Structural Reliability Analysis using Uncertainty Theory

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Abstract: The theory of structural reliability is widely applied in many fields, such as architectural design, and mechanical engineering. In this paper, the resident and the load of a structure are defined as uncertain variables and the reliability index is defined as the uncertain measure of the event that the resident is larger than the load. Furthermore, the reliability of series and parallel structure systems are considered, respectively. In these two modes, the failure of any rod will bring the entire system’s failure. According to this principle, the resistance of the structures are calculated and some basic structural reliability index theorems are given.

keywords: uncertainty theory; structural reliability; series structure; parallel structure

1 Introduction

The theory of structural reliability is widely applied in many fields, such as architectural design, civil engineering, mechanical engineering. In structural engineering, because of requiring long-term security commitments subject to various load, the analysis of the reliability is particularly important. The improvement of the structural reliability theory was gradually strengthened and enhanced along with the continuous development of mathematical sciences because a large number of mathematical tools was used in the study.

For a long time, the concept of “reliability” has been used to evaluate the quality of engineering structures. However, due to the uncertainty of material properties and load, and various types of errors of the structure in constructing and using, from the engineering point of view, a structural problem can be considered as “uncertain” when some lack of knowledge exists about the theoretical model which describes the structural system and its behavior, either with respect to the model itself, or to the value of its significant parameters. In the early twentieth century, probability theory and mathematical statistics are applied to the structural reliability analysis which can be marked that the theory of the structural reliability is created.

Freudenthal[1] was among the first in the world to develop structural reliability that is the application of probabilistic methods to evaluate the safety of structures that are made of various materials. From early 40s to the 60s of the twentieth century, the structural reliability theory is well advanced. Although applying the probability theory to the research of structural reliability theory was fruitful, there is still an obstacle for researching and putting it into practice due to stochastic analysis requires a lot of statistical data and the data in real life is sometimes difficult to obtain. Therefore, after Zadeh[2] proposed fuzzy set theory in 1965, many scholars began to apply the fuzzy theory to structural reliability analysis. Early studies in this area is Brown[3]. In recent years, Fabio et al.[4] also analyze the reliability of concrete structures under the fuzzy theory; Adduri et al.[6] studied the structural reliability problems in the environment of there are both fuzzy variables and random variables; Marano et al[5] proposed a new reliability index under the possibility theory. There are a number of other scholars studied this issue from different perspectives[7, 8, 9, 10, 11, 12, 13].

However, since fuzzy theory fails to explain many subjective uncertain phenomena, such that the computational result is not consistent with the real condition. Hence in 2007, based on normality, monotonicity and countable subadditivity, Liu[14] proposed uncertainty theory which is a powerful tool to interpret subject uncertainty. In 2010, Liu[18] proposed uncertain reliability and uncertain risk. The purpose of this paper is to study the problems of the structural reliability within the framework of uncertainty theory. For this purpose, this paper is organized as follows. Section 2 recalls some basic concepts and properties about uncertainty theory and uncertain reliability. In Section 3, a structural reliability index as the uncertain measure of the event is discussed, respectively. At the end of this paper, a brief summary about this paper is given.

2 Preliminaries

Uncertainty theory which is based on normality, monotonicity and countable subadditivity was founded in 2007 by Liu[14] and refined in 2010 by Liu[16]. In this section, some basic concepts about uncertainty theory such as uncertain measure, uncertain variable, uncertainty distribution and uncertain reliability are given.
Uncertainty Theory

Definition 2.1. (Liu [14]) Let \( \Gamma \) be a nonempty set. A collection \( \mathcal{L} \) of subsets of \( \Gamma \) is a \( \sigma \)-algebra. Each element \( \Lambda \) in the \( \sigma \)-algebra \( \mathcal{L} \) is called an event. If the function \( \mathcal{M} \) on \( \mathcal{L} \) which is subjected to
(1) \( \mathcal{M}\{\Gamma\} = 1 \)
(2) \( \Lambda_1 \subset \Gamma_2 \) whenever \( \mathcal{M}\{\Gamma_1\} \leq \mathcal{M}\{\Gamma_2\} \)
(3) \( \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1 \) for each event \( \Lambda \)
(4) For every countable sequence of events \( \{\Lambda_i\} \), we have
\[
\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\},
\]
then \( \mathcal{M} \) is an uncertain measure, \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is an uncertain space

In order to describe the uncertain phenomenon, Liu [14] gave the definition of uncertain variables.

Definition 2.2. (Liu [14]) An uncertain variable is a measurable function from an uncertain space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set
\[
\xi^{-1}(B) = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B \}
\]
is an event.

Definition 2.3. (Liu [15]) The uncertain variables \( \xi_1, \xi_2, \ldots, \xi_m \) are said to be independent if
\[
\mathcal{M}\left\{\bigcap_{i=1}^{m} \xi_i \in B_i\right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}
\]
for any Borel sets \( B_1, B_2, \ldots, B_m \) of real numbers.

Since we have the definition of uncertain variable and uncertain measure, we must consider the product measure and uncertain arithmetic. In 2009, Liu [15] proposed the product measure axiom.

Axiom 2.1. (Liu [15]) Let \( \Gamma_k \) be nonempty sets on which \( \mathcal{M}_k \) are uncertain measures, \( k = 1, 2, \ldots, k \), respectively. Then the product uncertain measure \( \mathcal{M} \) is an uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n \), satisfying
\[
\mathcal{M}\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\}.
\]
That is, for each event \( \Lambda \in \mathcal{L} \), we have

In order to characterize uncertain variables, in 2007, Liu [14] proposed the concept of uncertainty distribution. Then, in 2009, a sufficient and necessary condition for uncertainty distribution was proposed by Peng & Iwamura [17].

Definition 2.4. (Liu [14]) The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by
\[
\Phi(x) = \mathcal{M}\{\xi \leq x\}.
\]
for any real number \( x \).

Theorem 2.1. (Peng & Iwamura [17]) A function \( \Phi : \mathbb{R} \to [0, 1] \) is an uncertainty distribution if and only if it is an increasing function except \( \Phi(x) = 0 \) and \( \Phi(x) = 1 \).

Theorem 2.2. (Liu [16]) Let \( \Phi_i \) be uncertainty distributions of uncertain variables \( \xi_i, i = 1, 2, \ldots, m \), respectively, and \( \Phi \) the joint uncertainty distribution of uncertain vector \( (\xi_1, \xi_2, \ldots, \xi_m) \). If \( \xi_1, \xi_2, \ldots, \xi_m \) are independent, then we have
\[
\Phi(x_1, x_2, \ldots, x_m) = \min_{1 \leq i \leq m} \Phi_i(x_i).
\]
for any real numbers \( x_1, x_2, \ldots, x_m \).

Theorem 2.3. (Liu [16]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f : \mathbb{R}^n \to \mathbb{R} \) is a strictly increasing function, then
\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]
is an uncertain variable whose inverse uncertainty distribution is
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)),
\]
\( 0 < \alpha < 1 \).

Theorem 2.4. (Liu [16]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f : \mathbb{R}^n \to \mathbb{R} \) is a strictly decreasing function, then
\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]
is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha), \ldots, \Phi_n^{-1}(1 - \alpha)),
\]
\( 0 < \alpha < 1 \).

Theorem 2.5. (Liu [16]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If the function \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_m \) and strictly decreasing with \( x_{m+1}, x_{m+2}, \ldots, x_n \), then
\[
\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)
\]
is an uncertain variable with inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \ldots, \Phi_n^{-1}(1 - \alpha)).
\]
Uncertain Reliability

In 2010, Liu [18] proposed uncertain reliability analysis as a tool to deal with system reliability via uncertainty theory. Reliability index is defined as the uncertain measure that the system is working.

**Definition 2.5.** (Liu [18]) Assume a system contains uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$, and is working if and only if $R(\xi_1, \xi_2, \cdots, \xi_n) \geq 0$. Then the reliability index is

$$\text{Reliability} = \mu\{ R(\xi_1, \xi_2, \cdots, \xi_n) \geq 0 \}.$$  

(6)

**Theorem 2.6.** (Liu [18]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $R$ is a strictly increasing function. If some system is working if and only if $R(\xi_1, \xi_2, \cdots, \xi_n) \geq 0$, then the reliability index is

$$\text{Reliability} = \alpha$$  

(7)

where $\alpha$ is the root of

$$R(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) = 0.$$  

(8)

**Theorem 2.7.** (Liu [18]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $R$ is a strictly decreasing function. If some system is working if and only if $R(\xi_1, \xi_2, \cdots, \xi_n) \geq 0$, then the reliability index is

$$\text{Reliability} = \alpha$$  

(9)

where $\alpha$ is the root of

$$R(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) = 0.$$  

(10)

**Theorem 2.8.** (Liu [18]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and the function $R(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$. If some system is working if and only if $R(\xi_1, \xi_2, \cdots, \xi_n) \geq 0$, then the reliability index is

$$\text{Reliability} = \alpha$$  

(11)

where $\alpha$ is the root of

$$R(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) = 0.$$  

(12)

3 Structural Reliability

On the process of designing and using of engineering structures, there are a large number of uncertain factors and information that directly affect the safety and reliability of the structure. How to found the appropriate mathematical model of structural reliability under uncertain environment, is the main points of the structural reliability theory. The characteristics of reliability of engineering structures is that, besides corrosion and fatigue the structural system is not a gradual destruction, and even in some cases its intensity is enhancement, such as the strength of the concrete structure increases with age in a period of time. Therefore, this paper studies the the situation of the destruction of the structure for not to support too much load and be damaged immediately.

Loading capacity of the structure is related to many parameters. In addition to the shape of the structure itself, it is related to the loading capacity of each bars. In addition to determining the properties of materials at the bar, it also depends on the length of the material, cross-sectional area and so on. As the loading capacity is determined by a number of parameters it inevitably brings many errors in the actual construction process. Therefore, the loading capacity of rods is uncertainty. On the other hand, the load of the bars depend on many factors which can not be determined. In previous studies, these two variables are considered as random variables or fuzzy variables. However, both random variables and fuzzy variables would be insufficient. The following two examples will be used to illustrate the problem.

**Example 3.1.**

**Example 3.2.** It is assumed that the resistance of a rod is about $100N$. If $\text{about}100N$ is regarded as a fuzzy concept, then we may assign it a membership function, say

$$\mu(x) = \begin{cases} 
(x - 80)/20, & \text{if } 80 \leq x \leq 100 \\
(120 - x)/20, & \text{if } 100 \leq x \leq 120.
\end{cases}$$

This membership function represents a triangular fuzzy variable $(80, 100, 120)$. Please do not argue why I choose such a membership function because it is not important for the focus of debate. Based on this membership function, possibility theory (or credibility theory) will conclude the following proposition: The resistance of a rod is “exactly $100N$$’’ with belief degree 1 in possibility measure (or 0.5 in credibility measure). However, it is doubtless that the belief degree of “exactly $100N$$’’ is almost zero. Nobody is so naive to expect that “exactly $100N$$’’ is the true resistance of the rod. On the other hand, “exactly $100N$$’’ and “not $100N$$’’ have the same belief degree in either possibility measure or credibility measure. Thus we have to regard them “equally likely’’. It seems that no human being can accept this conclusion. This paradox shows that those imprecise quantities like “about $100N$$’’ cannot be quantified by possibility measure (or credibility measure) and then they are not fuzzy concepts. If those imprecise quantities are understood as uncertain variables, then the paradox will disappear immediately.

The structural reliability index is defined as the uncertain measure that the resistance is larger than the load. According to the meaning of structural reliability index, it is determined...
by the resistance and the load. For each rod, if it failed, then we say the structure failed. Now, some theorems of four basic structures’ reliability index are given below.

Theorem 3.1. The structure is shown in the figure 3.1. The gravity of the object is an uncertain variable γ, its distribution is Ψ. The resistance of each rods are β1, β2, · · · , βn, and the distributions of them are Φ1, Φ2, · · · , Φn, respectively. The resistance of the structure is β. Then the reliability index is α = α1 ∧ α2 ∧ · · · ∧ αn. Each α1, α2, · · · , αn is the root of the equations

\[
\Phi_1^{-1}(1 - \alpha_1) = \Psi^{-1}(\alpha_1), \\
\Phi_2^{-1}(1 - \alpha_2) = \Psi^{-1}(\alpha_2), \\
\vdots \\
\Phi_n^{-1}(1 - \alpha_n) = \Psi^{-1}(\alpha_n)
\]

, respectively.

Proof. According to the stress analysis, the load of each rod is γ, respectively. As we know, the resistance of the structure β is β = β1 ∧ β2 ∧ · · · ∧ βn, that is, there exist a function

\[R(\beta_1, \beta_2, \cdots, \beta_n, \gamma) = \beta_1 ∧ \beta_2 ∧ · · · ∧ \beta_n - \gamma,\]

if and only if \( R \geq 0 \) the system works. Then the reliability index α is the root of the equation

\[\Phi_1^{-1}(1-\alpha) ∧ \Phi_2^{-1}(1-\alpha) ∧ · · · ∧ \Phi_n^{-1}(1-\alpha) = \Psi^{-1}(\alpha). \quad (13)\]

The reliability of the ith rod is the root of the equation

\[\Phi_i^{-1}(1-\alpha_i) = \Psi^{-1}(\alpha_i). \]

According to the above two equations, the reliability index of the structure must be the reliability index of one of the rods, it means that there exist \( i, 1 \leq i \leq n, \) subject to \( \alpha = \alpha_i, \) and we have \( \Phi_i^{-1}(1-\alpha_i) = \Psi^{-1}(\alpha_i). \)

For any \( \alpha_k \geq \alpha_i, \) it satisfies \( \Phi_k^{-1}(1-\alpha_k) = \Psi^{-1}(\alpha_k), \) according to the property of the distribution function, \( \Phi_i^{-1} \) and \( \Psi^{-1} \) are all increasing function, then we have

\[\Psi^{-1}(\alpha_k) \geq \Psi^{-1}(\alpha_i) = \Phi_i^{-1}(1-\alpha_i) \geq \Phi_i^{-1}(1-\alpha_k),\]

that is

\[\Phi_1^{-1}(1-\alpha_k) ∧ \Phi_2^{-1}(1-\alpha_k) ∧ · · · ∧ \Phi_n^{-1}(1-\alpha_k) ≤ \Psi^{-1}(\alpha_k).\]

So \( \alpha_k \) is not the root of the equation(13). The reliability index α is the minimum of the reliability index of each rod, that is \( \alpha = \alpha_1 ∧ \alpha_2 ∧ · · · ∧ \alpha_n. \)  

Theorem 3.2. The structure is shown in the figure 3.2. The gravity of the object is an uncertain variable γ, its distribution is Ψ. The resistance of each rods are β1, β2, and the distributions of them are Φ1, Φ2, respectively. The resistance of the structure is β. Then the reliability index is α = α1 ∧ α2. α1 and α2 are the roots of the equations

\[\Phi_1^{-1}(1-\alpha_1) = \frac{\sin \theta_2}{\sin(\theta_1 + \theta_2)} \Psi^{-1}(\alpha_1)\]

and \( \Phi_2^{-1}(1-\alpha_2) = \frac{\sin \theta_1}{\sin(\theta_1 + \theta_2)} \Psi^{-1}(\alpha_2), \) respectively.

Proof. According to the stress analysis, the load of rods are γ sin \( \theta_2/\sin(\theta_1 + \theta_2) \) and γ sin \( \theta_1/\sin(\theta_1 + \theta_2) \), respectively. The reliability index of both rods α1, α2 are the roots of the equations

\[\Phi_1^{-1}(1-\alpha_1) = \frac{\sin \theta_2}{\sin(\theta_1 + \theta_2)} \Psi^{-1}(\alpha_1)\]

and \( \Phi_2^{-1}(1-\alpha_2) = \frac{\sin \theta_1}{\sin(\theta_1 + \theta_2)} \Psi^{-1}(\alpha_2). \)

For the structure, the resistance is

\[\beta \sin(\theta_1 + \theta_2) \land \beta \sin(\theta_1 + \theta_2), \]

that is, there exists a function

\[R = \frac{\beta \sin(\theta_1 + \theta_2)}{\sin \theta_2} \land \frac{\beta \sin(\theta_1 + \theta_2)}{\sin \theta_1} - \gamma, \]

if and only if \( R \geq 0 \) the system works. Then the reliability index α is the root of the equation

\[\frac{\sin(\theta_1 + \theta_2)}{\sin \theta_2} \Phi_1^{-1}(1-\alpha) \land \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \Phi_2^{-1}(1-\alpha) = \Psi^{-1}(\alpha). \quad (14)\]

According to the above two equations, the reliability index of the structure must be the reliability index of one of the rods, that is \( \alpha = \alpha_1 ∧ \alpha_2. \)  

Theorem 3.3. The structure is shown in the figure 3.3. The gravity of the object is an uncertain variable γ, its distribution is Ψ. The resistance of each rods are β1, β2 and β3, and the distributions of them are Φ1, Φ2 and Φ3, respectively. The resistance of the structure is β. Then the reliability index is α = α1 ∧ α2 ∧ α3. α1, α2 and α3 are the roots of the equations

\[\Phi_1^{-1}(1-\alpha_1) = \frac{1}{1 + 2 \cos^2 \theta} \Psi^{-1}(\alpha_1),\]

\[\Phi_2^{-1}(1-\alpha_2) = \frac{\cos^2 \theta}{1 + 2 \cos^2 \theta} \Psi^{-1}(\alpha_2),\]

\[\Phi_3^{-1}(1-\alpha_3) = \frac{\cos^2 \theta}{1 + 2 \cos^2 \theta} \Psi^{-1}(\alpha_3),\]

respectively.
Proof. According to the stress analysis, the load of rods are 
\[ \gamma/(1 + 2\cos^3 \theta), \gamma \cos^2 \theta/(1 + 2\cos^3 \theta), \gamma \cos^2 \theta/(1 + 2\cos^3 \theta), \text{ respectively.} \]
The reliability index of each rod is the root of the equation
\[ \Phi^{-1}_1(1 - \alpha_1) = \frac{1}{1 + 2\cos^3 \theta} \Psi^{-1}(\alpha_1), \]
\[ \Phi^{-1}_2(1 - \alpha_2) = \frac{\cos^2 \theta}{1 + 2\cos^3 \theta} \Psi^{-1}(\alpha_2), \]
\[ \Phi^{-1}_3(1 - \alpha_3) = \frac{\cos^2 \theta}{1 + 2\cos^3 \theta} \Psi^{-1}(\alpha_3). \]
For the structure, the resistance is
\[ (1 + 2\cos^3 \theta) \beta_1 \land \frac{1 + 2\cos^3 \theta}{\cos^2 \theta} \beta_2 \]
\[ \land \frac{1 + 2\cos^3 \theta}{\cos^2 \theta} \beta_3, \]
that is, there exist a function
\[ R(\beta_1, \beta_2, \beta_3, \gamma) = (1 + 2\cos^3 \theta) \beta_1 \land \frac{1 + 2\cos^3 \theta}{\cos^2 \theta} \beta_2 \]
\[ \land \frac{1 + 2\cos^3 \theta}{\cos^2 \theta} \beta_3 - \gamma, \]
if and only if \( R \geq 0 \) the system works. Then the reliability index \( \alpha \) is the root of the equation
\[ (1 + 2\cos^3 \theta) \Phi^{-1}_1(1 - \alpha) \land \frac{1 + 2\cos^3 \theta}{\cos^2 \theta} \Phi^{-1}_2(1 - \alpha) \]
\[ \land \frac{1 + 2\cos^3 \theta}{\cos^2 \theta} \Phi^{-1}_3(1 - \alpha) = \Psi^{-1}(\alpha). \]
According to the above two equations, the reliability index of the structure must be the reliability index of one of the rods, that is \( \alpha = \alpha_1 \land \alpha_2 \land \alpha_3. \]

4 Conclusions

The theory of structural reliability is widely applied in many fields, such as architectural design, and mechanical engineering. In this paper, the resident and the load of a structure are defined as uncertain variables and the reliability index is defined as the uncertain measure of the event that the resident is larger than the load. A structural reliability index as the uncertain measure of the event is discussed.

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