Analysis of k-out-of-n System with Uncertain Lifetimes

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Abstract: In this paper, a concept of order uncertain variable is introduced as a main tool in analysis of k-out-of-n system with uncertain lifetimes. Then, by the operational law of uncertain variable, the distribution of order uncertain variable is deduced. At last, the k-out-of-n system is studied, and the uncertainty distribution of k-out-of-n system lifetime is given.

Keywords: Uncertain variable, uncertainty distribution, order uncertain variable, operational law, k-out-of-n system, system reliability

1 Introduction

The real world is filled with uncertainty. In order to model uncertainty, randomness and fuzziness are introduced as hypotheses, behind which probability theory and credibility theory have been proposed. However, some practical systems behave neither random nor fuzzy, it cannot be modeled precisely by probability theory or credibility theory. In order to deal with this type of uncertainty, Liu [7] founded an uncertainty theory in 2007, which is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Since then, based on uncertainty theory, significant work both in theory and engineering has been done by many researchers. As an application of uncertainty theory, Liu [6] proposed a spectrum of uncertain programming which is a type of mathematical programming involving uncertain variables, and applied uncertain programming to system reliability design, facility location problem, vehicle routing problem, project scheduling problem, finance and so on. In addition, Li and Liu [5] presented uncertain logic in which the truth value is defined as the uncertain measure that the proposition is true. Furthermore, uncertain inference was pioneered by Liu [8] as a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertainty. Other researchers also have done a lot of theoretical work in uncertainty theory, such as Chen [1], Chen and Liu [2], Gao [3], Gao and Ralescu [4], Peng [11] [12], Qin [13], Qin and Kar [14], You [15] and Zhu [16], etc.

Problems of k-out-of-n system play an important role in system reliability design. Up to now, k-out-of-n system with random lifetimes has been researched in many books related to system reliability design. This paper aims to give some analysis of k-out-of-n system with uncertain lifetimes. Based on the uncertainty theory, this paper gives a new concept of order uncertain variable, which is used as a tool to analyze k-out-of-n system with uncertain lifetimes in this paper. It is similar to the order statistics (Balakrishnan and Cohen [10]) in mathematical statistics. However, order uncertain variable just requires that the initial uncertain variables are independent. Besides, the uncertainty distribution of order uncertain variable has a much simpler form than order statistics.

The remainder of this paper is organized as follows. In Section 2, basic concepts and properties regarding uncertain variables are reviewed. In Section 3, the concept of order uncertain variable is given. In Section 4, the uncertainty distribution of order uncertain variable is deduced. In Section 5, the order uncertain variable is used as a tool to analyze k-out-of-n system with uncertain lifetimes.

2 Preliminary

In this section, we introduce some foundational concepts and properties of uncertainty theory, which is used throughout this paper.

Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \in \mathcal{L} \) is assigned a number \( \mathcal{M}(\Lambda) \). In order to ensure that the number \( \mathcal{M}(\Lambda) \) has certain mathematical properties, Liu [7] presented the following four axioms:

**Axiom 1.** (Normality) \( \mathcal{M}(\Gamma) = 1 \).

**Axiom 2.** (Monotonicity) \( \mathcal{M}(\Lambda_1) \leq \mathcal{M}(\Lambda_2) \) whenever \( \Lambda_1 \subset \Lambda_2 \).

**Axiom 3.** (Self-Duality) \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any event \( \Lambda \).

**Axiom 4.** (Countable Subadditivity) For every countable sequence of events \( \{\Lambda_i\} \), we have

\[
\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).
\]

**Definition 2.1** (Liu [7]) An uncertain variable is a measurable function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set...
of real numbers, i.e., for any Borel set \( B \) of real numbers, the set
\[
\{ \xi \in B \} = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B \}
\]
is an event.

**Definition 2.2** (Liu [7]) The uncertainty distribution \( \Phi : \mathbb{R} \to [0,1] \) of an uncertain variable \( \xi \) is defined by
\[
\Phi(x) = \mathcal{M}\{ \gamma \in \Gamma \mid \xi(\gamma) \leq x \}. \tag{1}
\]

**Definition 2.3** (Liu [8]) The uncertain variables \( \xi_1, \xi_2, \ldots, \xi_n \) are said to be independent if
\[
\mathcal{M}\left\{ \bigcap_{i=1}^{n} \{ \xi_i \in B_i \} \right\} = \min_{1 \leq i \leq n} \mathcal{M}\{ \xi_i \in B_i \} \tag{2}
\]
for any Borel sets \( B_1, B_2, \ldots, B_n \) of real numbers.

**Definition 2.4** (Liu [7]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by
\[
E[\xi] = \int_{-\infty}^{+\infty} \mathcal{M}\{ \xi \geq r \} dr - \int_{-\infty}^{0} \mathcal{M}\{ \xi \leq r \} dr \tag{3}
\]
provided that at least one of the two integrals is finite.

The following theorem is given without proof.

**Theorem 2.1** (Liu [8]. Operational Law) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables, and \( f : \mathbb{R}^n \to \mathbb{R} \) a measurable function. Then \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) is an uncertain variable such that
\[
\mathcal{M}\{ \xi \in B \} = \left\{ \begin{array}{ll}
\sup_{f(B_1, B_2, \ldots, B_n) \subseteq B} \min_{1 \leq k \leq n} \mathcal{M}_k \{ \xi_k \in B_k \}, \\
\sup_{f(B_1, B_2, \ldots, B_n) \subseteq B} \min_{1 \leq k \leq n} \mathcal{M}_k \{ \xi_k \in B_k \} > 0.5 \\
1 - \min_{f(B_1, B_2, \ldots, B_n) \subseteq B} \sup_{1 \leq k \leq n} \mathcal{M}_k \{ \xi_k \in B_k \}, \\
0.5, \text{ otherwise}
\end{array} \right. \tag{4}
\]
where \( B, B_1, B_2, \ldots, B_n \) are Borel sets of real numbers.

**Lemma 2.1** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables taking values in \( \{0,1\} \), such that
\[
\mathcal{M}\{ \xi_i = 0 \} = a_i, \quad \mathcal{M}\{ \xi_i = 1 \} = 1 - a_i,
\]
where \( 0 \leq a_i \leq 1, i = 1, 2, \ldots, n \). Then for any \( 0 \leq k \leq n, k \in \mathbb{N} \), we have
\[
\mathcal{M}\left\{ \sum_{i=1}^{n} \xi_i \leq k \right\} = a_{k+1}. \tag{5}
\]

**Proof:** Because \( \xi_i \) takes values in \( \{0,1\}, i = 1, 2, \ldots, n \), according to the operational law, we can get
\[
\mathcal{M}\left\{ \sum_{i=1}^{n} \xi_i \leq k \right\} = \left\{ \begin{array}{ll}
\sup_{B_1 + B_2 + \cdots + B_n \subseteq \{ \infty, -\infty \}} \min_{1 \leq k \leq n} \mathcal{M}\{ \xi_i \in B_i \}, \\
\sup_{B_1 + B_2 + \cdots + B_n \subseteq \{ \infty, -\infty \}} \min_{1 \leq k \leq n} \mathcal{M}\{ \xi_i \in B_i \} > 0.5 \\
1 - \sup_{B_1 + B_2 + \cdots + B_n \subseteq \{k+1, +\infty \}} \min_{1 \leq k \leq n} \mathcal{M}\{ \xi_i \in B_i \}, \\
0.5, \text{ otherwise}
\end{array} \right.
\]

It is easy to verify that
\[
\mathcal{M}\{ \sum_{i=1}^{n} \xi_i \leq k \} = a_{k+1}
\]

3 **Order Uncertain Variable**

**Definition 3.1** Let \( f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, n \) be a series of measurable functions. We call \( f_1, f_2, \ldots, f_n \) the order functions of \( \xi \) if for any \( (x_{1}, x_{2}, \ldots, x_{n}) \in \mathbb{R}^n \),
\[
f_i(x_{1}, x_{2}, \ldots, x_{n}) = x_{(i)}, \quad i = 1, 2, \ldots, n
\]
where \( \{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\} \) is the the rearrangement of \( \{x_{1}, x_{2}, \ldots, x_{n}\} \) in ascending order of magnitude, that is, \( x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} \).

**Definition 3.2** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables, and \( f_1, f_2, \ldots, f_n \) order functions of \( \mathbb{R}^n \). Define
\[
\xi^{(i)} = f_i(\xi_1, \xi_2, \ldots, \xi_n), \quad i = 1, 2, \ldots, n.
\]
Then \( \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)} \) are called order uncertain variables of \( \xi_1, \xi_2, \ldots, \xi_n \), and each \( \xi^{(i)} \) is called the ith order uncertain variable of \( \xi_1, \xi_2, \ldots, \xi_n \).
To be more explicit, consider the case where \( n = 2 \) and the relationship between \( \xi_i \) and \( \xi^{(i)}(i = 1, 2) \) is
\[
\xi^{(1)} = \xi_1 \quad \text{and} \quad \xi^{(2)} = \xi_2 \quad \text{when} \quad \xi_1 < \xi_2, \\
\xi^{(1)} = \xi_2 \quad \text{and} \quad \xi^{(2)} = \xi_1 \quad \text{when} \quad \xi_2 < \xi_1.
\]
Similarly, for \( n = 3 \) the relationship between the respective uncertain variables is
\[
\xi^{(1)} = \xi_1, \quad \xi^{(2)} = \xi_2, \quad \text{and} \quad \xi^{(3)} = \xi_3, \quad \text{when} \quad \xi_1 < \xi_2 < \xi_3, \\
\xi^{(1)} = \xi_1, \quad \xi^{(2)} = \xi_3, \quad \text{and} \quad \xi^{(3)} = \xi_2, \quad \text{when} \quad \xi_1 < \xi_3 < \xi_2, \\
\vdots \\
\xi^{(1)} = \xi_3, \quad \xi^{(2)} = \xi_2, \quad \text{and} \quad \xi^{(3)} = \xi_1, \quad \text{when} \quad \xi_3 < \xi_2 < \xi_1.
\]

The most frequently encountered functions of order uncertain variable are \( \xi^{(1)} \) and \( \xi^{(n)} \). Obviously, \( \xi^{(1)} = \min_{1 \leq k \leq n} \xi_k \) and \( \xi^{(n)} = \max_{1 \leq k \leq n} \xi_k \).

4 Uncertainty Distribution of Order Uncertain Variable

In this section, the uncertainty distribution of the \( k \)-th order uncertain variable for \( k = 1, 2, \ldots, n \) will be given.

Theorem 4.1 Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with distributions \( \Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x) \), respectively. The uncertainty distribution of the \( k \)-th \((k = 1, 2, \cdots, n)\) order uncertain variable \( \xi^{(k)} \) is
\[
\Psi_k(x) = f_{n-k+1}(\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)) \quad (6)
\]
where \( f_k : \mathbb{R}^n \rightarrow \mathbb{R}, k = 1, 2, \cdots, n \) are order functions of \( \mathbb{R}^n \).

Proof: For any fixed \( x \in \mathbb{R} \), define a series of new variables
\[
\eta_i(x) = \begin{cases} 
0, & \text{if } \xi_i \leq x \\
1, & \text{if } \xi_i > x
\end{cases}
\]
where \( i = 1, 2, \cdots, n \). Then \( \eta_1(x), \eta_2(x), \cdots, \eta_n(x) \) are independent, and \( \mathcal{M}(\eta_i(x) = 0) = \mathcal{M}(\xi_i \leq x) = \Phi_i(x) \) for each \( i \). According to the definitions of uncertain variable and order uncertain variable, by Lemma 1, we have
\[
\Psi_k(x) = \mathcal{M}\{\xi^{(k)} \leq x\} = \mathcal{M}\{\text{at most } (n-k) \text{ variables from } \xi_1, \xi_2, \cdots, \xi_n \text{ are greater than } x\} \\
= \mathcal{M}\left\{ \sum_{i=1}^{n} \eta_i(x) \leq n-k \right\} \\
= f_{n-k+1}(\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x))
\]
where \( f_k : \mathbb{R}^n \rightarrow \mathbb{R}, k = 1, 2, \cdots, n \) are order functions of \( \mathbb{R}^n \). The theorem is proved.

Corollary 4.1 Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with the same uncertainty distribution \( \Phi_i(x) \), and let \( \xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(n)} \) be the corresponding order uncertain variables. Then for any \( k = 1, 2, \cdots, n \), the \( k \)-th order uncertain variable has uncertainty distribution \( \Phi(x) \).

Proof: It follows from Theorem 4.1 immediately.

Corollary 4.2 Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with the same uncertainty distribution \( \Phi_i(x) \), and let \( \xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(n)} \) be the corresponding order uncertain variables. Then for any \( k = 1, 2, \cdots, n \), the \( k \)-th order uncertain variable has expected value \( E[\xi] \) provided that \( E[\xi] \) exists.

Proof: It follows from Corollary 4.1 and Definition 2.4 immediately.

5 An Application

System reliability design with uncertain lifetimes was first studied by Liu [6]. The k-out-of-n system is a significant kind of problem in system reliability design. In this section, order uncertain variable is used as a tool in analysis of k-out-of-n system with uncertain lifetimes.

A system of \( n \) components is called a k-out-of-n system if it remains operational only if at least \( k \) components continue to function. Assume that the \( i \)-th element of this component has independent lifetime \( \xi_i \) with uncertainty distribution \( \Phi_i(x) \), \( i = 1, 2, \cdots, n \), respectively. Then what is the distribution \( \Phi(x) \) of this system lifetime \( \eta \)?

Let \( \xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(n)} \) be the order uncertain variables of \( \xi_1, \xi_2, \cdots, \xi_n \). Then, for any \( x > 0, x \in \mathbb{R} \), by Theorem 2
\[
\Phi(x) = \mathcal{M}\{\eta \leq x\} \\
= \mathcal{M}\{\text{at most } (k-1) \text{ components function at time } x\} \\
= \mathcal{M}\{\xi^{(n-k+1)} \leq x\} \\
= f_k(\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)) \quad (7)
\]
where \( f_k : \mathbb{R}^n \rightarrow \mathbb{R}, k = 1, 2, \cdots, n \) are order functions of \( \mathbb{R}^n \). Thus, we get the uncertainty distribution of \( \eta \).

The special cases \( k = 1 \) and \( k = n \) correspond respectively to parallel system and series system (Figure 1). According to formula (7), the uncertainty distribution \( \Phi_p(x) \) of parallel system lifetime \( \eta_p \) is
\[
\Phi_p(x) = \mathcal{M}\{\eta_p \leq x\} \\
= f_1(\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)) \\
= \Phi_1(x) \land \Phi_2(x) \land \cdots \land \Phi_n(x)
\]
and the uncertainty distribution \( \Phi_s(x) \) of series system lifetime \( \eta_s \) is
\[
\Phi_s(x) = \mathcal{M}\{\eta_s \leq x\} \\
= f_n(\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)) \\
= \Phi_1(x) \lor \Phi_2(x) \lor \cdots \lor \Phi_n(x).
\]
6 Conclusion

In order to analyze k-out-of-n system with uncertain lifetimes, we presented a new concept of order uncertain variable and gave uncertainty distribution of order uncertain variable in this paper. By using order uncertain variable, the uncertainty distribution of k-out-of-n system lifetime was deduced.

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References