Uncertain Satisfiability and Uncertain Entailment

Zhuo Wang, Xiang Li
Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
zwang0518@sohu.com, xiang-li04@mail.tsinghua.edu.cn

Abstract: Uncertain logic is a branch of multi-valued logic for dealing with subjective information, which explains formulas as uncertain variables and defines their truth values as uncertain measures. In this paper, uncertain satisfiability is proposed to verify the consistency of a set of truth values for the formulas. Furthermore, for a set of satisfiable formulas with given truth values, the problem of uncertain entailment is proposed to estimate the truth value of an additional formula. In order to solve this problem, a nonlinear optimization model is designed by applying the maximum entropy principle.

Keywords: Uncertain logic; Uncertain variable; Satisfiability; Entailment; Maximum entropy principle.

1 Introduction

Classical logic assumes that each proposition is either true or false, which makes it successful to deal with certain information. However, many applications in artificial intelligence require the ability to reason with uncertain information. For example, in experts systems, many of the rules obtained from experts are known with some degree of uncertainty. In order to deal with uncertain information, Lukasiewicz proposed a multi-valued logic by extending the range of truth values from \{0, 1\} to \[0, 1\], in which the central problem is how to define the truth values of formulas. In Lukasiewicz’s multi-valued logic system, this problem was solved by defining the truth value operational rules for basic connectives including negation, disjunction, conjunction and implication. As the development of multi-valued logic, many researchers revised Lukasiewicz’s rules and developed several different multi-valued logic systems [5, 6, 11, 16, 17]. For example, Kish [6] studied the noise-based logic, Li and Liu [11] proposed a credibilistic logic system based on credibility measure and Zadah [17] investigated the fuzzy logic system within the frame of fuzzy theory.

Although multi-valued logic theory was well developed and widely applied, Elkan [2] pointed out that it disobey the law of excluded middle, which is one of the most basic laws in classical logic. In order to overcome this problem, Nilsson [14] developed a probabilistic logic by considering formulas as random variables taking values 0 or 1 and defined their truth values as probability measures, which was proved to be consistent with the classical logic. That is, the law of excluded middle holds in probabilistic logic. Satisfiability and entailment are two central problems in probabilistic logic. Probabilistic satisfiability is used to verify the consistency of a set of truth values of formulas, which was well solved by a linear programming model [14]. A good survey about probabilistic satisfiability is provided by Hansen [4]. Probabilistic entailment is used to estimate the truth value of an additional formula based on a set of consistent formulas with the given truth values, which was first proposed and solved by Nilsson [14] in terms of maximum entropy principle, and developed by many researchers [1, 3, 15].

Recently, Li and Liu [12] introduced an uncertain logic by explaining the propositions and formulas as uncertain variables, and defined the truth values as uncertain measures. The law of excluded middle and the consistency between uncertain logic and classical logic are also proved. In fact, probabilistic logic and uncertain logic are all branches of multi-valued logic. The different is that the former is used to deal with objective information and the later is used to deal with subjective information.

The purpose of this paper is to study the problems of satisfiability and entailment in the framework of uncertain logic. For this purpose, this paper is organized as follows. Section 2 recalls some basic concepts and properties about probabilistic logic and uncertain logic. In section 3, the problem of uncertain satisfiability is proposed and the relations between this concept and probabilistic satisfiability are discussed. In section 4, the problem of uncertain entailment is discussed in the frame of uncertainty theory. At the end of this paper, a brief summary about this paper is given.

2 Preliminaries

In multi-valued logic, a proposition is defined as a statement with truth value belonging to \[0, 1\], and a formula is defined as a member of the minimal set \(\mathcal{S}\) of a finite sequence of primitive symbols \((\neg, \lor, p_1, p_2, \cdots)\) satisfying: (a) \(p \in \mathcal{S}\) for each proposition \(p\); (b) if \(X \in \mathcal{S}\), then \(\neg X \in \mathcal{S}\); (c) if \(X \in \mathcal{S}\) and \(Y \in \mathcal{S}\), then \(X \lor Y \in \mathcal{S}\). The connective symbol \(\neg\) means negation, and \(\lor\) means disjunction. For simplicity, the con-
nective symbols $\land$ and $\rightarrow$ are defined as
\[
X \land Y = \neg(\neg X \lor \neg Y), \quad X \rightarrow Y = \neg X \lor Y,
\]
which mean conjunction and implication, respectively. For example, let $X$ mean “Tom has a cold”, and let $Y$ mean “Tom goes to the cinema”, then $\neg X$ means “Tom does not have a cold”, and $\neg X \rightarrow Y$ means “If Tom does not have a cold, then Tom goes to the cinema.”

Let $X$ be a formula containing propositions $p_1, p_2, \ldots, p_n$. Its truth function is defined as a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ such that $f(x_1, x_2, \ldots, x_n) = 1$ if and only if $X = 1$ with respect to the assignment $p_1 = x_1, p_2 = x_2, \ldots, p_n = x_n$, where $X = 1$ means $X$ is true and $X = 0$ means $X$ is false. For example, the truth function of $p_1 \land p_2$ is
\[
f(1, 1) = 1, \quad f(1, 0) = 0, \quad f(0, 1) = 0, \quad f(0, 0) = 0.
\]

**Probabilistic Logic**

Probabilistic logic was proposed by Nilsson [14] as a branch of multi-valued logic for dealing with objective information. In probabilistic logic, the definitions of propositions and formulas are unchanged except that the truth values are explained as probability measure. For example,

“So Tom is in Beijing with truth value 0.6”

is a proposition, where “So Tom is in Beijing” is a statement, and its truth value is 0.6 in probability. Generally speaking, we use $\eta$ to express a proposition and use $p$ to express its probability measure. In fact, a proposition $\eta$ is a random variable defined as

\[
\eta = \begin{cases} 
1, & \text{with probability } p \\
0, & \text{with probability } 1 - p
\end{cases}
\]

where $\eta = 1$ means $\eta$ is true and $\eta = 0$ means $\eta$ is false.

Let $X$ be a formula containing propositions $\eta_1, \eta_2, \ldots, \eta_n$. If it has a truth function $f$, then it is essentially a random variable defined as $X = f(\eta_1, \eta_2, \ldots, \eta_n)$, where the symbols $\eta_1, \eta_2, \ldots, \eta_n$ are considered as random variables.

**Definition 2.1** (Nilsson [14]) For each formula $X$, its truth value is defined as

\[
T(X) = \Pr\{X = 1\}.
\]

**Probabilistic Satisfiability**

The purpose of probabilistic satisfiability is to verify whether a set of given truth values for formulas is consistent. For example, $T(p_1 \land p_2) = 0.3$ and $T(p_1) = 0.8$ are consistent. However, $T(p_1 \land p_2) = 0.8$ and $T(p_1) = 0.3$ are inconsistent because the monotonicity of probability measure implies that $T(p_1 \land p_2) \leq T(p_1)$.

In order to deal with the satisfiability problem, Nilsson [14] proposed a linear optimization model. First, define atoms as all the conjunctions of the propositions $p_1, p_2, \ldots, p_n$ or their negations such as $a_i = l_1 \land l_2 \land \ldots \land l_n, l_j = p_j$ or $\neg p_j$ for each $j$. Then it is easy to verify that there are $2^n$ atoms and the vector $(T(a_1), T(a_2), \ldots, T(a_{2^n}))$ is a mass function where $T(a_i)$ is the truth value of the $i$th atom, $i = 1, 2, \ldots, 2^n$, respectively. For simplicity, let $\Gamma$ represent the collection $\{a_1, a_2, \ldots, a_{2^n}\}$.

Let $L$ be the collection of formulas generated by some of the propositions $p_1, p_2, \ldots, p_n$ with connectives $\neg$, $\land$ and $\lor$. For each formula $X$ of $L$, according to the disjunctive normal form theorem, there is a unique set $A(X)$ of atoms such that $X \lor \bigvee_{a_i \in A(X)} a_i$ are equivalent. Then each formula $X$ is essentially defined as

\[
X(a) = \begin{cases} 
1, & \text{if } a \in A(X) \\
0, & \text{if } a \notin A(X).
\end{cases}
\]

It follows from Definition 2.1 that

\[
T(X) = \Pr\{X = 1\} = \Pr\{A(X)\}.
\]

Suppose that $T(a_i) = \rho_i$ for $i = 1, 2, \ldots, 2^n$, then for each $X \in L$, we have

\[
T(X) = \sum_{i=1}^{2^n} \rho_i X(a_i).
\]

Finally, if the data $T(X_j) = \alpha_j$ for $j = 1, 2, \ldots, m$ is given, the probabilistic satisfiability model is

\[
\begin{cases} 
\sum_{i=1}^{2^n} \rho_i X_j(a_i) = \alpha_j & \text{for } j = 1, 2, \ldots, m \\
\sum_{i=1}^{2^n} \rho_i = 1 \\
0 \leq \rho_i \leq 1 & \text{for all } i = 1, 2, \ldots, 2^n.
\end{cases}
\]

If the probabilistic satisfiability model has a feasible solution, then $L$ is satisfiable.

**Probabilistic Entailment**

Probabilistic entailment was first proposed by Nilsson [14], which aims to compute the truth value of an additional formula based on some given formulas whose truth values are known. Suppose $X_1, X_2, \ldots, X_m$ are consistent formulas with truth values $\alpha_1, \alpha_2, \ldots, \alpha_m$. In order to estimate the truth value of a new formula $X$, Nilsson [14] proposed the maximum entropy model. That is, if the feasible solution
of probabilistic satisfiability model is unique, for example, \((\rho_1^*, \rho_2^*, \ldots, \rho_n^*)\), then it is clear that

\[
T(X) = \sum_{i=1}^{2^n} \rho_i^* X(a_i).
\]

Otherwise, if the feasible solution is not unique, we select the one which maximizes the Shannon entropy. The maximize entropy model is defined as follows:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{2^n} \rho_i \ln \rho_i \\
\text{subject to:} & \quad \sum_{i=1}^{2^n} \rho_i X(a_i) = \alpha_i \quad \text{for} \quad j = 1, 2, \ldots, m \\
& \quad \sum_{i=1}^{2^n} \rho_i = 1 \\
& \quad 0 \leq \rho_i \leq 1 \quad \text{for all} \quad i = 1, 2, \ldots, 2^n.
\end{align*}
\]

**Uncertain Logic**

Within the framework of uncertainty theory (Liu [7]), uncertain logic was designed by Li and Liu [12] for dealing with subjective information. A brief knowledge about uncertainty theory may be found in the Appendix.

In uncertain logic, Li and Liu [12] considered the formulas as uncertain variables taking values in \([0, 1]\), and defined the truth value as uncertain measure.

**Definition 2.2** (Li and Liu [12]) For each formula \(X\), its truth value is defined as

\[
T(X) = \mathcal{M}\{X = 1\}.
\]

**Remark 2.1** The uncertain logic is proved to be well consistent with classical logic such as it obeys the basic laws in classical logic.

**Theorem 2.1** (Law of Excluded Middle [12]) For any formula \(X\), we have

\[
T(X \lor \neg X) = 1.
\]

**Theorem 2.2** (Law of Contradiction [12]) For any formula \(X\), we have

\[
T(X \land \neg X) = 0.
\]

**Theorem 2.3** (Law of Truth Conservation [12]) For any formula \(X\), we have

\[
T(\neg X) = 1 - T(X).
\]

### 3 Uncertain Satisfiability

The purpose of uncertain satisfiability is to verify whether a set of given truth values for formulas is consistent. In other words, whether there is an uncertain measure such that all the formulas are satisfied with the given truth values.

**Definition 3.1** For any \(\alpha_1, \alpha_2, \ldots, \alpha_m \in (0, 1]\), a set of formulas \(F = \{X_1, X_2, \ldots, X_m\}\) is said to be \((\alpha_1, \alpha_2, \ldots, \alpha_m)\)-satisfiable if there is an uncertain measure such that \(T(X_i) = \alpha_i\) for all \(i = 1, 2, \ldots, m\).

In order to deal with the satisfiability problem, we introduce a linear optimization model, which will be called the uncertain satisfiability model. Suppose that formulas \(X_1, X_2, \ldots, X_m\) are generated by \(p_1, p_2, \ldots, p_n\). Let \(\Gamma\) be the collection of atoms \(\{a_1, a_2, \ldots, a_n\}\), where \(a_i = l_1 \land l_2 \land \cdots \land l_n\) and let \(\mathcal{A}\) be the power set of \(\Gamma\). If \(\mathcal{M}\) is an uncertain measure on \(\mathcal{A}\), then \((\Gamma, \mathcal{A}, \mathcal{M})\) is an uncertainty space and each formula \(X\) is essentially an uncertain variable

\[
X(a) = \begin{cases} 
1, & \text{if } a \in A(X) \\
0, & \text{if } a \notin A(X).
\end{cases}
\]

It follows from Definition 2.2 that

\[
T(X) = \mathcal{M}\{X = 1\} = \mathcal{M}\{A(X)\}.
\]

If \((\rho_1, \rho_2, \ldots, \rho_n)\) is a second identification coefficient on \(\Gamma\), that is, \(T(a_i) = \rho_i\) for each \(i = 1, 2, \ldots, 2^n\), then for each formula \(X\) of \(F\), it follows from the equation (3) (see Appendix) that

\[
T(X) = \begin{cases} 
\sum_{i=1}^{2^n} \rho_i X(a_i), & \text{if } \sum_{i=1}^{2^n} \rho_i X(a_i) < 0.5 \\
1 - \sum_{i=1}^{2^n} \rho_i (1 - X(a_i)), & \text{if } \sum_{i=1}^{2^n} \rho_i (1 - X(a_i)) < 0.5 \\
0.5, & \text{otherwise}.
\end{cases}
\]

In order to verify the consistency of \(F = \{X_1, X_2, \ldots, X_m\}\), we may formulate the following uncertain satisfiability model:

\[
\begin{align*}
T(X_j) &= \alpha_j \quad \text{for} \quad j = 1, 2, \ldots, m \\
\sum_{i=1}^{2^n} \rho_i &= \beta \quad \text{for} \quad \beta \geq 1 \\
0 \leq \rho_i &\leq 1 \quad \text{for all} \quad i = 1, 2, \ldots, 2^n.
\end{align*}
\]
If the uncertain satisfiability model has a feasible solution, then \( F = (\alpha_1, \alpha_2, \cdots, \alpha_m) \)-satisfiable. Otherwise, it is said to be \((\alpha_1, \alpha_2, \cdots, \alpha_m)\)-unsatisfiable.

**Remark 3.1** In this model, the last two constraints are used to ensure that \( (\rho_1, \rho_2, \cdots, \rho_{2^n}) \) is an uncertainty mass function. Especially, if \( \beta = 1 \), then the maximum entropy model degenerates to the probabilistic one (Nilsson [14]).

**Remark 3.2** If it is known that \( T(X_i) = \alpha_i > 0.5, i = 1, 2, \cdots, m_1 \), \( T(Y_i) = \beta_i < 0.5, i = 1, 2, \cdots, m_2 \), \( T(Z_i) = \gamma_i = 0.5, i = 1, 2, \cdots, m_3 \), then the uncertain satisfiability model is changed to:

\[
\begin{align*}
1 - \sum_{j=1}^{2^n} \rho_j (1 - X_i(a_j)) &= \alpha_i & \text{for } i = 1, 2, \cdots, m_1 \\
\sum_{j=1}^{2^n} \rho_j Y_i(a_j) &= \beta_i & \text{for } i = 1, 2, \cdots, m_2 \\
\sum_{j=1}^{2^n} \rho_j Z_i(a_j) &\geq 0.5 & \text{for } i = 1, 2, \cdots, m_3 \\
\sum_{j=1}^{2^n} \rho_j (1 - Z_i(a_j)) &\geq 0.5 & \text{for } i = 1, 2, \cdots, m_3 \\
\sum_{j=1}^{2^n} \rho_j &= \beta & \text{for } \beta \geq 1 \\
0 &\leq \rho_j & \text{for } 1 \leq j \leq 2^n.
\end{align*}
\]

**Remark 3.3** Please note that if a set of formulas is satisfiable in probabilistic logic, it must be satisfiable in uncertain logic. Conversely, if a set of formulas is satisfiable in uncertain logic, it is not necessarily satisfiable in probabilistic logic. For example, if there are two propositions \( p \) and \( q \), then it is obvious that there are four atoms \( p \wedge q, \neg p \wedge q, p \wedge \neg q, \neg p \wedge \neg q \). If it is known that \( T(q) = 0.4 \), \( T(p \wedge q) = 0.4 \) and \( T(\neg p \wedge \neg q) = 0.3 \), then it is clear that \( \{ q, p \wedge \neg q, \neg p \wedge \neg q \} \) is unsatisfiable in probabilistic logic because the sum of the truth values is greater than one. However, it is easy to prove that \( \{ q, p \wedge \neg q, \neg p \wedge \neg q \} \) is satisfiable in uncertain logic and one of the feasible solutions of uncertain satisfiability model is \( T(p \wedge q) = 0.2 \), \( T(\neg p \wedge q) = 0.2 \), \( T(p \wedge \neg q) = 0.4 \) and \( T(\neg p \wedge \neg q) = 0.3 \).

4 Uncertain Entailment

In this section, we investigate the analogue of logical entailment for uncertain logic, which will be called uncertain entailment. The purpose of uncertain entailment is to estimate the truth value of an additional formula \( X \) based on the given truth values \( \{ \alpha_1, \alpha_2, \cdots, \alpha_m \} \) for a set of consistent formulas \( \{ X_1, X_2, \cdots, X_m \} \). It is clear that \( T(X) \) will be uniquely determined once we select a feasible uncertainty mass function from the uncertain satisfiability model. However, the feasible solution of the uncertain satisfiability model may be not unique. Then the problem is which one should be selected? In this section, we provide a maximum entropy model to solve this problem, which selects the feasible uncertain mass function that maximizes the Shannon entropy.

Let \( \Gamma = \{ \alpha_1, \alpha_2, \cdots, \alpha_{2^n} \} \) and let \( (\rho_1, \rho_2, \cdots, \rho_{2^n}) \) be an uncertainty mass function on \( \Gamma \). Then the maximum entropy model is

\[
\begin{align*}
\text{maximize} & \quad -\sum_{i=1}^{2^n} \rho_i \ln \rho_i \\
\text{subject to} & \quad T(X_j) = \alpha_j \quad \text{for all } j = 1, 2, \cdots, m \\
& \quad \sum_{i=1}^{2^n} \rho_i = \beta \quad \text{for } \beta \geq 1 \\
& \quad 0 \leq \rho_i \leq 1 \quad \text{for all } i = 1, 2, \cdots, 2^n \\
& \quad 1 - \sum_{i=1}^{2^n} \rho_i (1 - X_j(a_i)) < 0.5 \\
& \quad \text{if } 2^n \rho_i X_j(a_i) < 0.5 \\
& \quad \text{if } 2^n \rho_i (1 - X_j(a_i)) < 0.5 \\
& \quad 0.5, \text{ otherwise}.
\end{align*}
\]

**Example 4.1** Suppose that there are two propositions \( p \) and \( q \) with truth values \( \alpha_1 \) and \( \alpha_2 \), respectively. Let \( \Gamma = \{ p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q \} \) and let \( (\rho_1, \rho_2, \cdots, \rho_4) \) be an uncertainty mass function on \( \Gamma \) satisfying \( T(p \wedge q) = \rho_1, T(p \wedge \neg q) = \rho_2, T(\neg p \wedge q) = \rho_3 \) and \( T(\neg p \wedge \neg q) = \rho_4 \). It is obvious that \( p = (p \wedge q) \lor (p \wedge \neg q), q = (p \wedge q) \lor (\neg p \wedge \neg q) \) and \( p \lor q = (p \wedge q) \lor (p \wedge \neg q) \lor (\neg p \wedge \neg q) \). In order to estimate the truth value of \( p \lor q \), we apply the maximum entropy model. The argument breaks down into four cases:

Case 1. \( \alpha_1 < 0.5 \) and \( \alpha_2 < 0.5 \). In this case, the maximum entropy model attains its maximum at \( \rho_1 = \alpha_1 + \alpha_2 - \beta + (\beta - \alpha_1)(\beta - \alpha_2)/\beta, \rho_2 = \beta - \alpha_2 - (\beta - \alpha_1)(\beta - \alpha_2)/\beta, \rho_3 = \beta - \alpha_1 - (\beta - \alpha_1)(\beta - \alpha_2)/\beta \) and \( \rho_4 = (\beta - \alpha_1)(\beta - \alpha_2)/\beta \). Then according to equation (1), we
have

\[
T(p \lor q) = \begin{cases} 
\beta - \frac{(\beta - \alpha_1)(\beta - \alpha_2)}{\beta}, & \text{if } \beta - \frac{(\beta - \alpha_1)(\beta - \alpha_2)}{\beta} < 0.5 \\
1 - \frac{(\beta - \alpha_1)(\beta - \alpha_2)}{\beta}, & \text{if } \beta - \frac{(\beta - \alpha_1)(\beta - \alpha_2)}{\beta} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

Case 2. \( \alpha_1 < 0.5 \) and \( \alpha_2 \geq 0.5 \). In this case, the maximum entropy model attains its maximum at \( p_1 = \alpha_1 + \alpha_2 - 1 + (1 - \alpha_2)(\beta - \alpha_1)/\beta, p_2 = 1 - \alpha_2 - (1 - \alpha_2)(\beta - \alpha_1)/\beta, p_3 = \beta - \alpha_1 - (1 - \alpha_2)(\beta - \alpha_1)/\beta \) and \( p_4 = (1 - \alpha_2)(\beta - \alpha_1)/\beta \). Then according to equation (1), we have

\[
T(p \lor q) = 1 - \frac{(1 - \alpha_2)(\beta - \alpha_1)}{\beta}.
\]

Case 3. \( \alpha_1 \geq 0.5 \) and \( \alpha_2 < 0.5 \). In this case, the maximum entropy model attains its maximum at \( p_1 = \alpha_1 + \alpha_2 - 1 + (1 - \alpha_2)(\beta - \alpha_2)/\beta - 1, p_2 = \beta - \alpha_2 - (1 - \alpha_1)(\beta - \alpha_2)/\beta, p_3 = 1 - \alpha_1 - (1 - \alpha_1)(\beta - \alpha_2)/\beta \) and \( p_4 = (1 - \alpha_1)(\beta - \alpha_2)/\beta \). Then, according to equation (1), we have

\[
T(p \lor q) = 1 - \frac{(1 - \alpha_1)(\beta - \alpha_2)}{\beta}.
\]

Case 4. \( \alpha_1 \geq 0.5 \) and \( \alpha_2 \geq 0.5 \). In this case, the maximum entropy model attains its maximum at \( p_1 = \beta + \alpha_1 + \alpha_2 - 2 + (1 - \alpha_1)(1 - \alpha_2)/\beta, p_2 = 1 - \alpha_2 - (1 - \alpha_1)(1 - \alpha_2)/\beta, p_3 = 1 - \alpha_1 - (1 - \alpha_1)(1 - \alpha_2)/\beta \) and \( p_4 = (1 - \alpha_1)(1 - \alpha_2)/\beta \). Then, according to equation (1), we have

\[
T(p \lor q) = 1 - \frac{(1 - \alpha_1)(1 - \alpha_2)}{\beta}.
\]

Remark 4.1 If \( \beta = 1 \), then the above four cases coincide with the following formula

\[
T(p \lor q) = 1 - (1 - \alpha_1)(1 - \alpha_2),
\]

which is just the result of probabilistic entailment.

Example 4.2 Suppose there are two formulas \( p \) and \( p \rightarrow q \) with truth values \( \alpha_1 \) and \( \alpha_2 \), respectively. Let \( \Gamma = \{ p \land q, p \land \neg q, \neg p \land q, \neg p \land \neg q \} \) and let \( (p_1, p_2, \cdots, p_4) \) be an uncertainty mass function on \( \Gamma \) satisfying \( T(p \land q) = p_1, T(p \land \neg q) = p_2, T(\neg p \land q) = p_3 \) and \( T(\neg p \land \neg q) = p_4 \). The formulas \( p, p \rightarrow q \) and \( q \) can be presented in the following disjunction normal form \( p = (p \land q) \lor (p \land \neg q), p \rightarrow q = (p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q) \) and \( q = (p \land q) \lor (\neg p \land q) \). In order to estimate the truth value of \( q \), we apply the maximum entropy model. Since formulas \( p \) and \( p \rightarrow q \) are unsatisfiable in the case of \( \alpha_1 + \alpha_2 < 1 \), the argument breaks down into three cases:

Case 1. \( \alpha_1 < 0.5 \) and \( \alpha_2 > 0.5 \). In this case, the maximum entropy model attains its maximum at \( p_1 = \alpha_1 + \alpha_2 - 1, p_2 = \beta - \alpha_2 \) and \( p_3 = \beta = (\beta - \alpha_1)/2 \). According to equation (1), the truth value of \( q \) is calculated as

\[
T(q) = \begin{cases} 
\frac{\alpha_1}{2} + \alpha_2 - 1 + \frac{\beta}{2}, & \text{if } \frac{\alpha_1}{2} + \alpha_2 - 1 + \frac{\beta}{2} < 0.5 \\
\frac{\alpha_1}{2} + \alpha_2 - \frac{\beta}{2}, & \text{if } 1 - \frac{\alpha_1}{2} - \alpha_2 + \frac{\beta}{2} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

Case 2. \( \alpha_1 \geq 0.5 \) and \( \alpha_2 < 0.5 \). In this case, the maximum entropy model attains its maximum at \( p_1 = \beta + \alpha_1 + \alpha_2 - 2, p_2 = 1 - \alpha_2 - (1 - \alpha_1)(1 - \alpha_2)/\beta \) and \( p_3 = (1 - \alpha_1)(1 - \alpha_2)/\beta \). Then, according to equation (1), the truth value of \( q \) is

\[
T(q) = \begin{cases} 
\frac{\alpha_1}{2} + \alpha_2 + \beta - 1.5, & \text{if } \frac{\alpha_1}{2} + \alpha_2 + \beta - 1.5 < 0.5 \\
\frac{\alpha_1}{2} + \alpha_2 - 0.5, & \text{if } 1.5 - \alpha_2 - \frac{\alpha_1}{2} < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

Case 3. \( \alpha_1 > 0.5 \) and \( \alpha_2 < 0.5 \). In this case, the maximum entropy model attains its maximum at \( p_1 = \beta + \alpha_1 + \alpha_2 - 1, p_2 = \beta - \alpha_2 \) and \( p_3 = \beta = (1 - \alpha_1)/2 \). Then, the truth value of \( q \) is

\[
T(q) = \begin{cases} 
\frac{\alpha_1}{2} + \alpha_2 - 0.5, & \text{if } \frac{\alpha_1}{2} + \alpha_2 - 0.5 < 0.5 \\
\frac{\alpha_1}{2} + \alpha_2 + 0.5 - \beta, & \text{if } \beta + 0.5 - \frac{\alpha_1}{2} - \alpha_2 < 0.5 \\
0.5, & \text{otherwise.}
\end{cases}
\]

Remark 4.2 If \( \beta = 1 \), then the above three cases coincide with the following formula

\[
T(q) = \frac{\alpha_1}{2} + \alpha_2 - 0.5,
\]

which is the same as the result of probabilistic entailment.

5 Conclusions

Uncertain logic is a branch of multi-valued logic for dealing with subjective information. In this paper, the problems of uncertain satisfiability and uncertain entailment were proposed...
and solved. First, uncertain satisfiability was proposed to verify the consistency of a set of truth values for formulas. Furthermore, for a set of satisfiable truth values for formulas, uncertain entailment was proposed to estimate the truth value of an additional formula. This problem was then formulated to a nonlinear optimization model by applying the maximum entropy principle. Finally, two simple examples about uncertain entailment were presented.

6 Appendix: Uncertain Measure

In order to deal with the general subjective uncertain phenomena, Liu [7] defined an uncertain variable as a measurable function from an uncertainty space to the set of real numbers, and developed an uncertainty theory [7, 9, 10].

Let $\Gamma$ be a nonempty set, and let $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is called an event. In 2007, Liu [7] defined an uncertain measure as a set function $\mathcal{M}$ on $\mathcal{L}$ satisfying:

(i) (Normality) $\mathcal{M}\{\Gamma\} = 1$;
(ii) (Monotonicity) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subseteq \Lambda_2$;
(iii) (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for each event $\Lambda$;
(iv) (Countable Subadditivity) $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$,

for any countable consequence of events $\{\Lambda_i\}$.

If $\mathcal{M}$ is an uncertain measure, then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Furthermore, an uncertain variable $\xi$ was defined by Liu [8] as a measurable function from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.

In order to characterize uncertain variables, Liu [8] defined first identification functions and Liu [13] defined second identification functions. Take the second identification function for example, an uncertain variable $\xi$ is said to have a second identification function $\rho$ if

(i) $\rho(x)$ is a nonnegative and integrable function on $\mathbb{R}$ such that $\int_{\mathbb{R}} \rho(x)dx \geq 1$;
(ii) for any Borel set $B$ of real numbers, we have

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \int_{B} \rho(x)dx, & \text{if} \quad \int_{B} \rho(x)dx < 0.5 \\ 1 - \int_{B^c} \rho(x)dx, & \text{if} \quad \int_{B^c} \rho(x)dx < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Suppose that $\xi$ is a discrete uncertain variable taking values in $\{x_1, x_2, \cdots\}$. Then $\xi$ is said to have second identification coefficients $\{\rho_1, \rho_2, \cdots\}$ if and only if

(i) $\rho_i \geq 0$ for all $i$ and $\sum_{i=1}^{\infty} \rho_i \geq 1$;
(ii) for any set $B$ of real numbers, we have

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sum_{x_i \in B} \rho_i, & \text{if} \quad \sum_{x_i \in B} \rho_i < 0.5 \\ 1 - \sum_{x_i \in B^c} \rho_i, & \text{if} \quad \sum_{x_i \in B^c} \rho_i < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

References