Abstract: The variance of uncertain variables often appears in uncertain programming problems. Based on the definition of the variance of an uncertain variable, this paper deduces the variance formula for trapezoidal uncertain variables. This obtained formula is useful in investigating properties of uncertain programming problems.

Keywords: Uncertain Variable; Expected Value; Variance.

1 Introduction

Uncertainty theory [13] is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. The uncertainty theory has become a new tool to study subjective uncertainty. As an application of uncertainty theory, Liu [16] proposed a spectrum of uncertain programming which is a type of mathematical programming involving uncertain variables. In addition, Li and Liu [11] presented uncertain logic in which the truth value is defined as the uncertain measure that the proposition is true. Furthermore, uncertain inference was pioneered by Liu [15] as a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertainty. Gao and Ralescu [6] gave some expressions of Liu's inference rule via identification functions. Other researchers also have done a lot of theoretical work related to uncertainty theory, such as Chen and Ralescu [2], Gao [5], Qin et. al. [25], You [26], Peng [21], Peng and Iwamura [22] and Zhu [27], etc. For more details of uncertainty theory, the readers may consult Liu [17].

The variance of uncertain variable provides a measure of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value. The variance of uncertain variables is very important in economics, statistical analysis and other fields of uncertainty theory and applications. However, due to the difficulties involved in computing variance, studies usually focus on searching for good approximation and simulation methods to solve complex uncertain programming problems. Based on the definition of the variance of an uncertain variable [13], this paper deduces general formula for the variance of trapezoidal uncertain variables, which can be used much more conveniently in uncertain optimization problems.

The rest of this paper is organized as follows. Some preliminary concepts of uncertain variables are recalled in Section 2. In Section 3, we deduce the variance formula for trapezoidal uncertain variables. Finally, a brief summary is given.

2 Uncertain Variables

Let $\Gamma$ be a nonempty set, and let $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is called an event. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event a number which indicates the degree that the event will occur. In order to ensure that the number has certain mathematical properties, Liu [13] proposed the following four axioms:

**Axiom 1.** (Normality) $\mathcal{M}\{\Gamma\} = 1$.

**Axiom 2.** (Increasing) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$.

**Axiom 3.** (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^C\} = 1$ for any event $\Lambda$.

**Axiom 4.** (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$ 

Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

**Definition 1** (Liu [13]) An uncertain variable is defined as a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

**Definition 2** (Liu [13]) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{0}^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.
The variance of uncertain variable provides a measure of the spread of the distribution around its expected value.

**Definition 3 (Liu [13])** Let \( \xi \) be an uncertain variable with finite expected value \( c \). Then the variance of \( \xi \) is defined by 
\[
V[\xi] = E \left[ (\xi - c)^2 \right].
\]

The first identification function is one type of identification functions to characterize uncertain variables.

**Definition 4** An uncertain variable \( \xi \) has a first identification function \( \lambda \) if:

(i) \( \lambda(x) \) is a nonnegative function on \( \mathbb{R} \) such that 
\[
\sup_{x \neq y} (\lambda(x) + \lambda(y)) = 1;
\]

(ii) for any set \( B \) of real numbers, we have 
\[
M\{\xi \in B\} = \begin{cases} 
\sup \lambda(x), & \text{if } \sup \lambda(x) < 0.5 \\
1 - \sup \lambda(x), & \text{if } \sup \lambda(x) \geq 0.5. 
\end{cases}
\]

## 3 Variance Formula

By a *trapezoidal uncertain variable* we mean the uncertain variable fully determined by the quadruplet \( (a, b, c, d) \) of crisp numbers with \( a < b < c < d \), whose first identification function is 
\[
\lambda(x) = \begin{cases} 
x - a, & \text{if } a \leq x \leq b \\
\frac{2(b - a)}{2(b - a)}, & \text{if } b \leq x \leq c \\
x - d, & \text{if } c \leq x \leq d \\
\frac{2(c - d)}{2(c - d)}, & \text{otherwise}. 
\end{cases}
\]

The trapezoidal uncertain variable \( \xi = (a, b, c, d) \) has an expected value \( E[\xi] = (a + b + c + d)/4 \).

**Theorem 1** Let \( \xi = (a, b, c, d) \) be a trapezoidal uncertain variable. Then its variance is 
\[
V[\xi] = 4\alpha^2 + 3\alpha\beta + \beta^2 + 9\alpha\gamma + 3\beta\gamma + 6\gamma^2 + \frac{[(\alpha - \beta - 2\gamma)^+]^3}{384\alpha},
\]
where \( \alpha = (b - a) \lor (d - c), \beta = (b - a) \land (d - c) \) and \( \gamma = c - b \).

**Proof:** If \( \alpha - \beta \leq 2\gamma \), we have 
\[
M\{(\xi - c)^2 \geq r\} = \begin{cases} 
0.5, & \text{if } 0 \leq r \leq (\alpha - \beta + 2\gamma)^2/16 \\
\frac{8\beta}{\alpha} & \text{if } (\alpha - \beta + 2\gamma)^2/16 \leq r \leq (\alpha + \beta + 2\gamma)^2/16 \\
\frac{3\alpha + \beta + 2\gamma - 4\sqrt{r}}{8\alpha} & \text{if } (\alpha + \beta + 2\gamma)^2/16 \leq r \leq (3\alpha + \beta + 2\gamma)^2/16 \\
0, & \text{if } r \geq (3\alpha + \beta + 2\gamma)^2/16.
\end{cases}
\]

Thus the variance is 
\[
V[\xi] = \int_0^{\infty} M\{(\xi - c)^2 \geq r\} \, dr \\
= \int_0^{(\alpha - \beta + 2\gamma)^2/16} 0.5 \, dr + \int_{(\alpha - \beta + 2\gamma)^2/16}^{(3\alpha + \beta + 2\gamma)^2/16} \frac{8\beta}{\alpha} \, dr \\
+ \int_{(3\alpha + \beta + 2\gamma)^2/16}^{(\alpha + \beta + 2\gamma)^2/16} \frac{3\alpha + \beta + 2\gamma - 4\sqrt{r}}{8\alpha} \, dr \\
= \frac{4\alpha^2 + 3\alpha\beta + \beta^2 + 9\alpha\gamma + 3\beta\gamma + 6\gamma^2}{48}.
\]

If \( \alpha - \beta > 2\gamma \), we have 
\[
M\{(\xi - c)^2 \geq r\} = \begin{cases} 
1 - \frac{3\alpha + \beta + 2\gamma + 4\sqrt{r}}{8\alpha}, & \text{if } 0 \leq r \leq (\alpha - \beta - 2\gamma)^2/16 \\
0.5, & \text{if } (\alpha - \beta - 2\gamma)^2/16 \leq r \leq (\alpha - \beta + 2\gamma)^2/16 \\
\frac{8\beta}{\alpha} & \text{if } (\alpha - \beta + 2\gamma)^2/16 \leq r \leq (\alpha + \beta + 2\gamma)^2/16 \\
\frac{3\alpha + \beta + 2\gamma - 4\sqrt{r}}{8\alpha} & \text{if } (\alpha + \beta + 2\gamma)^2/16 \leq r \leq (3\alpha + \beta + 2\gamma)^2/16 \\
0, & \text{if } r \geq (3\alpha + \beta + 2\gamma)^2/16.
\end{cases}
\]
Hence
\[ V[\xi] = \int_{0}^{+\infty} \mathcal{M} \{ (\xi - r)^{2} \geq r \} \, dr \]
\[ = \frac{4\alpha^{2} + 3\alpha\beta + \beta^{2} + 9\alpha\gamma + 3\beta\gamma + 6\gamma^{2}}{48} \]
\[ + \frac{(\alpha - \beta - 2\gamma)^{3}}{384\alpha}. \]

The theorem is proved.

Remark 1: Especially, if \( \xi \) is a symmetric trapezoidal uncertain variable, i.e., \( (b - a) = (d - c) \), then its variance is
\[ V[\xi] = \frac{4(b - a)^{2} + 6(b - a)(c - b) + 3(c - b)^{2}}{24}. \]

Remark 2: Especially, if \( b = c \), when the trapezoidal uncertain variable degenerates into a triangular uncertain variable, then \( \gamma = 0 \), and its variance is
\[ V[\xi] = \frac{4\alpha^{2} + 3\alpha\beta + \beta^{2} + 9\alpha\gamma + 3\beta\gamma + 6\gamma^{2}}{48} + \frac{(\alpha - \beta)^{3}}{384\alpha} \]
\[ = \frac{33\alpha^{3} + 21\alpha^{2}\beta + 11\alpha\beta^{2} - \beta^{3}}{384\alpha}. \]

The formula of the variance for triangular uncertain variables is also obtained.

4 Conclusions

The variance of an uncertain variable is often contained in the objectives or constraints of uncertain optimization problems. In order to avoid the difficulty in computing the variance of uncertain variables, we established the variance formula for trapezoidal uncertain variables. This formula also applies to triangular uncertain variables.

Acknowledgments

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References