A Stock Model with Jumps for Uncertain Markets

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Abstract

Uncertain differential equation with jumps is a type of differential equation driven by two classes of uncertain processes, namely canonical process and renewal process. Based on uncertain differential equation with jumps, this paper proposes a stock model with jumps for uncertain financial markets. Furthermore, the European call and put option pricing formulas for the stock model are formulated and some mathematical properties of them are studied. Finally, some generalized uncertain stock models with jumps are discussed.

Keywords: Stock model, European option, Uncertain finance, Uncertain differential equation with jumps

1 Introduction

In 1900, Brownian motion was first introduced to finance by Bachelier [1]. However, it is clearly inadequate to use Brownian motion to describe a market price because it would predict negative stock price. In 1965, Samuelson [17, 18] proposed the argument that stock prices follow geometric Brownian motion. Following this argument, Black and Scholes [3] and Metron [15] independently used the geometric Brownian motion to determine the stock price in the early 1970s. After that, the Black-Scholes model has been widely used in pricing options. Hull and White [7] introduce Brownian motion to model interest rate, and Black and Karasinski [2] proposed a mean-reverting stochastic stock model. Interested reader may refer to Karatzas and Shreve [8, 9] for more information.

Most economical behavior incurs human uncertainty. In order to study uncertain phenomena in human systems, Liu [10] founded an uncertainty theory that is a branch of mathematics based on normality, self-duality, countable subadditivity, and product measure axioms. For the purpose of dealing with the evolution of uncertain phenomena, a concept of uncertain process was presented by Liu [11] in 2008. One of the most important uncertain process is canonical process, which is defined by Liu [12] in 2009. Based on this process, uncertain calculus was initialized by Liu [12] in 2009 to deal with differentiation and integration of functions of uncertain processes. Furthermore, uncertain differential equation driven

Uncertain renewal process is another kind of uncertain process. It was first proposed by Liu [11] in 2008. In 2011, Yao [19] proposed uncertain calculus with respect to uncertain renewal process. In daily life, the stock price is not always continuous because of economic crisis and war. In order to incorporate these sudden drift into stock models, we develop a stock model with jumps for uncertain financial market in this paper. The rest of this paper is organized as follows. The next section is intended to introduce some useful concepts of uncertain process as needed. The uncertain differential equation with jumps is presented in Section 3. A stock model with jumps for uncertain financial markets is proposed in Section 4. European option price on the proposed uncertain stock model with jumps are investigated in Section 5. Furthermore a multi-factor model with jumps is studied in Section 6. Finally, a brief conclusion is made in Section 7.

2 Preliminaries

Uncertainty theory was founded by Liu [10] to provide a mathematical model for dealing with uncertain phenomena in human system. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Some properties of uncertain measure have been studied by You [20] and Gao [6]. An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. An uncertain process is a sequence of uncertain variables indexed by time and space. The most important uncertain processes are canonical process and renewal process.

Definition 1. (Liu [11]) An uncertain process $C_t$ is said to be a canonical Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with uncertainty distribution

$$\Phi(x) = \left(1 + \exp \left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1}, \ x \in \mathbb{R}.$$ 

Definition 2. (Liu [12]) Let $X_t$ be an uncertain process and $C_t$ be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$
Then the uncertain integral of $X_t$ with respect to $C_t$ is defined by

$$
\int_a^b X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})
$$

provided that the limit exists almost surely and is finite.

**Definition 3.** (Liu [12]) Let $C_t$ be a canonical process and $Z_t$ be an uncertain process. If there exist uncertain processes $\mu_s$ and $\sigma_s$ such that

$$
Z_t = Z_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dC_s
$$

for any $t \geq 0$, then $Z_t$ is said to have an uncertain differential with respect to $C_t$, denoted by

$$
\, dZ_t = \mu_t \, dt + \sigma_t \, dC_t.
$$

**Definition 4.** (Liu [11]) Let $\xi_1, \xi_2, \cdots$ be iid positive uncertain variables. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process

$$
N_t = \max_{n \geq 0} \{ n | S_n \leq t \}
$$

is called a renewal process.

**Definition 5.** (Yao [19]) Let $X_t$ be an uncertain process and $N_t$ a renewal process. For any partition of a closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
$$

Then the uncertain integral of $X_t$ with respect to $N_t$ is

$$
\int_a^b X_t \, dN_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (N_{t_{i+1}} - N_{t_i})
$$

provided that the limit exists almost surely and is finite. For this case, the uncertain process $X_t$ is said to be integrable with respect to $N_t$.

**Definition 6.** (Yao [19]) Let $N_t$ be a renewal process and $Z_t$ an uncertain process. If there exist uncertain processes $\mu_s$ and $\gamma_s$ such that

$$
Z_t = Z_0 + \int_0^t \mu_s \, ds + \int_0^t \gamma_s \, dN_s
$$

for any $t \geq 0$, then $Z_t$ is said to have an uncertain differential with respect to $N_t$, denoted by

$$
\, dZ_t = \mu_t \, dt + \gamma_t \, dN_t.
$$
3 Uncertain Differential Equation with Jumps

In this section, we will introduce uncertain differential equation with jumps. First, we introduce uncertain differential with respect to both \( C_t \) and \( N_t \).

**Definition 7.** Let \( C_t \) be a canonical Liu process, \( N_t \) a renewal process and \( Z_t \) an uncertain process. If there exist uncertain processes \( \mu_s, \sigma_s \) and \( \gamma_s \) such that

\[
Z_t = Z_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dC_s + \int_0^t \gamma_s dN_s
\]

for any \( t \geq 0 \), then \( Z_t \) is said to have an uncertain differential

\[
dZ_t = \mu_t dt + \sigma_t dC_t + \gamma_t dN_t.
\]

For this case, \( Z_t \) is called a differentiable uncertain process with drift \( \mu_t \), diffusion \( \sigma_t \) and jump \( \gamma_t \).

**Example 1.** Let \( Z_t = \mu t + \sigma C_t + \gamma N_t \) be an uncertain process. Since

\[
Z_t = \int_0^t \mu ds + \int_0^t \sigma dC_s + \int_0^t \gamma dN_s,
\]

we obtain that the uncertain process \( Z_t \) is differentiable and has an uncertain differential

\[
dZ_t = \mu dt + \sigma dC_t + \gamma dN_t.
\]

**Example 2.** Let \( Z_t = tC_t N_t \) be an uncertain process. Since

\[
Z_t = \int_0^t C_s N_s ds + \int_0^t sN_s dC_s + \int_0^t sC_s dN_s,
\]

we obtain that the uncertain process \( Z_t \) is differentiable and has an uncertain differential

\[
dZ_t = C_t N_t dt + tN_t dC_t + tC_t dN_t.
\]

**Theorem 1.** (Yao [19], Fundamental Theorem) Let \( C_t \) be a canonical Liu process, \( N_t \) a renewal process, and \( h(t, C_t, N_t) \) a continuously differentiable function. Then the uncertain process \( Z_t = h(t, C_t, N_t) \) has an uncertain differential

\[
dZ_t = \frac{\partial h}{\partial t}(t, C_t, N_t) dt + \frac{\partial h}{\partial C_t}(t, C_t, N_t) dC_t + h(t, C_t, N_t) - h(t, C_t, N_{t-}).
\]

**Proof:** Since the function \( h(t, C_t, N_t) \) is continuous differentiable, by Taylor series expansion, we have

\[
\Delta Z_t = h(t, C_t, N_t) - h(t - \Delta t, C_{t-\Delta t}, N_{t-\Delta t})
\]

\[
= h(t, C_t, N_t) - h(t - \Delta t, C_{t-\Delta t}, N_{t-\Delta t}) + h(t, C_t, N_t) - h(t, C_t, N_t - \Delta t)
\]

\[
= \frac{\partial h}{\partial t}(t, C_t, N_t-\Delta t) \Delta t + \frac{\partial h}{\partial C_t}(t, C_t, N_t-\Delta t) \Delta C_t + h(t, C_t, N_t) - h(t, C_t, N_t-\Delta t) + o(\Delta t) + o(\Delta C_t).
\]

Letting \( \Delta t \to 0 \), we have

\[
dh(t, C_t, N_t) = \frac{\partial h}{\partial t}(t, C_t, N_t) dt + \frac{\partial h}{\partial C_t}(t, C_t, N_t) dC_t + h(t, C_t, N_t) - h(t, C_t, N_{t-}).
\]
Example 3. Consider the uncertain process $\mu t + \sigma C_t + \gamma N_t$. For this case, we have $h(t, c, n) = \mu t + \sigma c + \gamma n$.

It is clear that
$$\frac{\partial h}{\partial t}(t, c, n) = \mu, \quad \frac{\partial h}{\partial c}(t, c, n) = \sigma, \quad h(t, c, N_t) - h(t, c, N_{t-}) = \gamma dN_t.$$

It follows from the fundamental theorem that
$$d(\mu t + \sigma C_t + \gamma N_t) = \mu dt + \sigma dC_t + \gamma dN_t.$$

Now, we turn to uncertain differential equation with jumps.

Definition 8. (Yao [19]) Suppose $C_t$ is a canonical process, $N_t$ is a renewal process, and $f, g$ and $h$ are some given functions. Then
$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t + h(t, X_t)dN_t \tag{1}$$

is called an uncertain differential equation with jumps. A solution is an uncertain process $X_t$ that satisfies (1) identically in $t$.

Example 4. Let $\alpha, \beta$ and $\gamma$ be real numbers. Consider an uncertain differential equation with jumps
$$dX_t = \alpha dt + \beta dC_t + \gamma dN_t.$$

Integrate on both sides, and we have
$$X_t - X_0 = \int_0^t \alpha ds + \int_0^t \beta dC_s + \int_0^t \gamma dN_s.$$

Thus the uncertain differential equation has a solution
$$X_t = X_0 + \alpha t + \beta C_t + \gamma N_t.$$

Example 5. Let $\alpha, \beta$ and $\gamma$ be real numbers. Consider an uncertain differential equation with jumps
$$dX_t = \alpha X_t dt + \beta X_t dC_t + \gamma X_t dN_t.$$

It is easy to verify that the uncertain differential equation has a solution
$$X_t = X_0(1 + \gamma)^N_t \exp(\alpha t + \beta C_t).$$

4 An Uncertain Stock Model with Jumps

4.1 Liu’s Stock Model

Let $X_t$ be the bond price, and $Y_t$ the stock price. Assume that the stock price $Y_t$ follows a geometric canonical process. Then Liu [12] proposed a stock model as follows,
$$\begin{cases}
    dX_t = r X_t dt \\
    dY_t = e Y_t dt + \sigma Y_t dC_t
\end{cases}$$
where \( r \) is the riskless interest rate, \( e \) is the stock drift, and \( \sigma \) is the stock diffusion, and \( C_t \) is a canonical process. Liu [12] gave European option pricing formulas for Liu’s stock model, and Chen [5] gave American option pricing formulas.

### 4.2 Stock Model with Jumps

In many cases the stock price is not continuous because of economic crisis, war or other occasional accidents. In order to incorporate these occasional factors into stock model, we introduce uncertain differential equations with jumps to uncertain finance, and propose a stock model with jumps. Let \( X_t \) be the bond price, and \( Y_t \) the stock price. Assume that price \( Y_t \) follows a geometric canonical process with jumps. Then we propose an uncertain stock model as follows

\[
\begin{align*}
    dX_t &= rX_t\,dt \\
    dY_t &= \alpha Y_t\,dt + \beta Y_t\,dC_t + \gamma Y_t\,dN_t
\end{align*}
\]

(2)

where \( r \) is the riskless interest rate, \( \alpha, \beta \) and \( \gamma \) is the stock drift, diffusion and jump coefficients.

### 5 European Option

A European option is a financial instrument which gives the holder a right, but not the obligation, to trade the underlying asset at an expiry time for a prescribed strike price.

#### 5.1 European Call Option Pricing with Jumps

A European call option gives the holder the right, but not the obligation, to buy a stock at the expiry time \( t \) for the strike price \( K \). If \( Y_t \) is the final price of the underlying stock, then the payoff from buying a European call option is given by

\[
(Y_t - K)^+ = \begin{cases} 
Y_t - K, & \text{if } Y_t > K, \\
0, & \text{otherwise}.
\end{cases}
\]

Considering the time value of money resulted the bond, the present value of this payoff is \( \exp(-rt)(Y_t - K)^+ \). The European call option price should be the expected present value of the payoff. Therefore the European call option pricing formula is

\[
f_c(Y_0, K, \alpha, \beta, \gamma) = E[\exp(-rt)(Y_t - K)^+].
\]

**Theorem 2.** Assume that \( N_t \) is an uncertain renewal process with iid uncertain interarrivals \( \xi_1, \xi_2, \cdots \) whose uncertainty distribution is \( \Phi \). Then the European call option formula for uncertain stock model with jumps (2) is

\[
f_c = \exp(-rt)Y_0 \int_{K/Y_0}^{+\infty} \inf_{n \geq 0} \left( 1 + \exp \left( \frac{\pi}{\sqrt{3} \beta t} (\ln y - \alpha t - n \ln(1 + \gamma)) \right) \right)^{-1} \wedge \Phi \left( \frac{t}{n} \right) \,dy.
\]

(3)
Proof: By Example 5, the stock price $Y_i$ follows a geometric canonical process and an exponential renewal process, i.e., $Y_i = Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t)$. By the definition of expected value of uncertain variable, we have

$$f_c = \exp(-rt)E \left[ (Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t) - K)^+ \right]$$

$$= \exp(-rt) \int_0^{+\infty} M\{Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t) - K \geq x\} dx$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} M\{(1 + \gamma)^N \exp(\alpha t + \beta C_t) \geq y\} dy$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} M\{N_{i} \ln(1 + \gamma) + \alpha t + \beta C_t \geq \ln y\} dy$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} 1 - M\{\beta C_t + N_{i} \ln(1 + \gamma) \leq \ln y - \alpha t\} dy$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} 1 - M\left\{ \bigcup_{n=0}^{\infty} \{\beta C_t \leq \ln y - \alpha t - n \ln(1 + \gamma)\} \cap (N_{i} = n) \right\} dy$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} 1 - \sup_{n \geq 0} M\left\{ C_t \leq \frac{\ln y - \alpha t - n \ln(1 + \gamma)}{\beta} \right\} \wedge M\{N_{i} \leq n\} dy$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} \inf_{n \geq 0} \left\{ C_t \geq \frac{\ln y - \alpha t - n \ln(1 + \gamma)}{\beta} \right\} \vee M\{N_{i} \geq k\} dy$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} 1 - \sup_{n \geq 0} \left( 1 + \exp \left( \frac{-\pi}{\sqrt{3}bt} \left( \ln y - \alpha t - n \ln(1 + \gamma) \right) \right) \right)^{-1} \vee \left( 1 - \Phi \left( \frac{a}{n} \right) \right) dy.$$

$$= \exp(-rt) Y_0 \int_{K/Y_0}^{+\infty} \inf_{n \geq 0} \left( 1 + \exp \left( \frac{\pi}{\sqrt{3}bt} \left( \ln y - \alpha t - n \ln(1 + \gamma) \right) \right) \right)^{-1} \wedge \Phi \left( \frac{t}{n} \right) dy.$$

This yields the desired result and proves the form of the European call option pricing formula.

Theorem 3. European call option formula $f_c(Y_0, K, \alpha, \beta, \gamma) = E[\exp(-rt)(Y_t - K)^+]$ has the following properties:

(i) $f_c$ is an increasing and convex function of $Y_0$;

(ii) $f_c$ is a decreasing and convex function of $K$;

(iii) $f_c$ is an increasing function of $\alpha$;

(iv) $f_c$ is an increasing function of $\beta$;

(v) $f_c$ is an increasing of $\gamma$ if $\gamma > 0$;

(vi) $f_c$ is a decreasing function of $r$.

Proof: (i). This property means that if the other variables remain unchanged, then the option price is an increasing and convex function of the stock’s initial price. To prove it, first note that for any positive constant $b$, the $\exp(-rt)(Y_0b - K)^+$ is an increasing and convex function of $Y_0$. Consequently, the quantity $\exp(-rt) (Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t) - K)^+$ is increasing and convex in $Y_0$. Since the uncertainty distribution of $(1 + \gamma)^N \exp(\alpha t + \beta C_t)$ does not depend on $Y_0$, the desired result is verified.

(ii). This follows from the fact that $\exp(-rt) (Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t) - K)^+$ is decreasing and
convex in $K$. It means that European call option price is a decreasing and convex function of the stock’s strike price when the other variables remain unchanged.

(iii). It is obvious that $f_c(Y_0, K, \alpha, \beta, \gamma) = \exp(-rt)E\left[\left(Y_0(1 + \gamma)^{N_t}\exp(\alpha t + \beta C_t) - K\right)^+\right]$ is an increasing function of $\alpha$. This means that European call option price will increase with the stock drift.

(iv). This follows from the fact that $f_c(Y_0, K, \alpha, \beta, \gamma) = E\left[\exp(-rt)(Y_t - K)^+\right]$ is an increasing function of $\beta$ immediately. It means that European call option price will increase with the stock diffusion.

(v). This follows from the fact that $f_c(Y_0, K, \alpha, \beta, \gamma) = \exp(-rt)E\left[\left(Y_0(1 + \gamma)^{N_t}\exp(\alpha t + \beta C_t) - K\right)^+\right]$ is an increasing function of $\gamma$ if $\gamma > 0$ immediately. This property means that European call option price will increase with the stock renewal coefficient if $\gamma > 0$.

(vi). Since $f_c(Y_0, K, \alpha, \beta, \gamma) = \exp(-rt)E\left[\left(Y_t - K\right)^+\right]$ is a decreasing function of $r$ and the expected value is independent of $r$, the result is verified. This means European call option price will decrease with the riskless interest rate.

5.2 European Put Option Pricing Formula with Jumps

A European put option gives the holder the right, but not the obligation, to buy a stock at the expiry time $t$ for the strike price $K$. If $Y_t$ is the final price of the underlying stock, then the payoff from buying a European put option is given by

$$(K - Y_t)^+ = \begin{cases} 
K - Y_t, & \text{if } Y_t < K, \\
0, & \text{otherwise}.
\end{cases}$$

Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rt)(K - Y_t)^+$. Hence the European put option price is given by

$$f_p(Y_0, K, \alpha, \beta, \gamma) = E\left[\exp(-rt)(K - Y_t)^+\right].$$

Theorem 4. Assume that $N_t$ is an uncertain renewal process with iid uncertain interarrivals $\xi_1, \xi_2, \cdots$ whose uncertainty distribution is $\Phi$. Then the European put option formula for uncertain stock model with jumps (2) is

$$f_p = \exp(-rt)Y_0\int_0^{K/Y_0} \sup_{n\geq 0} \left(1 + \exp\left(\frac{\pi}{\sqrt{3}t} (at + n \ln(1 + \gamma) - \ln y)\right)\right)^{-1} \wedge \Phi\left(\frac{t}{n}\right) dy. \quad (4)$$

Proof: By Example 5, the stock price $Y_t$ follows a geometric canonical process and an exponential renewal process, i.e., $Y_t = Y_0(1 + \gamma)^{N_t}\exp(\alpha t + \beta C_t)$. By the definition of expected value of uncertain variable, we have

$$f_p = \exp(-rs)E\left[\left(K - Y_0(1 + \gamma)^{N_t}\exp(\alpha t + \beta C_t)\right)^+\right]$$

$$= \exp(-rt)\int_0^{+\infty} \mathcal{M}\{K - Y_0(1 + \gamma)^{N_t}\exp(\alpha t + \beta C_t) \geq x\} dx$$
\[
\begin{align*}
&= \exp(-rt) Y_0 \int_0^{K/Y_0} \mathcal{M}\{(1 + \gamma)^N \exp(\alpha t + \beta C_t) \leq y\} dy \\
&= \exp(-rt) Y_0 \int_0^{K/Y_0} \mathcal{M}\{N_t \ln(1 + \gamma) + \alpha t + \beta C_t \leq \ln y\} dy \\
&= \exp(-rt) Y_0 \int_0^{K/Y_0} \mathcal{M}\{\beta C_t + N_t \ln(1 + \gamma) \leq \ln y - \alpha t\} dy \\
&= \exp(-rt) Y_0 \int_0^{K/Y_0} \mathcal{M}\left\{\bigcup_{n=0}^{\infty} \{\beta C_t \leq \ln y - \alpha t - n \ln(1 + \gamma)\} \cap \{N_t = n\}\right\} dy \\
&= \exp(-rt) Y_0 \int_0^{K/Y_0} \sup_{n \geq 0} \mathcal{M}\left\{C_t \leq \frac{\ln y - \alpha t - n \ln(1 + \gamma)}{\beta}\right\} \land \mathcal{M}\{N_t \leq n\} dy \\
&= \exp(-rt) Y_0 \int_0^{K/Y_0} \sup_{n \geq 0} \left(1 + \exp\left(\frac{\pi}{\sqrt{3} \beta t} (\alpha t + n \ln(1 + \gamma) - \ln y)\right)\right)^{-1} \land \Phi\left(\frac{t}{\sqrt{n}}\right) dy.
\end{align*}
\]

This yields the desired result and completes the proof of the European put option pricing formula.

**Theorem 5.** European put option formula \( f_p(Y_0, K, \alpha, \beta, \gamma) = E[\exp(-rt)(K - Y_t)^+] \) has the following properties:

(i) \( f_p \) is a decreasing and convex function of \( Y_0 \);

(ii) \( f_p \) is an increasing and convex function of \( K \);

(iii) \( f_p \) is a decreasing function of \( \alpha \);

(iv) \( f_p \) is a decreasing function of \( \beta \);

(v) \( f_p \) is a decreasing function of \( \gamma \) if \( \gamma > 0 \);

(vi) \( f_p \) is a decreasing function of \( r \).

**Proof:**

(i). This property means that if the other variables remain unchanged, then the option price is a decreasing and convex function of the stock’s initial price. To prove it, first note that for any positive constant \( b \), the \( \exp(-rt)(K - Y_0b)^+ \) is a decreasing and convex function of \( Y_0 \). Consequently, the quantity \( \exp(-rt)(K - Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t))^+ \) is decreasing and convex in \( Y_0 \). Since the uncertainty distribution of \( (1 + \gamma)^N \exp(\alpha t + \beta C_t) \) does not depend on \( Y_0 \), the desired result is verified.

(ii). This follows from the fact that \( \exp(-rt)(K - Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t))^+ \) is increasing and convex in \( K \). It means that European put option price is an increasing and convex function of the stock’s strike price when the other variables remain unchanged.

(iii). It is obvious that \( f_p = E[\exp(-rt)(K - Y_t)^+] = \exp(-rt)E[K - Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t)]^+ \) is a decreasing function of \( \alpha \). This means that European put option price will decrease with the stock drift.

(iv). This follows from the fact that \( f_p(Y_0, K, \alpha, \beta, \gamma) = E[\exp(-rt)(K - Y_t)^+] \) is a decreasing function of \( \beta \) immediately. It means that European put option price will decrease with the stock diffusion.

(v). This follows from the fact that \( f_p(Y_0, K, \alpha, \beta, \gamma) = \exp(-rt)E[K - Y_0(1 + \gamma)^N \exp(\alpha t + \beta C_t)]^+ \) is a decreasing function of \( \gamma \) if \( \gamma > 0 \) immediately. This property means that European put option price will decrease with the stock renewal coefficient if \( \gamma > 0 \).
(vi). Since \( f_p(Y_0, K, \alpha, \beta, \gamma) = E[\exp(-rt)(K - Y_t)^+] \) is a decreasing function of \( r \) and the expected value is independent of \( r \), the result is verified. This means European put option price will decrease with the riskless interest rate.

6 Generalized Stock Models with Jumps

In stock models with jumps, there are three important constants: stock drift stock \( \alpha \), diffusion \( \beta \), stock renewal coefficient \( \gamma \), they play an important role in European option price which is a major concern to investors. Assume that three items of stock model are dynamic function with respect to time \( t \). Then the stock models with jumps naturally become a general stock model.

6.1 Stock Model with Uncertain Jumps

In order to adapt the above model to be more consistent with the general cases, we introduce an extension of the model with time-dependent parameters. The most general model with jumps satisfies the following uncertain differential equation

\[
\begin{align*}
    dX_t &= r_t X_t dt \\
    dY_t &= u(t, Y_t) dt + v(t, Y_t) dC_t + m(t, Y_t) dN_t
\end{align*}
\]

(5)

where \( r_t, u(t, Y_t), v(t, Y_t), m(t, Y_t) \) are deterministic functions of time \( t \).

6.2 Multi-factor Stock Model with Jumps

Assume that the stock price is driven by multiple canonical process and renewal process. Then the multi-factor stock model can be expressed as

\[
\begin{align*}
    dX_t &= r_t X_t dt \\
    dY_{it} &= u(t, Y_{it}) dt + \sum_{j=1}^{k} v_j(t, Y_{it}) dC_{jt} + \sum_{j=1}^{n} w_j(t, Y_{it}) dN_{jt}, i = 1, 2, \cdots, m
\end{align*}
\]

(6)

where \( r_t, u(t, Y_{it}), v_j(t, Y_{it}), w_j(t, Y_{it}) \) are deterministic functions of time \( t \).

7 Conclusion

This paper first studied uncertain differential with both canonical process and uncertain renewal process. Based on this, uncertain differential equation with jumps was proposed. It was used to model the stock price in uncertain finance markets, and a stock model with jumps was presented in this paper. This paper also derived its European option pricing formulas. At last, the stock model was extended to other stock model for better applications.
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References


