts of uncertainty theory and uncertain calculus with respect to single canonical process. In Section 3, an uncertain integral with respect to multiple canonical processes is proposed. In Section 4, the fundamental theorem of multifactor uncertain integral is proposed. Finally, some remarks are made in Section 5.

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2 Preliminary

Uncertainty theory is a branch of axiomatic mathematics to deal with human uncertainty. As a fundamental concept, uncertain measure is a set function satisfying normality, duality, subadditivity and production axioms.

**Definition 1** Let $L$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $M : L \rightarrow [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

1. **Axiom 1** (Normality Axiom) $M(\{\Gamma\}) = 1$ for the universal set $\Gamma$.
2. **Axiom 2** (Duality Axiom) $M(\Lambda) + M(\Lambda^{c}) = 1$ for any event $\Lambda$.
3. **Axiom 3** (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have
   \[ M \left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i). \]
4. **Axiom 4** (Product Axiom) Let $L_k$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure $M$ is an uncertain measure satisfying
   \[ M \left( \prod_{i=1}^{\infty} \Lambda_k \right) = \bigwedge_{k=1}^{\infty} M_k \{ \Lambda_k \} \]
   where $\Lambda_k$ are arbitrarily chosen events from $L_k$ for $k = 1, 2, \ldots$, respectively.

An uncertain variable is essential a measurable function on an uncertainty space. The formal definition of uncertain variable is given as follows.

**Definition 2** An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, L, M)$ to the set $\mathbb{R}$ of real numbers, i.e., for any Borel set $B$ of real numbers, the set
\[ \{ \xi \in B \} = \{ \gamma | \xi(\gamma) \in B \} \]
is an event.

**Definition 3** The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by
\[ \Phi(x) = M(\xi \leq x) \]
for any real number $x$.

The uncertainty distribution $\Phi$ is said to be regular if its inverse function $\Phi^{-1}$ exists and is unique for each $\alpha \in (0, 1)$. In this case, the inverse function $\Phi^{-1}$ is called the inverse uncertainty distribution, which plays an important role in the operation of uncertain variables.

**Theorem 1** Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution
\[ \Phi^{-1}(r) = f \left( \Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r) \right). \]

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. Canonical process is one of the most important uncertain processes.

**Definition 4** An uncertain process $C_t$ is said to be a canonical process if
1. $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
2. $C_t$ has stationary and independent increments,
3. every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution
\[ \Phi(x) = \left( 1 + \exp \left( \frac{-\pi x}{\sqrt{3}t} \right) \right)^{-1}, \quad x \in \mathbb{R}. \]
Note that $\Delta C_t$ and $\Delta t$ are infinitesimals with the same order. Based on canonical process, Liu \cite{7} defined an uncertain integral, thus founding an uncertain calculus theory.

**Definition 5** \cite{7} Let $X_t$ be an uncertain process and $C_t$ be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$  

Then the uncertain integral of $X_t$ is defined by

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be integrable.

For example, a continuous function $f(t)$ is integrable, and

$$\int_0^s f(t) dC_t \sim \left(0, \int_0^s |f(t)| dt \right)$$

is a normal uncertain variable at any time $s$.

**Definition 6** \cite{7} Let $C_t$ be a canonical process and $Z_t$ be an uncertain process. If there exist uncertain processes $\mu_s$ and $\sigma_s$ such that

$$Z_t = Z_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dC_s$$

for any $t \geq 0$, then $Z_t$ is said to have an uncertain differential

$$dZ_t = \mu_t dt + \sigma_t dC_t.$$  

In this case, the uncertain process $Z_t$ is called a differentiable uncertain process with drift $\mu_t$ and diffusion $\sigma_t$.

Liu \cite{7} verified the fundamental theorem of uncertain calculus, i.e., for a canonical process $C_t$ and a continuous differentiable function $h(t, c)$, the uncertain process $Z_t = h(t, C_t)$ is differentiable and has an uncertain differential

$$dZ_t = \frac{\partial h}{\partial t}(t, C_t) dt + \frac{\partial h}{\partial c}(t, C_t) dC_t.$$  

Based on the fundamental theorem, Liu \cite{7} proved the chain rule, i.e., for two continuously differentiable functions $f$ and $g$, the uncertain process $f(g(C_t))$ has an uncertain differential

$$df(g(C_t)) = f'(g(C_t)) g'(C_t) dC_t,$$

and the integration by parts theorem, i.e., for two differentiable uncertain processes $X_t$ and $Y_t$, the uncertain process $X_t Y_t$ has an uncertain differential

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t.$$  

### 3 Multifactor Uncertain Integral

**Definition 7** Let $X_{1t}, X_{2t}, \ldots, X_{nt}$ be integrable uncertain processes, and $C_{1t}, C_{2t}, \ldots, C_{nt}$ be canonical processes. Then

$$Z_t = \sum_{i=1}^n \int_0^t X_{is} dC_{is}$$

is called an uncertain integral of $X_{1t}, X_{2t}, \ldots, X_{nt}$ with respect to multiple canonical processes $C_{1t}, C_{2t}, \ldots, C_{nt}$.
Theorem 2 Assume that $Z_t$ is an uncertain integral of $X_{1t}, X_{2t}, \ldots, X_{nt}$ with respect to multiple canonical processes $C_{1t}, C_{2t}, \ldots, C_{nt}$. Then $Z_t$ is a sample-continuous uncertain process.

Proof: It follows immediately from the definition of multifactor uncertain integral that

$$|Z_t(\gamma) - Z_r(\gamma)| = \left| \sum_{i=1}^{n} \int_{0}^{t} X_{is}(\gamma) dC_{is}(\gamma) - \sum_{i=1}^{n} \int_{0}^{r} X_{is}(\gamma) dC_{is}(\gamma) \right| = \left| \sum_{i=1}^{n} \int_{r}^{t} X_{is}(\gamma) dC_{is}(\gamma) \right| \to 0$$

for each $\gamma \in \Gamma$ as $r \to t$. Thus $Z_t$ is sample-continuous, and the theorem is proved.

Theorem 3 (Linearity of Uncertain Integral) Assume $X_{1t}, X_{2t}, \ldots, X_{nt}$ and $Y_{1t}, Y_{2t}, \ldots, Y_{nt}$ are integrable uncertain processes with respect to $C_{1t}, C_{2t}, \ldots, C_{nt}$ on $[a, b]$. Then for any given real numbers $\alpha$ and $\beta$, we have

$$\sum_{i=1}^{n} \int_{a}^{b} (\alpha X_{it} + \beta Y_{it}) dC_{it} = \alpha \sum_{i=1}^{n} \int_{a}^{b} X_{it} dC_{it} + \beta \sum_{i=1}^{n} \int_{a}^{b} Y_{it} dC_{it}.$$ 

Proof: It follows immediately from the linearity of uncertain integral with respect to single canonical process that

$$\sum_{i=1}^{n} \int_{a}^{b} (\alpha X_{it} + \beta Y_{it}) dC_{it} = \sum_{i=1}^{n} \left( \alpha \int_{a}^{b} X_{it} dC_{it} + \beta \int_{a}^{b} Y_{it} dC_{it} \right) = \alpha \sum_{i=1}^{n} \int_{a}^{b} X_{it} dC_{it} + \beta \sum_{i=1}^{n} \int_{a}^{b} Y_{it} dC_{it}.$$ 

4 Multifactor Uncertain Differential

Definition 8 Let $C_{1t}, C_{2t}, \ldots, C_{nt}$ be canonical processes and let $Z_t$ be an uncertain process. If there exist uncertain processes $\mu_t$ and $\sigma_{1t}, \sigma_{2t}, \ldots, \sigma_{nt}$ such that

$$Z_t = Z_0 + \int_{0}^{t} \mu_s ds + \sum_{i=1}^{n} \int_{0}^{t} \sigma_{is} dC_{is}$$

(1)

for any $t \geq 0$, then we say $Z_t$ has an uncertain differential

$$dZ_t = \mu_t dt + \sum_{i=1}^{n} \sigma_{it} dC_{it}.$$ 

(2)

In this case, $Z_t$ is called a differentiable uncertain process with drift $\mu_t$ and diffusions $\sigma_{1t}, \sigma_{2t}, \ldots, \sigma_{nt}$.

The following theorem gives the uncertain differential of a function of multiple canonical processes.

Theorem 4 (Fundamental Theorem) Let $C_{1t}, C_{2t}, \ldots, C_{nt}$ be canonical processes. If $h(t, c_1, c_2, \ldots, c_n)$ is a continuously differentiable function, then the uncertain process $Z_t = h(t, C_{1t}, C_{2t}, \ldots, C_{nt})$ is differentiable and has an uncertain differential

$$dZ_t = \frac{\partial h}{\partial t}(t, C_{1t}, C_{2t}, \ldots, C_{nt}) dt + \sum_{i=1}^{n} \frac{\partial h}{\partial c_i}(t, C_{1t}, C_{2t}, \ldots, C_{nt}) dC_{it}.$$ 

Proof: Since the function $h$ is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of $Z_t$ has a first-order approximation

$$\Delta Z_t = \frac{\partial h}{\partial t}(t, C_{1t}, C_{2t}, \ldots, C_{nt}) \Delta t + \sum_{i=1}^{n} \frac{\partial h}{\partial c_i}(t, C_{1t}, C_{2t}, \ldots, C_{nt}) \Delta C_{it}.$$ 

Thus the theorem is proved.
Example 1 Let us calculate the uncertain differential of \( C_{1t} + C_{2t} \). In this case, we have \( h(t, c_1, c_2) = c_1 + c_2 \). It is clear that
\[
\frac{\partial h}{\partial t}(t, c_1, c_2) = 0, \quad \frac{\partial h}{\partial c_1}(t, c_1, c_2) = 1, \quad \frac{\partial h}{\partial c_2}(t, c_1, c_2) = 1.
\]
It follows from the fundamental theorem of uncertain calculus that
\[
d(C_{1t} + C_{2t}) = dC_{1t} + dC_{2t}.
\]

Example 2 Let us calculate the uncertain differential of \( \exp(C_{1t} + C_{2t}) \). In this case, we have \( h(t, c_1, c_2) = \exp(c_1 + c_2) \). It is clear that
\[
\frac{\partial h}{\partial t}(t, c_1, c_2) = 0, \quad \frac{\partial h}{\partial c_1}(t, c_1, c_2) = \exp(c_1 + c_2), \quad \frac{\partial h}{\partial c_2}(t, c_1, c_2) = \exp(c_1 + c_2).
\]
It follows from the fundamental theorem of uncertain calculus that
\[
d \exp(C_{1t} + C_{2t}) = \exp(C_{1t} + C_{2t})dC_{1t} + \exp(C_{1t} + C_{2t})dC_{2t}.
\]

Example 3 Let us calculate the uncertain differential of \( tC_{1t} + C_{2t} \). In this case, we have \( h(t, c_1, c_2) = tc_1c_2 \). It is clear that
\[
\frac{\partial h}{\partial t}(t, c_1, c_2) = c_1c_2, \quad \frac{\partial h}{\partial c_1}(t, c_1, c_2) = tc_2, \quad \frac{\partial h}{\partial c_2}(t, c_1, c_2) = tc_1.
\]
It follows from the fundamental theorem of uncertain calculus that
\[
d(tC_{1t} + C_{2t}) = C_{1t}C_{2t}dt + tC_{2t}dC_{1t} + tC_{1t}dC_{2t}.
\]

Example 4 Let us calculate the uncertain differential of \( t \sin C_{1t} \sin C_{2t} \). In this case, we have \( h(t, c_1, c_2) = t \sin c_1 \sin c_2 \). It is clear that
\[
\frac{\partial h}{\partial t}(t, c_1, c_2) = \sin c_1 \sin c_2, \quad \frac{\partial h}{\partial c_1}(t, c_1, c_2) = t \cos c_1 \sin c_2, \quad \frac{\partial h}{\partial c_2}(t, c_1, c_2) = t \sin c_1 \cos c_2.
\]
It follows from the fundamental theorem of uncertain calculus that
\[
d t \sin C_{1t} \sin C_{2t} = \sin C_{1t} \sin C_{2t}dt + t \cos C_{1t} \sin C_{2t}dC_{1t} + t \sin C_{1t} \cos C_{2t}dC_{2t}.
\]

5 Conclusion

This paper presented the concepts of uncertain integral and uncertain differential of uncertain processes with respect to multiple canonical processes. Besides, it proved the sample-continuity and linearity of uncertain integral, as well as the fundamental theorem of uncertain calculus.

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References


