Developments of Mean-Variance Model for Portfolio Selection in Uncertain Environment

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Abstract

This paper presents portfolio selection problems with ambiguous returns assumed as "return is about ξ" which is neither estimated by randomness nor fuzziness. Portfolio selection problems in uncertain environment are formulated as nonlinear programming models based on uncertain programming approaches. Since there is no efficient solution method to solve these problems directly, original problems are transformed into equivalent deterministic programming problems using expected value and variance via identification function of uncertain variables and finally to find the optimal solution of each problem is constructed. To illustrate the proposed models, some examples of portfolio selection problems in uncertain environment as a generalization of random environment and fuzzy environment are provided.

Keywords: Portfolio selection, Mean-variance model, Uncertain variable, Uncertain measure, Uncertain programming

1 Introduction

The portfolio selection problem is based on a single period model of investment. The investor has to choose and allocate his available capital among various securities such that the investment goal can be achieved. Markowitz [27][28] initialized the problem by mean-variance methodology and that has been serving as the basic of modern financial theory. The mathematical formulation of the Markowitz's portfolio selection problem is the trade-off between risk and return which combines probability theory and optimization theory to model the behavior of the economic agents. This classical mean-variance model is valid if the return is multi-variate normally distributed and the investor is averse to risk and always prefers lower risk, or it is valid if for any given return which is multi-variate distributed the investor has quadratic objective function.

After the introduction of Markowitz's model, numerous portfolio selection models have been developed to improve the mean-variance model. In the last fifty years since then, portfolio theory has been improved and completed in several directions. Several extensions to mean-variance models has been proposed. Some models have been developed to minimize semivariance in different cases such as [13][29], while other researchers (e.g. [15][24][30]) added the skewness in consideration for portfolio selection.

Most of the above research works the common assumptions are that the investor have enough historical data and that the situation of asset markets in future can be correctly reflected by asset data in past. However, it is not always ensured such two kinds of assumptions. For example, when new stocks are listed in the stock market, there is no past information for these securities. Random, fuzzy and random fuzzy optimization models proved some useful methods for investors to tackle the uncertainty. Some authors [5][17][34] (Osti et al. 2002) use fuzzy numbers to replace uncertain returns of the securities and they define the portfolio selection as a mathematical programming problem in order to select the best alternative. Tanaka and Guo [32][33] used possibilistic distributions to model uncertainty in returns. Arenas-Parra et al. [3] introduced vague goals for return rate, risk and liquidity based on expected intervals. Bilbao-Terol et al. [4] formulated a fuzzy compromise programming to the mean-variance portfolio selection problem. Huang [12] measured portfolio risk by credibility measure and proposed two credibility based mean-variance models. Huang [13] also proposed a mean-semivariance model for describing the asymmetry of fuzzy returns. Huang extended the risk definition of variance and risk definition of chance to a random fuzzy environment and formulates optimization models where security returns are fuzzy random variables. But in reality, sometimes investors have to deal with the uncertainty which acts neither randomness nor fuzziness. In order to deal with such type of uncertainty, Liu [20] founded uncertainty theory as a branch of mathematics. Subsequently, Liu [21] proposed uncertain process and uncertain differential equation to deal with dynamic uncertain phenomena. In addition, uncertain calculus was introduced by Liu [22] to describe the function of uncertain processes, uncertain inference was introduced by Liu [22] via the tool of conditional uncertainty and uncertain logic was proposed Li and Liu [18] to deal with uncertain knowledge. Liu [23] proposed an uncertain programming including expected value model, chance constrained programming and dependent-chance programming to model several optimization problems. Till now, several research works [8][18][22][23][35] have been done in this area, but none has considered the portfolio selection problems in the framework of risk-return trade-off in the uncertain environment.

It is worth pointing out that Liu [23] gave a formulation of mean-variance model for portfolio selection. This paper further extends the work with uncertain returns an extension of Markowitz's mean-variance model in general uncertain environment. Here the security returns are uncertain variables which are characterized by their identification functions. We use the uncertainty measure to describe an uncertain event instead of probability/credibility measure. Finally, as a generalization of random portfolio selection and fuzzy portfolio selection, a general uncertain portfolio selection model will be introduced by using the definition of expected value and variance of uncertain variable based on uncertainty theory [20][23].

2 Preliminaries

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Assume that $\mathcal{M}$ is a set function over $\mathcal{L}$. Then $\mathcal{M}$ is called an uncertain measure by Liu [20] if it satisfies the following four axioms

Axiom 1. (Normality) $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2. (Monotonicity) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$;

Axiom 3. (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 4. (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 1 *(Liu [20]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.*

It is known that a random variable can be characterized by a probability density function and a fuzzy variable can be characterized by a membership function. Similarly, an uncertain variable may be characterized by some identification function. Next, we introduce two kinds of identification functions.

Definition 2 *(Liu [23]) An uncertain variable ξ is said to have a first identification function λ if*
(i) $\lambda(x)$ is a nonnegative function on $\Re$ such that $\sup_{x\neq y}(\lambda(x)+\lambda(y)) = 1$;
(ii) for any set B of real numbers, we have

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \sup\limits_{x\in B}\lambda(x), & \text{if } \sup\limits_{x\in B}\lambda(x) < 0.5, \\ 1 - \sup\limits_{x\in B^c}\lambda(x), & \text{if } \sup\limits_{x\in B}\lambda(x) \ge 0.5. \end{cases}$$

Definition 3 *(Liu [26]) An uncertain variable ξ is said to have a second identification function ρ if*
(i) $\rho(x)$ is a nonnegative and integrable function on $\Re$ such that $\int_{\Re}\rho(x)\mathrm{d}x \ge 1$;
(ii) for any set B of real numbers, we have

$$\mathcal{M}\{\xi \in B\} = \begin{cases} \displaystyle\int_B \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_B \rho(x)\mathrm{d}x < 0.5, \\ 1 - \displaystyle\int_{B^c}\rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{B^c}\rho(x)\mathrm{d}x \ge 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Next, we introduce the expected value and variance of uncertain variable. In a sense, expected value is the average of uncertain variable and represents its size. Especially, we can rank uncertain variables by using expected value operator.

Definition 4 *(Liu [20]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty}\mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r \tag{1}$$

provided that at least one of the two integrals is finite.

Definition 5 *(Liu [23]) Let ξ be an uncertain variable with finite expected value. Then its variance is defined as*

$$V[\xi] = E[(\xi - E[\xi])^2]. \tag{2}$$

3 Uncertain Portfolio Selection Models

The mean-variance model of portfolio selection with uncertain returns is presented by Liu [23] as extension of Markowitz's mean-variance model [27] in uncertain environment. In this section, we further develop the work by Liu [23] and give the variations of mean-variance model in uncertain environment.

Let r_i be the uncertain returns of the ith security. In general, r_i is defined as $(p_i' + d_i - p_i)/p_i$ in which p_i is the closing price of the ith security at present, p_i' its estimated closing price in the next period, and d_i its estimated dividends during the coming period. Since both p_i' and d_i are uncertain, it is reasonable to estimate r_i by uncertain variables ξ_i.

Assume that there are n securities with return rates $\xi_i(i = 1, 2, \cdot, n)$ and denoted by x_i (with $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$) the proportion of total amount of funds invested in the ith security. The performances of individual securities were considered as uncertain variables. The return of the portfolio was quantified as the mean and the risk was quantified as variance.

In the case of maximizing the return at a desired level of risk, the investor may employ the following model,

$$\begin{cases} \max & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{s.t.} & V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \leq d \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3}$$

where where E denotes the expected value operator, V the variance operator and d is the maximum risk level the investors can tolerate. Model (3) is just the mean-variance model introduced by Liu [23].

However, in some situations, the investors would like to minimize investment risk subject to a given expected return r. This leads to the following programming problem:

$$\begin{cases} \min & V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{s.t.} & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \geq r \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{4}$$

The first constraint indicates that the expected return is no less than the target r. The second constraint ensures that all the capital will be invested to n securities. In addition, $x_i \geq 0$ implies that the short-selling or borrowing of security i is not allowed.

If the investor does not know how to set return and/or risk level, then he/she may employ the following he bi-objective programming problem,

$$\begin{cases} \max & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \min & V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{s.t.} & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0 \quad i = 1, 2, \cdots, n. \end{cases} \tag{5}$$

This model may be formulated as a single objective programming as follows,

$$\begin{cases} \max & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] - \lambda V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{s.t.} & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0 \quad i = 1, 2, \cdots, n, \end{cases} \tag{6}$$

where λ represents the degree of risk aversion of the investors.

4 Special Cases

In this section we establish some important results to estimate the uncertain variable for portfolio selection problems in uncertain environment.

4.1 Trapezoidal Uncertain Variables

A security return ξ is a trapezoidal uncertain variable if it has the following first identification function

$$\lambda(x) = \begin{cases} \dfrac{x-a}{2(b-a)}, & \text{if } a \le x \le b \\ 0.5, & \text{if } b \le x \le c \\ \dfrac{x-d}{2(c-d)}, & \text{if } c \le x \le d \end{cases} \tag{7}$$

where a, b, c and d are real numbers with $a < b < c < d$. For simplicity, write $\xi = (a, b, c, d)$.

Assume that $x_1, x_2, \cdots, x_n$ are n real numbers. The function $f(x_1, x_2, \cdots, x_n)$ is defined as

$$f(x_1, x_2, \cdots, x_n) = 4\alpha^2 + 3\alpha\beta + 4\beta^2 + 6\gamma(2\alpha + \beta + 2\gamma) + \frac{(\beta - 2\gamma)^3 + (\beta - 2\gamma)^2|\beta - 2\gamma|}{4(\alpha + \beta)}$$

where

$$\alpha = \sum_{i=1}^{n} x_i(b_i - c_i + d_i - a_i), \ \beta = \sum_{i=1}^{n} x_i(b_i + c_i - d_i - a_i) \text{ and } \gamma = \sum_{i=1}^{n} x_i(c_i - b_i),$$

and a_i, b_i, c_i, d_i are given real numbers for $i = 1, 2, \cdots, n$.

Theorem 1 *Suppose that the ith security return $\xi_i = (a_i, b_i, c_i, d_i)$ for $i = 1, 2, \cdots, n$. Then mean-variance model (4) can be converted into the following deterministic programming,*

$$\begin{cases} \min & f(x_1, x_2, \cdots, x_n) \\ s.t. & \displaystyle\sum_{i=1}^{n} x_i(a_i + b_i + c_i + d_i) \ge 4r \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{8}$$

Proof: It follows from operational law of uncertain variables that

$$\sum_{i=1}^{n} x_i\xi_i = \left(\sum_{i=1}^{n} x_i a_i, \sum_{i=1}^{n} x_i b_i, \sum_{i=1}^{n} x_i c_i, \sum_{i=1}^{n} x_i d_i \right)$$

which is also a trapezoidal uncertain variable. According to the definitions of expected value and variance of uncertain variable, we have

$$E\left[\sum_{i=1}^{n} x_i\xi_i\right] = \frac{1}{4}\sum_{i=1}^{n} x_i(a_i + b_i + c_i + d_i),$$

$$V\left[\sum_{i=1}^{n} x_i\xi_i\right] = \frac{1}{96} f(x_1, x_2, \cdots, x_n).$$

Since $x_1 + x_2 + \cdots + x_n = 1$ and $x_i \geq 0$ for each i, minimizing $f(x_1, x_2, \cdots, x_n)$ is the value of variance $V[\sum_{i=1}^n x_i\xi_i]$. Therefore, the conclusion is obtained and the theorem is proved.

Remark 1: In Theorem 1, if $b_i - a_i = d_i - c_i$ for $i = 1, 2, \cdots, n$, then we have the following optimization model,

$$\begin{cases} \min & \left(\sum_{i=1}^n x_i(d_i - a_i)\right)^2 + \left(\sum_{i=1}^n x_i(d_i - a_i)\right)\left(\sum_{i=1}^n x_i(c_i - b_i)\right) + \left(\sum_{i=1}^n x_i(c_i - b_i)\right)^2 \\ \text{s.t.} & \sum_{i=1}^n x_i(a_i + b_i + c_i + d_i) \geq 4r \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{9}$$

Remark 2: In Theorem 1, if $\xi_i = (a_i, b_i, c_i)$ for $i = 1, 2, \cdots, n$, then we have the following optimization model,

$$\begin{cases} \min & \dfrac{6(p^2 + q^2)(p + q + |p - q|) + 11(p + q)^2|p - q|}{p + q + |p - q|} \\ \text{s.t.} & \sum_{i=1}^n x_i(a_i + 2b_i + c_i) \geq 4r \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{10}$$

where $p = \sum_{i=1}^n x_i(b_i - a_i)$ and $q = \sum_{i=1}^n x_i(c_i - b_i)$.

Remark 3: In Theorem 1, if $\xi_i = (a_i, b_i, c_i)$ is symmetrical for $i = 1, 2, \cdots, n$, i.e., $b_i - a_i = c_i - b_i$, then model (4) is equivalent to the following optimization model,

$$\begin{cases} \min & x_1(c_1 - a_1) + x_2(c_2 - a_2) + \cdots + x_n(c_n - a_n) \\ \text{s.t.} & x_1b_1 + x_2b_2 + \cdots + x_nb_n \geq r \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{11}$$

which is a linear programming problem.

4.2 Normal Uncertain Variables

An uncertain variable ξ is called normal if it has normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ x \in \Re \tag{12}$$

denoted by $N(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$. It is easy to verify that the normal uncertain variable ξ has expected value e, i.e., $E[\xi] = e$. In addition, we have $\sigma^2 \leq V[\xi] \leq \sigma^2$. If a scalar variance is needed, then Liu [23] took the maximum value, i.e., $V[\xi] = \sigma^2$.

Theorem 2 *Suppose that ith security returns are normally distributed uncertain variables with expected value e_i and variance σ_i^2 for $i = 1, 2, \cdots, n$. Then mean-variance model (4) can be converted into the following* *deterministic programming,*

$$\begin{cases} \mathit{min} & x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n \\ \mathit{s.t.} & x_1e_1 + x_2e_2 + \cdots + x_ne_n \geq r \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{13}$$

Proof: It is known from [23] that a weighted sum of normally distributed uncertain variables is also normally distributed. That is, $x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$ is a normally distributed uncertain variable with expected value $x_1e_1 + x_2e_2 + \cdots + x_ne_n$ and variance $(x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n)^2$. Since $\sigma_i > 0$, and $x_1 + x_2 + \cdots + x_n = 1$ and $x_i \geq 0$, $i = 1, 2, \cdots, n$, we get $x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n > 0$. Thus, maximizing the value of $x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n$ is equivalent to $(x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n)^2$. The theorem is proved.

Corollary 1: Suppose that ith security returns are normally distributed uncertain variables with expected value e_i and variance σ_i^2 for $i = 1, 2, \cdots, n$. Then mean-variance model (3) can be converted into the following linear programming,

$$\begin{cases} \max & x_1e_1 + x_2e_2 + \cdots + x_ne_n \\ \text{s.t.} & x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n \leq \sqrt{d} \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{14}$$

Corollary 2: Suppose that ith security returns are normally distributed uncertain variables with expected value e_i and variance σ_i^2 for $i = 1, 2, \cdots, n$. Then mean-variance model (5) can be converted into the following linear programming,

$$\begin{cases} \max & x_1e_1 + x_2e_2 + \cdots + x_ne_n \\ \min & x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n \\ \text{s.t.} & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{15}$$

Theorem 1 and Corollary 4.2 imply that if all the security returns are normally distributed uncertain variables, then models (4) and (3) are both equivalent to linear programming problem.

Example 1: Suppose that there are two risky securities which are normal uncertain variables with expected value e_i and σ_i^2 for $i = 1, 2$. With loss of generality, we assume $e_1 > e_2$. Then we have

$$\begin{cases} \min & (\sigma_1 - \sigma_2)x + \sigma_2 \\ \text{s.t.} & (e_1 - e_2)x + e_2 \geq r \\ & 0 \leq x \leq 1. \end{cases} \tag{16}$$

The argument breaks down into cases.

Case 1: $r \leq e_2$. Then we have $(r - e_2)/(e_1 - e_2) \leq 0$. Thus, $x \in [0, 1]$. That is to say, the optimal portfolio selection is $(x_1^*, x_2^*) = (x, 1 - x)$ for any $x \in [0, 1]$.

Case 2: $e_2 < r < e_1$. Then we have $(r - e_2)/(e_1 - e_2) < r \leq 1$. If $\sigma_1 > \sigma_2$, then

$$(x_1^*, x_2^*) = \left(\frac{r - e_2}{e_1 - e_2}, \frac{e_1 - r}{e_1 - e_2}\right).$$

If $\sigma_1 = \sigma_2$, then $(x_1^*, x_2^*) = (x, 1-x)$ for any $x \in [(r-e_2)/(e_1-e_2), 1]$. If $\sigma_1 < \sigma_2$, then $(x_1^*, x_2^*) = (0, 1)$.

Case 3: $r = e_1$. Then $(x_1^*, x_2^*) = (1, 0)$.

Case 4: $r > e_1$. Then there is no optimal solution.

5 Conclusions

In this paper, we have considered mean-variance portfolio selection problems involving expected uncertain returns and proposed three different models. To describe uncertain events, we provides the portfolio selection models based on uncertainty measures. The uncertain programming problems are converted into equivalent deterministic programming problems using identification function of uncertain variables. Some numerical examples are shown for better illustration of our models.

In future, we will apply there general uncertain portfolio selection problems to other asset allocation problem, multi-period problems and combinational optimization models. The proposed models can also be extended to more complex portfolio selection models considering higher moments.

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