Some Properties of Optimistic and Pessimistic Values of Uncertain Variables

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Abstract

Based on uncertain measure, the pessimistic value and optimistic value of uncertain variables have been introduced for handling optimization problems in uncertain environments. In this paper, some new properties of critical values of uncertain variables are investigate. equations was given.

Keywords: Fuzzy variable, fuzzy differential equation, existence and uniqueness theorem

1 Preliminary

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is assigned a number $M(\Lambda)$. In order to ensure that the number $M(\Lambda)$ has certain mathematical properties, Liu [5] presented the following four axioms:

Axiom 1. (Normality) $M(\Gamma) = 1$.

Axiom 2. (Monotonicity) $M(\Lambda_1) \leq M(\Lambda_2)$ whenever $\Lambda_1 \subset \Lambda_2$.

Axiom 3. (Self-Duality) $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$.

Axiom 4. (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\left(\bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).$$

Definition 1 (Liu[5]) An set function $M$ is called an uncertain measure if it satisfies the normality, monotonicity, self-duality, and countable subadditivity axioms.

Definition 2 (Liu [5]) An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$M(\{\xi \in B\}) = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 3 (Liu [7]) The uncertain variables $X_1, X_2, \cdots, X_m$ are said to be independent if

$$M\left(\bigcap_{i=1}^{m} X_i \in B_i \right) = \min_{1 \leq i \leq m} M(\{X_i \in B_i\})$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.
2 Uncertain Critical Value

Definition 4 Let \( \xi \) be an uncertain variable, and \( \alpha \in (0, 1] \). Then
\[
\xi_{\sup} = \sup \{ r \mid M \{ \xi \geq r \} \geq \alpha \}
\]
is called the \( \alpha \)-optimistic value to \( \xi \), and
\[
\xi_{\inf} = \inf \{ r \mid M \{ \xi \leq r \} \geq \alpha \}
\]
is called the \( \alpha \)-pessimistic value to \( \xi \).

Example 1 Let \( \xi \) be a rectangular uncertain variable on \((a, b)\). Then its \( \alpha \)-optimistic and \( \alpha \)-pessimistic values are
\[
x_{\sup}(\alpha) = \begin{cases} 
  b, & \text{if } \alpha \leq 0.5 \\
  a, & \text{if } \alpha > 0.5
\end{cases}
\]
\[
x_{\inf}(\alpha) = \begin{cases} 
  b, & \text{if } \alpha \leq 0.5 \\
  a, & \text{if } \alpha > 0.5
\end{cases}
\]

Example 2 Let \( \xi = (a, b, c) \) be a triangular uncertain variable. Then its \( \alpha \)-optimistic and \( \alpha \)-pessimistic values are
\[
x_{\sup}(\alpha) = \begin{cases} 
  2ab + (1 - 2a)c, & \text{if } \alpha \leq 0.5 \\
  (2\alpha - 1)a + (2 - 2\alpha)b, & \text{if } \alpha > 0.5
\end{cases}
\]
\[
x_{\inf}(\alpha) = \begin{cases} 
  (1 - 2\alpha)a + 2ab, & \text{if } \alpha \leq 0.5 \\
  (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha > 0.5
\end{cases}
\]

Example 3 Let \( \xi = (a, b, c, d) \) be a trapezoidal uncertain variable. Then its \( \alpha \)-optimistic and \( \alpha \)-pessimistic values are
\[
x_{\sup}(\alpha) = \begin{cases} 
  2ac + (1 - 2a)d, & \text{if } \alpha \leq 0.5 \\
  (2\alpha - 1)a + (2 - 2\alpha)b, & \text{if } \alpha > 0.5
\end{cases}
\]
\[
x_{\inf}(\alpha) = \begin{cases} 
  (1 - 2\alpha)a + 2ab, & \text{if } \alpha \leq 0.5 \\
  (2 - 2\alpha)c + (2\alpha - 1)d, & \text{if } \alpha > 0.5
\end{cases}
\]

Example 4 Let \( \xi \) be an exponential uncertain variable \( \mathcal{E}(a, b) \). Then its \( \alpha \)-optimistic and \( \alpha \)-pessimistic values are
\[
x_{\inf}(\alpha) = \begin{cases} 
  a \ln \frac{a}{a - \alpha b}, & \text{if } \alpha < 0.5 \\
  a \ln \frac{a}{(1 - \alpha)b}, & \text{if } \alpha \geq 0.5
\end{cases}
\]
\[
x_{\sup}(\alpha) = \begin{cases} 
  a \ln \frac{a}{a - \alpha b}, & \text{if } \alpha < 0.5 \\
  a \ln \frac{a}{a - (1 - \alpha)b}, & \text{if } \alpha \geq 0.5
\end{cases}
\]

Example 5 Let \( \xi \) be a bell-like uncertain variable \( \mathcal{B}(a, b) \). Then its \( \alpha \)-optimistic and \( \alpha \)-pessimistic values are
\[
x_{\inf}(\alpha) = \begin{cases} 
  a \sqrt{\frac{2}{\pi}} \Phi^{-1} \left( \frac{\alpha b}{a} \right) + e, & \text{if } \alpha < 0.5 \\
  a \sqrt{\frac{2}{\pi}} \Phi^{-1} \left( \frac{a - (1 - \alpha)b}{a} \right) + e, & \text{if } \alpha \geq 0.5
\end{cases}
\]
\[
x_{\sup}(\alpha) = \begin{cases} 
  a \sqrt{\frac{2}{\pi}} \Phi^{-1} \left( \frac{a - \alpha b}{a} \right) + e, & \text{if } \alpha < 0.5 \\
  a \sqrt{\frac{2}{\pi}} \Phi^{-1} \left( \frac{(1 - \alpha)b}{a} \right) + e, & \text{if } \alpha \geq 0.5
\end{cases}
\]

Example 6 An uncertain \( \xi \) is called normal if it has a distribution function
\[
\Phi(x) = \left( 1 + \exp \left( \frac{\pi(e - x)}{\sqrt{3\sigma}} \right) \right), \quad x \in \mathbb{R}
\]
denoted by $\mathcal{N}(\varepsilon, \sigma)$. Then its $\alpha$-optimistic value is

$$\xi_{\text{sup}}(\alpha) = \varepsilon - \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1 - \alpha};$$

its $\alpha$-pessimistic value is

$$\xi_{\text{inf}}(\alpha) = \varepsilon + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

**Theorem 1** Let $\xi$ be a uncertain variable. Then we have
(a) $\xi_{\text{inf}}(\alpha)$ is an increasing and left-continuous function of $\alpha$;
(b) $\xi_{\text{sup}}(\alpha)$ is an decreasing and left-continuous function of $\alpha$.

**Proof:** (a) It is obvious that $\xi_{\text{inf}}(\alpha)$ is an increasing function of $\alpha$. Let $\{\alpha_i\}$ be an arbitrary sequence of positive numbers such that $\alpha_i \uparrow \alpha$. Then $\xi_{\text{inf}}(\alpha)$ is an increasing sequence and

$$\lim_{i \to +\infty} \xi_{\text{inf}}(\alpha_i) \leq \xi_{\text{inf}}(\alpha).$$

If the inequality holds, there exists a number $t$ such that $\lim_{i \to +\infty} \xi_{\text{inf}}(\alpha_i) < t < \xi_{\text{inf}}(\alpha)$. Thus $M\{\xi \leq t\} \geq \alpha_i$ for each $i$. Letting $i \to +\infty$, we get $M\{\xi \leq t\} \geq \alpha$. Hence $t \geq \xi_{\text{inf}}(\alpha)$. Therefore the contradiction proves the left-continuity of $\xi_{\text{inf}}(\alpha)$ with respect to $\alpha$.

The part (b) may be proved similarly.

**Theorem 2** Suppose that $\xi$ and $\eta$ are independent uncertain variables. Then for any $\alpha \in (0, 1]$, we have

$$\left(\xi + \eta\right)_{\text{sup}}(\alpha) = \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha), \quad (\xi + \eta)_{\text{inf}}(\alpha) = \xi_{\text{inf}}(\alpha) + \eta_{\text{inf}}(\alpha),$$

$$(\xi \vee \eta)_{\text{sup}}(\alpha) = \xi_{\text{sup}}(\alpha) \vee \eta_{\text{sup}}(\alpha), \quad (\xi \vee \eta)_{\text{inf}}(\alpha) = \xi_{\text{inf}}(\alpha) \vee \eta_{\text{inf}}(\alpha),$$

$$(\xi \wedge \eta)_{\text{sup}}(\alpha) = \xi_{\text{sup}}(\alpha) \wedge \eta_{\text{sup}}(\alpha), \quad (\xi \wedge \eta)_{\text{inf}}(\alpha) = \xi_{\text{inf}}(\alpha) \wedge \eta_{\text{inf}}(\alpha),$$

**Proof:** For any given number $\varepsilon > 0$, since $\xi$ and $\eta$ are independent uncertain variables, we have

$$M\{\xi + \eta \geq \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha) - \varepsilon\}$$

$$\geq M\{\{\xi \geq \xi_{\text{sup}}(\alpha) - \varepsilon/2\} \cap \{\eta \geq \eta_{\text{sup}}(\alpha) - \varepsilon/2\}\}$$

$$= M\{\xi \geq \xi_{\text{sup}}(\alpha) - \varepsilon/2\} \wedge M\{\eta \geq \eta_{\text{sup}}(\alpha) - \varepsilon/2\}$$

$$\geq \alpha$$

which implies

$$\left(\xi + \eta\right)_{\text{sup}}(\alpha) \geq \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha) - \varepsilon.$$

On the other hand, by the independence, we have

$$M\{\xi + \eta \geq \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha) + \varepsilon\}$$

$$\leq M\{\{\xi \geq \xi_{\text{sup}}(\alpha) + \varepsilon/2\} \cup \{\eta \geq \eta_{\text{sup}}(\alpha) + \varepsilon/2\}\}$$

$$= M\{\xi \geq \xi_{\text{sup}}(\alpha) + \varepsilon/2\} \vee M\{\eta \geq \eta_{\text{sup}}(\alpha) + \varepsilon/2\} < \alpha$$

which implies

$$\left(\xi + \eta\right)_{\text{sup}}(\alpha) \leq \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha) + \varepsilon.$$ 

Hence

$$\xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha) + \varepsilon \geq (\xi + \eta)_{\text{sup}}(\alpha) \geq \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha) - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $(\xi + \eta)_{\text{sup}}(\alpha) = \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha)$. The other equalities may be proved similarly.

**Theorem 3** Let $\xi$ be a continuous uncertain variable and $\alpha \in (0, 1]$ be given, then

$$M\{\xi \leq \xi_{\text{inf}}(\alpha)\} \geq \alpha, \quad M\{\xi \geq \xi_{\text{sup}}(\alpha)\} \geq \alpha.$$

**Theorem 4** Let $\alpha \in (0, 1]$ be given and $\xi, \xi_1, \xi_2, \cdots$ be uncertain variables.
(a) If $\xi \to \xi$ in distribution, then $(\xi_n)_{\text{inf}}(\alpha) \to \xi_{\text{inf}}(\alpha)$ as $n \to \infty$;
(b) If $\xi \to \xi$ in distribution, then $(\xi_n)_{\text{sup}}(\alpha) \to \xi_{\text{sup}}(\alpha)$ as $n \to \infty$. 
3 Conclusion

In this paper, we discussed some new properties of optimistic and pessimistic values. Some useful examples are provided to formulate the optimistic and pessimistic values. These properties of uncertain variables can be used to analyze the properties of the solutions in uncertain programming.

Acknowledgments

This work was supported by National Natural Science Foundation of China Grant No.60874067.

References


