Barrier Options Pricing in Uncertain Financial Market

Jianqiang Xu, Jin Peng
Institute of Uncertain Systems, Huanggang Normal University, Hubei 438000, China
College of Mathematics and Science, Shanghai Normal University, Shanghai 200234, China
peng@hgnu.edu.cn

Abstract: In modern finance market, the option pricing problem is one of the most important contents. A barrier option is a derivative contract that is activated or extinguished when the price of the underlying asset passes a predetermined level. In this paper, pricing formulas for barrier options are defined in uncertain financial market and their pricing formulas concerning uncertain stock model are investigated.

Keywords: finance, barrier options, uncertain process, canonical process, option pricing formula.

1 Introduction

Brownian motion was introduced to finance by Bachelier [1]. Samuelson [19] [20] proposed the argument that geometric Brownian motion is a good model for stock prices. In the early 1970s, Black and Scholes [3] and, independently, Metron [15] used the geometric Brownian motion to determine the prices of stock options. Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. Different from randomness, fuzziness is another type of uncertainty in the real world. In order to deal with the change of fuzzy phenomena with time, Liu [12] proposed Liu process, Liu formula and Liu integral, just like Brownian motion, Ito formula and Ito integral. As a different doctrine, Liu [12] presented an alternative assumption that stock price follows geometric Liu process. Moreover, a basic stock model for fuzzy financial market was also proposed by Liu [12]. We call it Liu’s stock model which is a counterpart of Black-Scholes stock model. Some researches surrounding the subject have been made by Peng [16], Qin and Li [17], You [21], etc.

However, the real life decisions are usually made in the state of uncertainty other than randomness or fuzziness. In order to cope with this kind of complicated uncertainty, Liu [11] founded an uncertainty theory that had become a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Based on Liu’s uncertainty theory, uncertain process and uncertain differential equation have been proposed by Liu [11]. An uncertain process is essentially a sequence of uncertain variables indexed by time or space. Uncertain differential equation is a type of differential equation driven by canonical process. The canonical process and geometric canonical process which are different from Brownian motion and geometric Brownian motion are fundamental and important uncertain processes.

In finance market, barrier options are the most important weakly path-dependent options which come in many flavours and forms, but they have two key features: A knock-out feature causes the option to immediately terminate if the underlying asset reaches a specified barrier level; A knock-in feature causes the option to become effective only if the underlying asset first reaches a specified barrier level. Merton [15] provided the first analytical formula for a down-and-out call option which was followed by the more detailed paper the formulas for all eight types of barriers. This paper focuses exclusively on one-dimensional, single barrier options in uncertain financial market, which include up (down)-and-in (out) call (put) options.

2 Preliminaries

In this section, we recalls the fundamental knowledge of uncertainty theory.

2.1 Uncertain Variable

Definition 2.1 (Liu [12]) An uncertain variable is a measurable function ξ from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set

\[ \{ξ \in B\} = \{γ \in Γ | ξ(γ) ∈ B\} \]

is an event.

An uncertain variable ξ can be used to characterize the uncertain phenomenon.

2.2 First Identification Function

Definition 2.2 (Liu [12]) An uncertain variable ξ is said to have a first identification function λ if

\[ (i) \ λ(x) \ is \ a \ nonnegative \ function \ on \ ℝ \ such \ that \ \\
\[ (ii) \ \sup_{x \neq y} (\lambda(x) + \lambda(y)) \]

(ii) For any set B of real numbers, we have

\[ M\{\xi \in B\} = \begin{cases} \sup_{x \in B} \lambda(x), & \text{if } \sup_{x \in B} \lambda(x) < 0.5 \\ 1 - \sup_{x \in B} \lambda(x), & \text{if } \sup_{x \in B} \lambda(x) \geq 0.5. \end{cases} \]  

**Theorem 2.1** (Liu [12]) (First Measure Inversion Theorem)
Let \( \xi \) be an uncertain variable with first identification \( \lambda \). Then for any set \( B \) of real numbers, \( M\{\xi \in B\} \) can be expressed by (3).

### 2.3 Expected Value

**Definition 2.3** (Liu [12]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by

\[ E[\xi] = \int_{-\infty}^{+\infty} M\{\xi \geq r\} \, dr - \int_{-\infty}^{0} M\{\xi \leq r\} \, dr \]  

provided that at least one of the two integrals is finite.

**Theorem 2.2** (Liu [12]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value \( E[\xi] \) exists, then

\[ E[\xi] = \int_{-\infty}^{+\infty} (1 - \Phi(x)) \, dx - \int_{-\infty}^{0} \Phi(x) \, dx. \]  

### 2.4 Uncertain Process

**Definition 2.4** (Liu [14]) Given an index set \( T \) and an uncertainty space \( (\Gamma, L, M) \). An uncertain process is a measurable function from \( T \times (\Gamma, L, M) \) to the set of real numbers.

That is to say an uncertain process \( X(t, \gamma) \) is a function of two variables such that the function \( X(t^*, \gamma) \) is an uncertain variable for each \( t^* \). For simplicity, sometimes we simply use the symbol \( X_t \) instead of longer notation \( X(t^*, \gamma) \).

**Definition 2.5** (Liu [14]) An uncertain process \( X_t \) is said to have independent increments if

\[ X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}} \]

are independent uncertain variables for any times \( t_0 < t_1 < \cdots < t_k \). An uncertain process \( X_t \) is said to have stationary increments if, for any given \( t > 0 \), the \( X_{s+t} - X_s \) are identically distributed uncertain variables for all \( s > 0 \).

### 2.5 Canonical Process

**Definition 2.6** (Liu [14]) An uncertain process \( C_t \) is said to be a canonical process if
(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous, and
(ii) \( C_t \) has stationary and independent increments, and
(iii) every increment \( C_{s+t} - C_s \) is a normal uncertain variable with expected value 0 and variance \( t^2 \), whose uncertainty distribution is

\[ \Phi(x) = \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3t}} \right) \right)^{-1}, \quad x \in \mathbb{R}. \]  

The canonical process plays the role of a counterpart of Brownian motion.

**Definition 2.7** (Liu [14]) Let \( C_t \) be a canonical process. Then the uncertain process

\[ G_t = \exp(\epsilon t + \sigma C_t) \]

is called a geometric canonical process, where \( \epsilon \) is called the log-drift and \( \sigma \) is called the log-diffusion.

The geometric canonical process is expected to model stock prices in an uncertain environment.

### 2.6 Uncertain Integral

**Definition 2.8** (Liu [14]) Let \( X_t \) be an uncertain process and let \( C_t \) be a canonical process. Then the uncertain integral of \( X_t \) with respect to \( C_t \) is

\[ \int_{a}^{b} X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \]  

provided that the limit exists almost surely and is an uncertain variable.

### 2.7 Chain Rule

**Theorem 2.3** (Liu [14]) Let \( C_t \) be a canonical process, and let \( h(t, C_t) \) be a continuously differentiable function. Define \( X_t = h(t, C_t) \). Then it holds the following chain rule

\[ dX_t = \frac{\partial h}{\partial t}(t, C_t) \, dt + \frac{\partial h}{\partial C}(t, C_t) \, dC_t. \]  

### 3 Liu’s Stock Model

Let \( X_t \) be the bond price, and \( Y_t \) the stock price. Assume that stock price follows a geometric canonical process. Then Liu [14] characterizes the uncertain price dynamics as follows, 

\[
\begin{align*}
  dX_t &= rX_t \, dt \\
  dY_t &= \epsilon Y_t \, dt + \sigma Y_t \, dC_t
\end{align*}
\]

where \( r \) is the riskless interest rate, \( \epsilon \) is the stock drift, \( \sigma \) is the stock diffusion, and \( C_t \) is the canonical process.
4 Knock-Out Barrier Options

Barrier options are the most popular path-dependent options traded in exchanges worldwide and also in over-the-counter markets. They have two features: A knock-out feature causes the option to immediately terminate if the underlying asset reaches a specified barrier level before the expiration date; A knock-in feature causes the option to become effective only if the underlying asset first reaches a specified barrier level before the expiration date.

4.1 Up-and-Out Call Option

Definition 4.1 The price of up-and-out call option with the strike price \( K \) and maturity time \( T \) is defined as

\[
f = E[\exp(-rT)(Y_T - K)^+] \cdot M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\}.
\]

where \( a \) is the knock-out barrier.

Theorem 4.1 Suppose that \( X_t \) and \( Y_t \) satisfy the price dynamics described by the Liu's stock model (9), then the up-and-out call option pricing formula is given by

\[
f = Y_0 \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} 
\int_0^\infty M \{ eT + \sigma C_T \geq \ln u \} du
\]

Proof.

\[
E[\exp(-rT)(Y_T - K)^+] M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\}
= \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} 
E \left[ (Y_0 \exp(eT + \sigma C_T) - K)^+ \right]
= \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} 
\int_0^\infty M \{ Y_0 \exp(eT + \sigma C_T) - K \geq x \} dx
= Y_0 \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} 
\int_0^\infty \frac{1}{1 + \exp \left( \frac{\pi(\ln u - T + \delta T)}{\sqrt{3}rT} \right)} du.
\]

Example 1: Suppose that a stock is presently selling for a price of \( Y_0 = 35 \), the riskless interest rate is \( r = 0.08 \) per annum, the stock drift \( \varepsilon \) is 0.06 and the stock diffusion \( \sigma \) is 0.30. The barrier level \( a \) is 38. Suppose that \( \max_{0 \leq t \leq T} Y_t \leq a \) is equivalent to \( Y_T \leq a \). We would like to find up-and-out call barrier option price that expires in half a year and has a strike price of \( K = 40 \).

To calculate this barrier option price, the following MATLAB codes may be employed in a personal computer:

```matlab
syms x;
y='(35*exp(-0.08*0.50)./(1+exp(pi*(log(x)-0.06+0.50))./(sqrt(3)*0.30+0.50))))
*(1-(1+exp(pi*(log(38/35)-0.06+0.25))/(sqrt(3)*0.30+0.25))))';
f = quad(y,40/35,100);
The calculation result shows that \( f = 0.4694 \). This means the appropriate call option price in the example is about 47 cents.

4.2 Up-and-Out Put Option

Definition 4.2 The price of up-and-out put option with the strike price \( K \) and maturity time \( T \) is defined as

\[
f = E[\exp(-rT)(K - Y_T)^+] \cdot M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\}
\]

where \( a \) is the knock-out barrier.

Theorem 4.2 Suppose that \( X_t \) and \( Y_t \) satisfy the price dynamics described by the Liu’s stock model (9), then the up-and-out put option pricing formula is given by

\[
f = Y_0 \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} 
\int_0^\infty \frac{1}{1 + \exp \left( \frac{\pi(\ln u - T + \delta T)}{\sqrt{3}rT} \right)} du.
\]
Proof.

\[
E[\exp(-rT)(K - Y_T)^+] \cdot M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} \\
= \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} .
\]

\[
E \left[ (K - Y_0 \exp(eT + \sigma C_T))^+ \right] = \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} .
\]

\[
\int_0^{+\infty} M \left\{ K - Y_0 \exp(eT + \sigma C_T) \geq x \right\} dx \\
= Y_0 \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} .
\]

\[
\int_{-\infty}^{\frac{K}{Y_0}} M \left\{ eT + \sigma C_T \leq u \right\} du \\
= Y_0 \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \leq a \right\} .
\]

\[
\int_0^{\frac{K}{Y_0}} \frac{1}{1 + \exp \left( \frac{\sigma(T - \ln(u))}{\sqrt{3σT}} \right)} du.
\]

Example 2: Suppose that a stock is presently selling for a price of \( Y_0 = 40 \), the riskless interest rate is \( r = 0.08 \) per annum, the stock drift \( e \) is 0.06 and the stock diffusion \( \sigma \) is 0.30. The barrier level \( a \) is 38. Suppose that \( \max_{0 \leq t \leq T} Y_t \leq a \) is equivalent to \( Y_T \leq a \). We would like to find \( \text{up-and-out put barrier option price} \) that expires in half a year and has a strike price of \( K = 35 \).

To calculate this barrier option price, the following MATLAB codes may be employed in a personal computer:

```matlab
syms x;
y =’(40+exp(-0.08*0.50)./(1+exp(pi*(0.06+0.50-\log(x)))./(sqrt(6)+0.30+0.50)))*((1/(1+exp(pi*(log(38/40)-0.06+0.25)/(sqrt(3)+0.30+0.25))))
’;
f = quad(y,40/35,100)
```

The calculation result shows that \( f = 0.1525 \). This means the appropriate put option price in the example is about 15 cents.

### 4.3 Down-and-Out Call Option

**Definition 4.3** The price of down-and-out call option with the strike price \( K \) and maturity time \( T \) is defined as

\[
f = E[\exp(-rT)(Y_T - K)^+] \cdot M \left\{ \min_{0 \leq t \leq T} Y_t \geq a \right\} .
\]

where \( a \) is the knock-out barrier.

**Theorem 4.3** Suppose that \( X_t \) and \( Y_t \) satisfy the price dynamics described by the Liu’s stock model (9), then the down-and-out call option pricing formula is given by

\[
f = Y_0 \exp(-rT) M \left\{ \min_{0 \leq t \leq T} Y_t \geq a \right\} .
\]

\[
\int_0^{+\infty} \frac{1}{1 + \exp \left( \frac{\sigma(T - \ln(u))}{\sqrt{3σT}} \right)} du.
\]

**Example 3:** Suppose that a stock is presently selling for a price of \( Y_0 = 35 \), the riskless interest rate is \( r = 0.08 \) per annum, the stock drift \( e \) is 0.06 and the stock diffusion \( \sigma \) is 0.30. The barrier level \( a \) is 38. Suppose that \( \min_{0 \leq t \leq T} Y_t \geq a \) is equivalent to \( Y_T \geq a \). We would like to find up-and-out call Option barrier Option price that expires in half a year and has a strike price of \( K = 40 \).

To calculate this barrier option price, the following MATLAB codes may be employed in a personal computer:

```matlab
syms x;
y =’(35+exp(-0.08*0.50)./(1+exp(pi*(log(x)-0.06+0.50))/(sqrt(3)+0.30+0.50)))*((1/(1+exp(pi*(log(38/35)-0.06+0.25))/(sqrt(3)+0.30+0.25))))’;
f = quad(y,40/35,100)
```

The calculation result shows that \( f = 0.4069 \). This means the appropriate call option price in the example is about 41 cents.

### 4.4 Down-and-Out Put Option

**Definition 4.4** The price of down-and-out put option with the strike price \( K \) and maturity time \( T \) is defined as

\[
f = E[\exp(-rT)(K - Y_T)^+] \cdot M \left\{ \min_{0 \leq t \leq T} Y_t \geq a \right\} .
\]

where \( a \) is the knock-out barrier.

**Theorem 4.4** Suppose that \( X_t \) and \( Y_t \) satisfy the price dynamics described by the Liu’s stock model (9), then the down-and-out put option pricing formula is given by

\[
f = Y_0 \exp(-rT) M \left\{ \min_{0 \leq t \leq T} Y_t \geq a \right\} .
\]

\[
\int_0^{+\infty} \frac{1}{1 + \exp \left( \frac{\sigma(T - \ln(u))}{\sqrt{3σT}} \right)} du.
\]
Example 4: Suppose that a stock is presently selling for a
price of $Y_0 = 40$, the riskless interest rate is $r$ is 0.08 per
annum, the stock drift $c$ is 0.06 and the stock diffusion $\sigma$ is
0.30. The barrier level $a$ is 38. Suppose that $\min_{0 \leq t \leq T} Y_t \geq a$ is
equivalent to $Y_T \geq a$. We would like to find up-and-out put
barrier option price that expires in half a year and has a strike
price of $K = 35$.

To calculate this barrier option price, the following
MATLAB codes may be employed in a personal computer:

```matlab
syms x;
y = (40*exp(-0.08*0.50)/(1+exp(pi*0.06+0.50-
log(x)))/(sqrt(3)*0.30+0.50)));
+(1/(1+exp(pi*(log(38./40)-
0.06+0.25)/(sqrt(3)*0.30+0.50))))';
f = quad(y,0,35/40)
The calculation result shows that $f = 0.1796$. This means the
appropriate put option price in the example is about 18 cents.

5 Knock-in Barrier Options

5.1 Up-and-In Call Option

Definition 5.1 The price of up-and-in call option with the
strike price $K$ and maturity time $T$ is defined as

$$f = E[\exp(-rT)(Y_T - K)^+] \cdot M \left\{ \max_{0 \leq t \leq T} Y_t \geq a \right\}$$

(18)

where $a$ is the knock-in barrier.

Theorem 5.1 Suppose that $X_t$ and $Y_t$ satisfy the price

dynamics described by the Liu’s stock model (9), then the up-
and-in call option pricing formula is given by

$$f = Y_0 \exp(-rT) M \left\{ \max_{0 \leq t \leq T} Y_t \geq a \right\} \cdot \int_{\pi(\infty - cT) \sqrt{30T}}^k \frac{1}{1 + \exp \left( \frac{\pi(x - eT)}{\sqrt{30T}} \right)} \, dx.$$ 

(19)

5.2 Up-and-In Put Option

Definition 5.2 The price of up-and-in put option with the
strike price $K$ and maturity time $T$ is defined as

$$f = E[\exp(-rT)(K - Y_T)^+] \cdot M \left\{ \min_{0 \leq t \leq T} Y_t \leq a \right\}$$

(20)

where $a$ is the knock-in barrier.

Theorem 5.2 Suppose that $X_t$ and $Y_t$ satisfy the price
dynamics described by the Liu’s stock model (9), then the up-
and-in put option pricing formula is given by

$$f = Y_0 \exp(-rT) M \left\{ \min_{0 \leq t \leq T} Y_t \leq a \right\} \cdot \int_{\pi(\infty - cT) \sqrt{30T}}^k \frac{1}{1 + \exp \left( \frac{\pi(x - eT)}{\sqrt{30T}} \right)} \, dx.$$ 

(21)

5.3 Down-and-In Call Option

Definition 5.3 The price of down-and-in call option with the
strike price $K$ and maturity time $T$ is defined as

$$f = E[\exp(-rT)(Y_T - K)^+] \cdot M \left\{ \min_{0 \leq t \leq T} Y_t \leq a \right\}$$

(22)

where $a$ is the knock-in barrier.

Theorem 5.3 Suppose that $X_t$ and $Y_t$ satisfy the price

dynamics described by the Liu’s stock model (9), then the down-
and-in call option pricing formula is given by

$$f = Y_0 \exp(-rT) M \left\{ \min_{0 \leq t \leq T} Y_t \leq a \right\} \cdot \int_{\pi(\infty - cT) \sqrt{30T}}^k \frac{1}{1 + \exp \left( \frac{\pi(x - eT)}{\sqrt{30T}} \right)} \, dx.$$ 

(23)

5.4 Down-and-In Put Option

Definition 5.4 The price of down-and-out put option with the
strike price $K$ and maturity time $T$ is defined as

$$f = E[\exp(-rT)(K - Y_T)^+] \cdot M \left\{ \min_{0 \leq t \leq T} Y_t \leq a \right\}$$

(24)

where $a$ is the knock-in barrier.

Theorem 5.4 Suppose that $X_t$ and $Y_t$ satisfy the price

dynamics described by the Liu’s stock model (9), then the down-
and-out put option pricing formula is given by

$$f = Y_0 \exp(-rT) M \left\{ \min_{0 \leq t \leq T} Y_t \leq a \right\} \cdot \int_{\pi(\infty - cT) \sqrt{30T}}^k \frac{1}{1 + \exp \left( \frac{\pi(x - eT)}{\sqrt{30T}} \right)} \, dx.$$ 

(25)

6 Conclusions

In this paper, one-dimensional, single barrier options pricing
formulas, which include eight possible types, were defined.
And we investigated their pricing problems based on the Liu’s
stock model in uncertain financial market. Ground on the re-
results, some potential applications of uncertain stock models
will be an interesting topic of future research.
Acknowledgements

This work is supported by the National Natural Science Foundation (Grant No.70671050), the Major Research Program (Grant No.Z20082701) of Hubei Provincial Department of Education, China.

References