Fixed Charge Transportation Problem and Its Uncertain Programming Model

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ABSTRACT

In this paper, we study the fixed charge transportation problem with uncertain variables. The fixed charge transportation problem has two kinds of costs: direct cost and fixed charge. The direct cost is the cost associated with each source-destination pair, and the fixed charge occurs when the transportation activity takes place in the corresponding source-destination pair. The uncertain fixed charge transportation problem is modeled on the basis of uncertainty theory. According to inverse uncertainty distribution, the model can be transformed into a deterministic form. Finally, in order to solve the uncertain fixed charge transportation problem, a numerical example is given to show the application of the model and algorithm.

Keywords: Transportation Problem, Uncertainty Theory, Uncertain Variable, Uncertain Measure, Uncertain Programming

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1. INTRODUCTION

The fixed charge transportation problem (FCTP) is an extension of the (linear) transportation problem (TP). Since the fixed charge problem was initialized by Hirsch and Dantzig (1968), it has been widely applied in many decision making and optimization problems. Interested readers may refer to Sun et al. (1998) and Gottlieb and Paulmann (1998). The TP reflects the situation of transporting amounts of a single commodity from a set of plants to a set of customers. The capacities of the plants and the demands of the customers are known in advance, and a feasible transportation plan must obey these restrictions. The goal is to minimize the overall shipping cost with the transportation cost between plants and customers depending linearly on the transported amount of the commodity. Although this is a useful model, in practice fixed costs occur once a transport is estimated between a plant and a customer. The FCTP takes these fixed charges into account, so that the TP is a FCTP with equal fixed costs 0 for all routes.

It is well-known that the TP is one of the important traditional optimization problems, and it has also been widely applied in real-life such as distributing systems, job assignment and transportation. In the traditional TP, there are two kinds of constraints: source constraint and destination constraint. There are many efficient algorithms for solving the traditional transportation problems such as Balinski (1961) and Srinivasan and Thompson (1972). As an extension of the traditional TP, the solid transportation problem (STP) was introduced by Haley (1962) and Li et al. (1997). Another important extension of the traditional TP is FCTP (Jimenez and Verdegay, 1999, 1998; Chanas et al., 1984; Bit et al., 1996). If the cost parameters of the TP are uncertain variables, they are called as uncertain cost transportation problem (UCTP).
Along with the development of global economy, produc-
tion and demand have a growing importance. Trans-
portation model in logistics and supply management
plays an important role in reducing costs and improving
the service quality (Klingman et al., 1974). In reality,
due to changes in market supply and demand, weather
conditions, road conditions and other uncertainty factors,
uncertainty TP is particularly important. Therefore, stu-
dying uncertain TP has important theoretical and practi-
cal significance. This paper proposes an UCTP model.

This paper consists of 6 sections, and its frame is
organized as follows: In section 2, some basic concepts
and properties in uncertainty theory used throughout this
paper are introduced. In section 3, an uncertain FCTP
model is constructed. In section 4, according to uncer-
tainty theory, the model can be transformed to its deter-
mistic form, and then we can use the classical method
to find its solution. A numerical example is given in
section 5. A brief summary is presented in section 6.

2. PRELIMINARIES

In order to deal with human uncertainty, an uncer-
tainty theory was established by Liu (2007), and refined
by Liu (2010b), which is a branch of mathematics based
on normality, duality, subadditivity, and product meas-
ure axioms. Since then significant work has been done
by researchers based on the uncertainty theory both in
theoretical and practical aspects. In theoretical aspect,
Liu (2008) proposed an uncertain process which is a
sequence of uncertain variables indexed by time or space
and uncertain differential equation which is a type of
differential equation by canonical process in 2008. Liu
(2009a) established uncertain calculus to deal with dy-
namic uncertain phenomena in 2009. Besides, the con-
cept of uncertain logic was proposed by Li and Liu
(2009) to describe uncertain knowledge, uncertain infer-
ence was introduced by Liu (2010a) via conditional un-
certain measure. Meanwhile, some other theoretical pro-
PERTIES of uncertain measure are studied (Gao, 2009; You,
2009). In practical aspect, Liu (2009b) designed uncertain
programming involving uncertain variables, which has been
used to model system reliability design, project scheduling
problem, vehicle routing problem and facility location
problem. Now, the uncertainty theory has become a
mathematical tool to model the indeterminate phenome-
on in our real world. In this paper, the uncertainty fixed
transportation problem is particularly important. Therefore, stu-
dying uncertain TP has important theoretical and practi-
cal significance. This paper proposes an UCTP model.

Theorem 1: (Liu, 2010b) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independ-
ent uncertain variables with uncertainty distribution \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f \) is a strictly increas-
ing function, then

\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).
\]

Theorem 2: (Liu, 2010b) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be inde-
pendent uncertain variables with uncertainty distribution \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f \) is a strictly decreas-
ing function, then

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]

is an uncertain variable with inverse uncertainty distri-
butions

\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).
\]

Theorem 3: (Liu, 2007) Let \( \xi \) and \( \eta \) be independent
uncertain variables with expected values. Then for any
real numbers \( a, b \), we have

\[
E[a\xi + b\eta] = aE[\xi] + bE[\eta].
\]

3. FIXED CHARGE TRANSPORTATION
MODEL

Let \( m \) be the number of origin of the number, let \( n \) be
the number of destination, the direct cost and the fixed
charge with respect to the transportation from source \( i \)
to destination \( j \) are denoted by \( c_{ij} \) and \( f_{ij} \), respec-
tively, where \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \). The capacity of source
and that of destination \( j \) are denoted by \( a_i \) and \( b_j \), respec-
tively, where \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \). Let \( x_{ij} \) be
the quantity to be transported from source \( i \) to destination
\( j \) and \( y_j \) be a 0-1 variable to describe the transporta-
tion activity from source \( i \) to destination \( j \) and is defined as

\[
y_j = \begin{cases} 1, & \text{if } x_{ij} > 0 \\ 0, & \text{otherwise.} \end{cases}
\]

Then the FCTP can be formulated as follows:

\[
\begin{align*}
\min \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} x_{ij} + f_{ij} y_j) \\
\text{subject to: } \\
\sum_{j=1}^{n} x_{ij} \leq a_i, \ i = 1, 2, \ldots, m \\
\sum_{i=1}^{m} x_{ij} \leq b_j, \ j = 1, 2, \ldots, n \\
x_{ij} \geq 0, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.
\end{align*}
\]
In this model, the first constraint implies that the total amount transported from source \( i \) is no more than \( a_i \), and the second constraint implies that the total amount transported from \( j \) sources should satisfy the demand of destination \( b_j \). The above model is constructed under certain conditions, that is, the parameters in the model are all fixed quantities. But due to the complex nature of the real world, we may always meet uncertain phenomena in constructing a mathematical model.

In this paper, we assume that the direct cost, the fixed charge, the capacity of each source and each destination are all uncertain variables and denoted by \( \xi_i, \eta_j, a_i \) and \( b_j \) respectively. For such conditions, we generally add the uncertain variables to the model. In order to describe the problems conveniently, we denote the cost function of model (1) as

\[
f(x, \xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_i x_{ij} + \eta_j y_{ij})
\]

Where \( x, \xi, \) and \( \eta \) denote the vectors consisting of \( x_i, \xi_i, \) and \( \eta_j, \eta_j \), \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \) respectively.

The main idea of the uncertain programming model is to optimize the expected value of objective function under the chance constraints.

**Definition 1:** Assume that \( f(x, \xi, \eta) \) is an objective function, and \( g_j(x, \xi, \eta) \) are constraints functions, \( j = 1, 2, \ldots, p \). A solution \( x \) is feasible if and only if

\[
M \{ g_j(x, \xi, \eta) \leq 0 \} \geq \alpha_j
\]

for \( j = 1, 2, \ldots, p \).

A solution \( x^* \) is an optimal solution to the uncertain programming model if

\[
E[f(x^*, \xi, \eta)] \leq E[f(x, \xi, \eta)]
\]

for any feasible solution \( x \).

We may construct the model of the uncertain fixed charge transportation problem as follows:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_i x_{ij} + \eta_j y_{ij}) \\
\text{subject to:} & \\
M \{ \sum_{j=1}^{m} x_{ij} - a_i \leq 0 \} & \geq \beta_i, \quad i = 1, 2, \ldots, m \\
M \{ \sum_{j=1}^{m} x_{ij} - b_j \geq 0 \} & \geq \gamma_j, \quad j = 1, 2, \ldots, n \\
x_{ij} & \geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.
\end{align*}
\]

In the above model \( \beta_i, \gamma_j, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \) are predetermined confidence levels. The objective function implies optimizing the expected value of the total transportation cost, the first constraint implies that total amount transported from \( i \) source should be no more than its supply capacity at the predetermined confidence level \( \beta_i \); the second constraint implies that the total amount transported from \( i \) sources should satisfy the requirement of destination \( j \) at the predetermined confidence level \( \gamma_j \).

### 4. CRISP EQUIVALENCE OF MODEL

We can see that there are many uncertain variables in the above models. In order to find the best solution, we must compute the expected value or uncertain measure. If the uncertain variables are complex, computing objective value is actually a time-consuming work. So it is better for us to transfer the model into its crisp equivalence. In this section, we shall induce the crisp equivalence of model under some special conditions.

**Theorem 4:** Assume that \( \xi_i, \eta_j, a_i \) and \( \beta_i \) are independent uncertain variables with uncertainty distributions \( \Phi_{\xi_i}, \Phi_{\eta_j}, \Phi_{a_i}, \Phi_{\beta_i} \) respectively, then the model (2) is converted to equivalence model:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} \sum_{j=1}^{m} \Phi_{\xi_i}^{-1}(\alpha)dx_{ij} + \eta_j \int_{\Phi_{\beta_i}}^{\Phi_{\alpha_i}} \Phi_{\eta_j}^{-1}(\alpha)d\alpha \\
\text{subject to:} & \\
\sum_{j=1}^{m} x_{ij} - \Phi_{\alpha_i}^{-1}(1 - \beta_i) & \leq 0, \ i = 1, 2, \ldots, m \\
\sum_{i=1}^{n} x_{ij} - \Phi_{\gamma_j}^{-1}(\gamma_j) & \geq 0, \ j = 1, 2, \ldots, n \\
x_{ij} & \geq 0, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.
\end{align*}
\]

**Proof:** By using the linearity of expected value operator, we can easily prove this theorem, i.e., since \( \xi_i, \eta_j, a_i \) and \( \beta_i \) are independent uncertain variables with uncertainty distributions \( \Phi_{\xi_i}, \Phi_{\eta_j}, \Phi_{a_i}, \Phi_{\beta_i} \) respectively, then

\[
\begin{align*}
E[\sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_i x_{ij} + \eta_j y_{ij})] &= \sum_{i=1}^{n} \sum_{j=1}^{m} E[\xi_i x_{ij}] + E[\eta_j y_{ij}] \\
M \{ \sum_{j=1}^{m} x_{ij} - a_i \leq 0 \} & \geq \beta_i \iff \sum_{j=1}^{m} x_{ij} - \Phi_{\alpha_i}^{-1}(1 - \beta_i) \leq 0, \ i = 1, 2, \ldots, m, \\
M \{ \sum_{i=1}^{n} x_{ij} - b_j \geq 0 \} & \geq \gamma_j \iff \sum_{i=1}^{n} x_{ij} - \Phi_{\gamma_j}^{-1}(\gamma_j) \geq 0, \ j = 1, 2, \ldots, n,
\end{align*}
\]

where

\[
\begin{align*}
E[\xi_i] &= \int_{\Phi_{\xi_i}}^{\Phi_{\alpha_i}} \Phi_{\xi_i}^{-1}(\alpha)d\alpha, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n. \\
E[\eta_j] &= \int_{\Phi_{\eta_j}}^{\Phi_{\beta_i}} \Phi_{\eta_j}^{-1}(\alpha)d\alpha, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.
\end{align*}
\]
So the model (3) is converted to equivalence model:

\[
\begin{align*}
\min & \sum_{i=1}^{k} \sum_{j=1}^{s} \left[ x_{ij} \Phi_{\xi_{ij}}(\alpha) d\alpha + y_{ij} \Phi_{\eta_{ij}}(\alpha) d\alpha \right] \\
\text{subject to } & \sum_{j=1}^{m} x_{ij} - \Phi_{\xi_{ij}}(1 - \beta_i) \leq 0, \quad i = 1, 2, \ldots, k \\
& \sum_{i=1}^{k} x_{ij} - \Phi_{\eta_{ij}}(\gamma_j) \geq 0, \quad j = 1, 2, \ldots, s \\
& x_{ij} \geq 0, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, s.
\end{align*}
\]

(4)

Note that model (4) is a deterministic programming model.

5. NUMERICAL EXPERIMENT

In order to show the application of the model, we shall present an example of coal TP in this section. Coal is a crucial energy source in the development of economy and society. Accordingly, how to transport the coal from mines to the different areas economically is also an important issue in the coal transportation. For the convenience of description, we summarize the problem as follows. Suppose that there are four coal mines to supply the coal for six cities. During the process of transportation, two kinds of conveyances are available to be selected, i.e., train and cargo ship. Now, the task for the decision-maker is to make the transportation plan for the next month. At the beginning of this task, the decision maker needs to obtain the basic data, such as supply capacity, demand, transportation cost of unit product, and so on. In fact, since the transportation plan is made in advance, we generally cannot get these data exactly. For this condition, the usual way is to obtain the uncertain data by means of experience evaluation or expert advice. Assume that all uncertain variables are normal uncertain variables, in which we suppose:

\[
\begin{align*}
\xi_{ij} & \sim N(\mu_{ij}, \sigma_{ij}), \quad i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4, 5, 6. \\
\eta_{ij} & \sim N(\mu_{ij}, \sigma_{ij}), \quad i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4, 5, 6. \\
a_{ij} & \sim N(\mu_{ij}, \sigma_{ij}), \quad i = 1, 2, 3, 4. \\
b_{ij} & \sim N(\mu_{ij}, \sigma_{ij}), \quad j = 1, 2, 3, 4, 5, 6.
\end{align*}
\]

then the model (4) is equivalent to the following model:

\[
\begin{align*}
\min & \sum_{i=1}^{k} \sum_{j=1}^{s} \left[ x_{ij} \mu_{ij} + y_{ij} \mu_{ij} \right] \\
\text{subject to } & \sum_{j=1}^{m} x_{ij} - \frac{\sigma_{ij} \sqrt{\frac{\pi}{2}} \ln \frac{1 - \beta_i}{\beta_i}}{\beta_i} \leq 0, \quad i = 1, 2, 3, 4 \\
& \sum_{i=1}^{k} x_{ij} - \frac{\sigma_{ij} \sqrt{\frac{\pi}{2}} \ln \frac{\gamma_j}{1 - \gamma_j}}{\gamma_j} \geq 0, \quad i = 1, 2, 3, 4, 5, 6 \\
& x_{ij} \geq 0, \quad i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4, 5, 6.
\end{align*}
\]

(5)

And the associated uncertain data are listed in the following Tables 1-4,

<table>
<thead>
<tr>
<th>Table 1. Parameters of normal distribution $N(\mu_{ij}, \sigma_{ij})$ of direct costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu_{ij}, \sigma_{ij})$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2. Parameters of normal distribution $N(\mu_{ij}, \sigma_{ij})$ of fixed charges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu_{ij}, \sigma_{ij})$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3. Parameters of normal distribution $N(\mu_{ij}, \sigma_{ij})$ of supplies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu_{ij}, \sigma_{ij})$</td>
</tr>
<tr>
<td>(23, 1.5)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4. Parameters of normal distribution $N(\mu_{ij}, \sigma_{ij})$ of demands</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu_{ij}, \sigma_{ij})$</td>
</tr>
<tr>
<td>(10, 1.5)</td>
</tr>
</tbody>
</table>

We also suppose the confidence level is $\beta_i = 0.9$, $\gamma_j = 0.9$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5, 6$. Then we can get transportation problem cost is 941.1438. And the corresponding transportation plan is:

\[
\begin{align*}
& x_{13} = 1.325328, \quad x_{16} = 11.21139, \quad x_{22} = 13.21139, \\
& x_{33} = 12.97152, \quad x_{31} = 11.81709, \quad x_{33} = 8.239877, \\
& x_{44} = 7.520246, \quad x_{44} = 11.81709, \quad x_{45} = 13.88607.
\end{align*}
\]

6. CONCLUSION

This paper mainly investigated the uncertain fixed charge transportation problem based on an uncertainty theory, and an uncertain model was constructed. By applying the uncertainty theory, the uncertain model can be transformed to a deterministic form, and then we can find its solution by the classical method. Finally, as an application of the model and algorithm, we presented a coal transportation problem as example, and the result showed that the proposed uncertain cost transportation problem model works well.
REFERENCES


