Fuzzy Arithmetic for LR Fuzzy Numbers with Applications to Fuzzy Programming

Jian Zhou, Fan Yang, Ke Wang*  
School of Management, Shanghai University, Shanghai 200444, China

Abstract

LR fuzzy numbers are quite common in real world applications involving the triangular fuzzy number, the Gaussian fuzzy number, and the Cauchy fuzzy number as special cases. In this paper, we first prove that a fuzzy number is an LR fuzzy number if and only if its credibility distribution is strictly increasing. After that, based on credibility measure, an operational law on independent LR fuzzy numbers is proposed for fuzzy arithmetic, providing a novel approach to analytically and exactly calculating the inverse credibility distribution of some specific arithmetical operations. As an application of the operational law, an equivalent form of the expected value operator as well as a theorem for computing the expected value of strictly monotone functions is suggested. Finally, we utilize the operational law to construct a solution framework of fuzzy programming with parameters of LR fuzzy numbers, and such type of fuzzy programming problems can be handled by the operational law as the classic deterministic programming without any particular solving techniques.

Keywords: LR fuzzy number, fuzzy arithmetic, operational law, expected value, fuzzy programming

1. Introduction

In many research fields, such as optimal control and operations research, some problems are usually described as a programming model or other mathematical relationships with vague information (e.g., demand, time, distance, etc.). To model the imprecision in these problems, the fuzzy set theory introduced by Zadeh [38] can be employed, in which different types of fuzzy numbers and the corresponding fuzzy arithmetic for them are necessary in the modeling process.

In 1975, Zadeh [39] presented the extension principle for fuzzy basic operations of addition, subtraction, multiplication, division and so on. Since implementing the Zadeh’s extension principle is equal to addressing a nonlinear programming, Kaufmann and Gupta [20] alternatively initialized an interval method for triangular and trapezoidal fuzzy numbers based on α-cut, which is easy to

*Corresponding author, Tel.: +86-21-66137696-804, Fax:+86-21-66134284.  
Email address: ke@shu.edu.cn (Ke Wang)
perform with low complexity for simple operations. However, their method may lead to higher powers of α when there are more terms being multiplied. For example, if there are n terms, the result of multiplication would be an nth-order polynomial in α. Further, all fuzzy numbers in the arithmetic procedure are treated as independent fuzzy numbers although most of them are not when based on the extension principle or the interval method. To tackle this problem, some researchers defined requisite constraints or proposed some novel approaches (see, e.g., [17, 21]).

Since the arithmetic based on above approaches are difficult to evaluate and computationally expensive, some approximation methods were introduced. For instance, Dubios and Prade [8] extended usual algebraic operations on real numbers to fuzzy numbers, and suggested a standard approximation to fuzzy arithmetic with efficient computation. Nevertheless, they reminded that frequent uses of the standard approximation for multiplication may lead to wrong results. Consequently Giachetti and Young [13] discussed the error of the standard approximation, and developed a new approximation for triangular and trapezoidal fuzzy numbers to reduce errors. In [14], they next proposed a form using six parameters to define fuzzy numbers, and provided the methods for performing fuzzy arithmetic with better accuracy and similar computational speed with the standard approximation. Guerra and Stefanini [15] used piecewise monotonic interpolations to approximate and represent a fuzzy number, and derived a procedure to control the absolute error associated with the arithmetic operations on fuzzy numbers.

Besides, there were some other discussions about fuzzy arithmetic. For instance, Chutia et al. [32] developed a generalised method to find the membership function for functions of triangular fuzzy numbers and to deal the basic operations. Liu and Iwamura [26] initialized fuzzy simulation, a Monte Carlo simulation-like method, to handle the possibility constraints. Cheng [4], Eslamipoor et al. [11, 12] and Haji et al. [16] proposed different methods to rank fuzzy numbers based on distance method. Furthermore, fuzzy arithmetic was also applied to fuzzy linear programming [18], fuzzy regression [31], risk analysis [29] and so on.

The LR fuzzy number initialized by Dubios and Prade [8] is a commonly used type of fuzzy numbers, especially its special case of triangular fuzzy numbers, in various problems over the past few decades (see, e.g., [2, 3, 7, 6]). An LR fuzzy number can be represented in the so-called LR-type by its mean value (most likely value), left and right spreads, and shape functions. In view of its properties on the shape, experts may easily specify the most likely value, the lower bound and the upper bound to formulate an LR fuzzy number base on their experience or knowledge. On the other hand, the fuzzy arithmetic for LR fuzzy numbers has been also investigated in some literature. For example, Wang and Kuo [35] proposed an alternative operation of fuzzy arithmetic on LR fuzzy numbers by three parameters, and developed a new learning algorithm of a fully fuzzified neural network based on the proposed approximation method. Sorini and Stefanini [34] suggested a parametrization for LR fuzzy numbers, and the parametric representations can be used to model the shapes of the membership functions and obtain operators for the fuzzy arithmetic operations.
Chou [5] presented an inverse function arithmetic principle on triangular fuzzy numbers, which was easy to interpret the multiplication operation with the membership functions of fuzzy numbers.

In this paper, we focus on the fuzzy arithmetic for LR fuzzy numbers. First, we give the equivalent condition for LR fuzzy numbers if its credibility distribution is strictly increasing. And the operational law is presented based on the inverse credibility distribution, which allows fuzzy arithmetic to be calculated exactly instead of approximation or simulation. Based on the proposed operational law, we give an equivalent definition and a simpler proof of linearity for the expected value operator of LR fuzzy numbers. Finally, a fuzzy programming model with expected objective and chance constraints is formulated. We show that this model can be transferred to a crisp equivalent linear programming model by the operational law, and then solved by LINGO or MATLAB. A numerical example about purchasing planning problem is also given at last.

The rest of this paper is organized as follows. In Section 2, the concepts of LR fuzzy number are introduced. In Section 3, we discuss the credibility distribution of LR fuzzy numbers based on the credibility measure. In Section 4, a novel operational law on LR fuzzy numbers is put forward. In Section 5, an equivalent form of the expected value operator for LR fuzzy numbers is presented. In Section 6, the operational law is used to construct a solution framework of fuzzy programming, and then we illustrate it by an example of purchasing planning problem.

2. LR Fuzzy Numbers

Dubios and Prade [8] initialized the well-known LR type of representation for fuzzy numbers, where L and R respectively denote the left and right shape functions which can be defined as follows.

**Definition 1.** (Dubios and Prade [10]) A shape function L (or R) is a function from $\mathbb{R}^+ \rightarrow [0,1]$ such that

1. $L(0) = 1$;
2. $L(x) < 1$, $\forall x > 0$;
3. $L(x) > 0$, $\forall x < 1$;
4. $L(1) = 0$ [or $L(x) > 0$, $\forall x$ and $L(+\infty) = 0$];
5. $L(x)$ is decreasing on the open interval $\{x | 0 < L(x) < 1\}$.

**Example 2.1:** Different functions can be chosen for $L(x)$ (or $R(x)$). For instance, as mentioned by Dubios and Prade [10], $L(x) = \max\{0, 1 - x^p\}$ with $p > 0$; $L(x) = e^{-x}$; and $L(x) = 1/(1 + x^2)$.

**Remark 1:** Generally, most shape functions L (or R) are strictly decreasing in practical applications, such as the shape functions of triangular fuzzy number and the Gaussian fuzzy number, i.e.,

$L(x) = R(x) = \max\{0, 1 - x\}$ and $L(x) = R(x) = e^{-x^2}$, respectively.
Definition 2. (Dubios and Prade [10]) A fuzzy number $\xi$ is of LR-type if there exist shape functions $L$ (for left) and $R$ (for right), and scalers $\alpha > 0$, $\beta > 0$ with membership function

$$
\mu_\xi(x) = \begin{cases} 
L\left(\frac{m-x}{\alpha}\right), & \text{if } x \leq m \\
R\left(\frac{x-m}{\beta}\right), & \text{if } x \geq m 
\end{cases}
$$

where the real number $m$ is called the mean value or peak of $\xi$, and $\alpha$ and $\beta$ are called the left and right spreads, respectively. Symbolically, $\xi$ is denoted by $(m, \alpha, \beta)_{LR}$.

Remark 2: If the mean value $m$ is not a real number but an interval $[m, \bar{m}]$, then $\xi$ is treated as a fuzzy interval (see [42]) or a generalized LR fuzzy number (see, e.g., [33]). In this paper, we only consider the situation that $\underline{m} = \bar{m}$.

Example 2.2: Let $L(x) = \max\{0, 1-x\}$, $R(x) = e^{-x}$, $\alpha = 2$, $\beta = 3$, and $m = 4$. Then $(4, 2, 3)_{LR}$ denotes an LR fuzzy number with membership function (see Fig. 1)

$$
\mu(x) = \begin{cases} 
0, & \text{if } x \leq 2 \\
x/2 - 1, & \text{if } 2 < x \leq 4 \\
e^{(4-x)/3}, & \text{if } x \geq 4.
\end{cases}
$$

Figure 1: The membership function of $(4, 2, 3)_{LR}$

Figure 2: The membership function of a triangular fuzzy number

Example 2.3: If $L(x)$ and $R(x)$ are both linear functions on the domains $\{x|0 < L(x) < 1\}$ and $\{x|0 < R(x) < 1\}$, the corresponding LR fuzzy number is a triangular fuzzy number. A triangular fuzzy number $\xi$ determined by the triplet $(a, b, c)$ of crisp numbers with $a < b < c$ has a membership function (see Fig. 2)

$$
\mu(x) = \begin{cases} 
\frac{x-a}{b-a}, & \text{if } a \leq x < b \\
\frac{c-x}{c-b}, & \text{if } b \leq x \leq c \\
0, & \text{otherwise},
\end{cases}
$$

Figure 2: The membership function of a triangular fuzzy number

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which can be denoted as \((b, b - a, c - b)_{LR}\), where the shape functions \(L\) and \(R\) are

\[
L(x) = R(x) = \max\{0, 1 - x\}. \tag{4}
\]

**Example 2.4:** A fuzzy number \(\xi\) is called a Gaussian fuzzy number if it has a membership function (see Fig. 3)

\[
\mu(x) = e^{-\left(\frac{x - a}{b}\right)^2}, \quad x \in \mathbb{R}, b > 0,
\]

which can be denoted as \((a, b, b)_{LR}\), where the shape functions \(L\) and \(R\) are

\[
L(x) = R(x) = e^{-x^2}. \tag{6}
\]

![Figure 3: The membership function of a Gaussian fuzzy number](image1)

**Example 2.5:** A fuzzy number \(\xi\) is called a Cauchy fuzzy number if it has a membership function (see Fig. 4)

\[
\mu(x) = \frac{1}{1 + \left(\frac{x - p}{q}\right)^2}, \quad x \in \mathbb{R}, q > 0,
\]

which can be denoted as \((p, q, q)_{LR}\), where the shape functions \(L\) and \(R\) are

\[
L(x) = R(x) = 1/(1 + x^2). \tag{8}
\]

**3. Credibility Distribution of LR Fuzzy Numbers**

In this section, we discuss the credibility distribution of LR fuzzy numbers based on the credibility measure. Firstly, some notions on the credibility measure are reviewed briefly.

Suppose that \(\xi\) is a fuzzy number with the membership function \(\mu\), and \(r\) is a real number. The possibility [40] and the necessity [41] of a fuzzy event \(\xi \leq r\) are expressed as follows,

\[
\text{Pos}\{\xi \leq r\} = \sup_{x \leq r} \mu(x), \quad \text{Nec}\{\xi \leq r\} = 1 - \sup_{x > r} \mu(x) \tag{9}
\]

However, it is not suitable that the possibility measure or necessity measure is singly utilized to measure a fuzzy event in decision-making system because of the absence of the self-duality. For example, we assume the profits of a company as a triangular fuzzy variable \(\xi = (7, 8, 9)\).
Then the possibility of the fuzzy event that the profits are not less than 7.7 is \( \text{Pos}\{\xi \geq 7.7\} = 1 \). However, this "high" confidence level is not justifiable, since the possibility value of the event that the profits are less than 7.7 is \( \text{Pos}\{\xi < 7.7\} = 0.7 \). On the other hand, if we use the necessity to measure this event, the \( \text{Nec}\{\xi \geq 7.7\} = 0.3 \) and \( \text{Nec}\{\xi < 7.7\} = 0 \). The above results will confuse the decision-maker. To overcome this deficiency, the credibility measure was proposed in [27] as follows,

\[
\text{Cr}\{\xi \leq r\} = \frac{1}{2} (\text{Pos}\{\xi \leq r\} + \text{Nec}\{\xi \leq r\})
\]

which has been proved that the credibility measure is increasing and self-dual [27], and

\[
\text{Cr}\{\xi \leq r\} = \frac{1}{2} \left( \sup_{x \leq r} \mu(x) + 1 - \sup_{x > r} \mu(x) \right).
\]

In the above example, the credibility of the event that the profits are not less than 7.7 is 0.65, i.e., \( \text{Cr}\{\xi \geq 7.7\} = 0.65 \), while the event that the profits are less than 7.7 occurs with credibility 0.35, that is \( \text{Cr}\{\xi < 7.7\} = 0.35 \). Hence, the credibility will be used to measure fuzzy events in this paper.

Let \( \Theta \) be a nonempty set representing the sample space, \( \mathcal{P} \) the power set of \( \Theta \), and \( \text{Cr} \) a credibility measure on \( \mathcal{P} \). Then \((\Theta, \mathcal{P}(\Theta), \text{Cr})\) is called a credibility space. A fuzzy variable is defined in [24] as a function from a credibility space \((\Theta, \mathcal{P}(\Theta), \text{Cr})\) to the set of real numbers. Furthermore, Liu [22] defined the credibility distribution for fuzzy variables as follows.

**Definition 3.** *(Liu [22], Credibility Distribution)* The credibility distribution \( \Phi : \mathbb{R} \to [0, 1] \) of a fuzzy variable \( \xi \) is defined by

\[
\Phi(x) = \text{Cr}\{\theta \in \Theta \mid \xi(\theta) \leq x\}.
\]

Here, \( \Phi(x) \) is the credibility that the fuzzy variable \( \xi \) takes a value less than or equal to \( x \). Liu [23] proved that the credibility distribution \( \Phi \) is a nondecreasing function on \( \mathbb{R} \) with \( \Phi(-\infty) = 0 \) and \( \Phi(+\infty) = 1 \).

**Example 3.1:** On the basis of (2), (11) and (12), the credibility distribution of the LR fuzzy number in Example 2.2 can be figured out as

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 2 \\
(x - 2)/4, & \text{if } 2 < x \leq 4 \\
1 - \frac{1}{2} e^{(4-x)/3}, & \text{if } x > 4,
\end{cases}
\]

which has been depicted in Fig. 5.

**Example 3.2:** A triangular fuzzy number determined by the triplet \((a, b, c)\) of crisp numbers with
The credibility distribution plays a key role when studying fuzzy variables just like the probability distribution for random variables. Based on this notion, we advise the regular credibility distribution and the inverse credibility distribution as follows.

**Definition 4.** *(Regular Credibility Distribution)* A credibility distribution \( \Phi \) is said to be regular if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0,1) \).

**Definition 5.** *(Inverse Credibility Distribution)* Let \( \xi \) be a fuzzy number with regular credibility distribution \( \Phi \). Then the inverse function \( \Phi^{-1} \) is called the inverse credibility distribution of \( \xi \).

Note that the inverse credibility distribution \( \Phi^{-1} \) is well defined on the open interval \((0,1)\). If required, we may extend the domain via

\[
\Phi^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha).
\]

**Example 3.3:** The inverse credibility distribution of triangular fuzzy number \( \xi = T(a,b,c) \) (see Fig. 7) is

\[
\Phi^{-1}(\alpha) = \begin{cases} 
(2b-2a)\alpha + a, & \text{if } \alpha < 0.5 \\
(2c-2b)\alpha + 2b - c, & \text{if } \alpha \geq 0.5.
\end{cases}
\]
Example 3.4: A Gaussian fuzzy number with the membership function in (5) has a regular credibility distribution (see Fig. 8)

\[
\Phi(x) = \begin{cases} 
\frac{1}{2} e^{-\left(\frac{x-a}{\sigma}\right)^2}, & \text{if } x \leq a \\
1 - \frac{1}{2} e^{-\left(\frac{x-a}{\sigma}\right)^2}, & \text{if } x > a 
\end{cases}
\]  

(17)

denoted by \( N(a, b) \), where \( a \) and \( b > 0 \) are real numbers. The inverse credibility distribution of a Gaussian fuzzy number (see Fig. 9) is

\[
\Phi^{-1}(\alpha) = \begin{cases} 
\alpha - b\sqrt{-\ln(2\alpha)}, & \text{if } \alpha \leq 0.5 \\
\alpha + b\sqrt{-\ln(2 - 2\alpha)}, & \text{if } \alpha > 0.5.
\end{cases}
\]  

(18)

Example 3.5: A Cauchy fuzzy number with the membership function in (7) has a regular credibility distribution (see Fig. 10)

\[
\Phi(x) = \begin{cases} 
\frac{1}{2 + 2 \left(\frac{x-p}{q}\right)^2}, & \text{if } x \leq q \\
1 - \frac{1}{2 + 2 \left(\frac{x-p}{q}\right)^2}, & \text{if } x > q
\end{cases}
\]  

(19)
denoted by $C(p, q)$, where $p$ and $q > 0$ are real numbers. The inverse credibility distribution of a Cauchy fuzzy number (see Fig. 11) is

$$
\Phi^{-1}(\alpha) = \begin{cases} 
p - q \sqrt{(1 - 2\alpha)/2\alpha}, & \text{if } \alpha \leq 0.5 
p + q \sqrt{(2\alpha - 1)/(2 - 2\alpha)}, & \text{if } \alpha > 0.5.
\end{cases}
$$

(20)

From Examples 3.3 to 3.5, it can be seen that the credibility distributions of the LR fuzzy numbers involving triangular fuzzy numbers, Gaussian fuzzy numbers, and Cauchy fuzzy numbers are all regular. Moreover, we verify this conclusion together with some equivalent conditions for all the LR fuzzy numbers via the following theorem.

**Theorem 1. (Equivalent Conditions of LR Fuzzy Number)** A fuzzy number is an LR fuzzy number if and only if any one of the following assertions holds,

(i) The credibility distribution of $\xi$ is regular;

(ii) The credibility distribution $\Phi$ is strictly increasing on $\{x|0 < \Phi(x) < 1\}$;

(iii) The inverse function of $\Phi$ exists and is strictly increasing on $(0, 1)$.

**Proof:** It is easy to verify that part (ii) and part (iii) are equivalent to part (i) according to the definition of regular credibility distribution. Hence we only prove part (i).

On the one hand, if a fuzzy number $\xi$ is an LR fuzzy number, then its credibility distribution $\Phi$ is strictly increasing on $\{x|0 < \Phi(x) < 1\}$, which can be verified in three cases as follows.

Assume that the membership function of $\xi$ is

$$
\mu(x) = \begin{cases} 
L \left( \frac{m - x}{a} \right), & \text{if } x \leq m, \ a > 0 
R \left( \frac{x - m}{b} \right), & \text{if } x \geq m, \ b > 0
\end{cases}
$$

(21)

where $L(x)$ and $R(x)$ are both decreasing functions on $\mathbb{R}^+$ and strictly decreasing on the open intervals $\{x|0 < L(x) < 1\}$ and $\{x|0 < R(x) < 1\}$. Denote $L^{-1}(0) = \lim_{\alpha \downarrow 0} L^{-1}(\alpha)$ and $R^{-1}(0) = \lim_{\alpha \downarrow 0} R^{-1}(\alpha)$. 

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Firstly, if \( x_1 < x_2 \leq m \), according to (10) and (12), we have

\[
\Phi(x_1) = \frac{1}{2}(\text{Pos}(\xi \leq x_1) + \text{Nec}(\xi \leq x_1)) = \frac{1}{2} \mu(x_1) = \frac{1}{2} L \left( \frac{m-x_1}{a} \right)
\]

(22)

and

\[
\Phi(x_2) = \frac{1}{2}(\text{Pos}(\xi \leq x_2) + \text{Nec}(\xi \leq x_2)) = \frac{1}{2} \mu(x_2) = \frac{1}{2} L \left( \frac{m-x_2}{a} \right).\]

(23)

Since \( L(x) \) is monotone decreasing on \( \mathbb{R}^+ \) and strictly decreasing on \( \{x | 0 < L(x) < 1\} \), we have

\[
\Phi(x_1) \leq \Phi(x_2), \quad \forall x_1 < x_2 \leq m
\]

(24)

and

\[
\Phi(x_1) < \Phi(x_2), \quad \forall m - aL^{-1}(0) < x_1 < x_2 \leq m.
\]

(25)

Secondly, if \( m \leq x_1 < x_2 \), similarly, we have

\[
\Phi(x_1) \leq \Phi(x_2), \quad \forall m \leq x_1 < x_2
\]

(26)

and

\[
\Phi(x_1) < \Phi(x_2), \quad \forall m \leq x_1 < x_2 < m + bR^{-1}(0).
\]

(27)

Finally, if \( x_1 < m < x_2 \), we have

\[
\Phi(x_1) = \frac{1}{2} L \left( \frac{m-x_1}{a} \right) < \frac{1}{2} L(0) = \frac{1}{2}
\]

(28)

and

\[
\Phi(x_2) = 1 - \frac{1}{2} R \left( \frac{x_2-m}{b} \right) > 1 - \frac{1}{2} R(0) = \frac{1}{2}.
\]

(29)

Then \( \Phi(x_1) < \Phi(x_2) \).

In summary, we obtain \( \Phi(x_1) < \Phi(x_2) \) for \( m - aL^{-1}(0) < x_1 < x_2 < m + bR^{-1}(0) \), which means \( \Phi(x) \) is strictly increasing on \( (m - aL^{-1}(0), m + bR^{-1}(0)) \), i.e., \( \{x | 0 < \Phi(x) < 1\} \). Therefore, \( \xi \) has a regular credibility distribution.

On the other hand, if we assume that a fuzzy number \( \xi \) has a regular credibility distribution, then its credibility distribution \( \Phi(x) \) is strictly increasing on \( \{x | 0 < \Phi(x) < 1\} \).

Firstly, we can verify that there is at most one point with membership 1 in \( (-\infty, +\infty) \). Otherwise, if there exist two points \( x_1 < x_2 \) with \( \mu(x_1) = \mu(x_2) = 1 \), then for any \( x_0 \in (x_1, x_2) \), we have

\[
\Phi(x_0) = \frac{1}{2}(\text{Pos}(\xi \leq x_0) + \text{Nec}(\xi \leq x_0))
\]

\[
= \frac{1}{2} \left( \sup_{x \leq x_0} \mu(x) + 1 - \sup_{x > x_0} \mu(x) \right)
\]

(30)

\[
= \frac{1}{2} (1 + 1 - 1) = \frac{1}{2},
\]

which is contrary to the assumption that \( \Phi(x) \) is strictly increasing on \( \{x | 0 < \Phi(x) < 1\} \). Therefore, there is only one point with membership 1, denoted by \( m \). That is, \( \mu(m) = 1 \).
For $x_0 < m$, according to (10) and (12), we have

$$\Phi(x_0) = \frac{1}{2} \left( \sup_{x \leq x_0} \mu(x) + 1 - \sup_{x > x_0} \mu(x) \right)$$

$$= \frac{1}{2} \left( \sup_{x \leq x_0} \mu(x) + 1 - \mu(m) \right)$$

$$= \frac{1}{2} \sup_{x \leq x_0} \mu(x).$$

(31)

Since $\Phi(x)$ is strictly increasing on $\{x|0 < \Phi(x) < 1\}$, thus $\mu(x)$ is strictly increasing on $\{x|g(0) < x < m\}$, where $g(0) = \lim_{\alpha \uparrow 0} \Phi^{-1}(\alpha)$.

Similarly, for $x_0 > m$, we have

$$\Phi(x_0) = \frac{1}{2} \left( \sup_{x \leq x_0} \mu(x) + 1 - \sup_{x > x_0} \mu(x) \right)$$

$$= \frac{1}{2} \left( \mu(m) + 1 - \sup_{x > x_0} \mu(x) \right)$$

$$= 1 - \frac{1}{2} \sup_{x > x_0} \mu(x).$$

(32)

Since $\Phi(x)$ is strictly increasing on $\{x|0 < \Phi(x) < 1\}$, we have $\mu(x)$ is strictly decreasing on $\{x|m < x < g(1)\}$, where $g(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha)$.

In summary, for the membership function $\mu(x)$ of $\xi$, we have

$$\mu(x) = \begin{cases} 
0, & \text{if } x \leq g(0) \\
\text{is strictly increasing,} & \text{if } g(0) < x < m \\
1, & \text{if } x = m \\
\text{is strictly decreasing,} & \text{if } m < x < g(1) \\
0, & \text{if } x \geq g(1).
\end{cases}$$

(33)

It is clear that $\mu(x)$ can be represented in the LR-type with shape functions $L(x)$ and $R(x)$, where $L(x)$ and $R(x)$ are strictly decreasing on $\{x|0 < L(x) < 1\}$ and $\{x|0 < R(x) < 1\}$. Therefore, $\xi$ is an LR fuzzy number.

4. Operational Law

This section gives the operational law for calculating the credibility distribution of strictly monotone function of independent LR fuzzy numbers. Here we only consider LR fuzzy numbers with strictly monotone shape functions for simplicity in this paper, and the notions of independence of fuzzy numbers and strictly monotone function will be recalled firstly.
The independence of fuzzy numbers has been studied by many researchers from different angles, such as Zadeh [40], Nahmias [30], Yager [37], Liu [23], and Liu and Gao [25], who presented lots of equivalent conditions of independence. Here we introduce the definition and its equivalent theorem given by Liu and Gao [25].

**Definition 6.** (Liu and Gao [25]) The fuzzy variables \( \xi_1, \xi_2, \cdots, \xi_n \) are said to be independent if

\[
\operatorname{Cr}\{\xi_i \in B_i, i = 1, 2, \cdots, n\} = \min_{1 \leq i \leq n} \operatorname{Cr}\{\xi_i \in B_i\}
\]

(34)

for any Borel sets \( B_1, B_2, \cdots, B_n \) of real numbers.

**Theorem 2.** (Liu and Gao [25]) The fuzzy variables \( \xi_1, \xi_2, \cdots, \xi_n \) are independent if and only if

\[
\operatorname{Cr}\left\{ \bigcup_{i=1}^{n} \{\xi_i \in B_i\} \right\} = \max_{1 \leq i \leq n} \operatorname{Cr}\{\xi_i \in B_i\}
\]

(35)

for any Borel sets \( B_1, B_2, \cdots, B_n \) of real numbers.

**Definition 7.** A real-valued function \( f(x_1, x_2, \cdots, x_n) \) is said to be strictly monotone if it is strictly increasing with respect to \( x_1, x_2, \cdots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \cdots, x_n \), that is,

\[
f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \leq f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n)
\]

(36)

whenever \( x_i \leq y_i \) for \( i = 1, 2, \cdots, m \) and \( x_i \geq y_i \) for \( i = m+1, \cdots, n \), and

\[
f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) < f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n)
\]

(37)

whenever \( x_i < y_i \) for \( i = 1, 2, \cdots, m \) and \( x_i > y_i \) for \( i = m+1, \cdots, n \).

**Example 4.1:** The following are strictly monotone functions,

\[
f(x_1, x_2) = x_1 - x_2, \quad f(x_1, x_2) = x_1/x_2, \quad x_1, x_2 > 0.
\]

**Example 4.2:** If \( f(x_1, x_2, \cdots, x_n) \leq f(y_1, y_2, \cdots, y_n) \) whenever \( x_i \leq y_i \) for all \( i \), and \( f(x_1, x_2, \cdots, x_n) < f(y_1, y_2, \cdots, y_n) \) whenever \( x_i < y_i \) for all \( i \), the function is said to be strictly increasing. The following are strictly increasing functions,

\[
f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n,
\]

\[
f(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n, \quad x_1, x_2, \cdots, x_n \geq 0.
\]

**Example 4.3:** If \( f(x_1, x_2, \cdots, x_n) \geq f(y_1, y_2, \cdots, y_n) \) whenever \( x_i \leq y_i \) for all \( i \), and \( f(x_1, x_2, \cdots, x_n) > f(y_1, y_2, \cdots, y_n) \) whenever \( x_i < y_i \) for all \( i \), the function is said to be strictly decreasing.
functions. The following are strictly decreasing functions,
\[ f(x) = -x_1 - x_2 \cdots - x_n, \]
\[ f(x) = \exp(-x), \]
\[ f(x) = \frac{1}{x}, \quad x > 0. \]

Based on the notions of independence and strictly monotone functions, we present the operational law of LR fuzzy numbers as follows.

**Theorem 3. (Operational Law)** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent LR fuzzy numbers with credibility distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If the function \( f(x_1, x_2, \cdots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \cdots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \cdots, x_n \), then
\[
\xi = f(\xi_1, \cdots, \xi_m, \xi_{m+1}, \cdots, \xi_n)
\]
is an LR fuzzy number with inverse credibility distribution
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).
\]

**Proof:** For simplicity, we only prove the case of \( m = 1 \) and \( n = 2 \). That is, suppose that
\[
\xi = f(\xi_1, \xi_2),
\]
and \( f \) is strictly increasing with respect to \( \xi_1 \) and strictly decreasing with respect to \( \xi_2 \). Besides, denote
\[
G(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))
\]
where \( \Phi_1^{-1} \) and \( \Phi_2^{-1} \) are the inverse credibility distributions of \( \xi_1 \) and \( \xi_2 \), respectively. Based on (40) and (41), we always have
\[
\{\xi \leq G(\alpha)\} \equiv \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\}. \tag{42}
\]

On the one hand, since \( f \) is a strictly monotone function, from Definition 7, we can deduce that
\[
\xi_1 \leq \Phi_1^{-1}(\alpha) \text{ and } \xi_2 \geq \Phi_2^{-1}(1-\alpha) \implies f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha)) \tag{43}
\]
which implies that
\[
\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\} \subseteq \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\}. \tag{44}
\]

Then it follows from (42) and (44) that
\[
\{\xi \leq G(\alpha)\} \supseteq \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}. \tag{45}
\]
Since the credibility measure \( \text{Cr} \) is an increasing set function (Liu [22]), we have
\[
\text{Cr}\{\xi \leq G(\alpha)\} \geq \text{Cr}\{\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}\}. \tag{46}
\]
By using the definition of independence (see Definition 6) and the independence of \( \xi_1 \) and \( \xi_2 \), we get
\[
\text{Cr} \left\{ \{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \cap \{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \} \right\} = \text{Cr}\{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \land \text{Cr}\{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \} = \alpha \land \alpha = \alpha.
\]
According to (46) and (47), we obtain
\[
\text{Cr}\{ \xi \leq G(\alpha) \} \geq \alpha.
\]

On the other hand, since \( f \) is strictly increasing with respect to \( \xi_1 \) and strictly decreasing with respect to \( \xi_2 \), we can deduce that
\[
f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha)) \quad \Rightarrow \quad \xi_1 \leq \Phi_1^{-1}(\alpha) \text{ or } \xi_2 \geq \Phi_2^{-1}(1 - \alpha)
\]
which implies that
\[
\{ f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha)) \} \subseteq \{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \cup \{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \}.
\]
Then it follows from (42) and (50) that
\[
\{ \xi \leq G(\alpha) \} \subseteq \{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \cup \{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \}.
\]
Since the credibility measure \( \text{Cr} \) is an increasing set function, we have
\[
\text{Cr}\{ \xi \leq G(\alpha) \} \leq \text{Cr}\{ \{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \cup \{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \} \}.
\]
By using Theorem 2 and the independence of \( \xi_1 \) and \( \xi_2 \), we get
\[
\text{Cr}\{ \{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \cup \{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \} \} = \text{Cr}\{ \xi_1 \leq \Phi_1^{-1}(\alpha) \} \lor \text{Cr}\{ \xi_2 \geq \Phi_2^{-1}(1 - \alpha) \} = \alpha \lor \alpha = \alpha.
\]
According to (52) and (53), we obtain
\[
\text{Cr}\{ \xi \leq G(\alpha) \} \leq \alpha.
\]
Finally, it follows from (48) and (54) that \( \text{Cr}\{ \xi \leq G(\alpha) \} = \alpha \). From the definition of inverse credibility distribution in Definition 5, we know that \( G(\alpha) \) is just the inverse credibility distribution of \( \xi \). That is, the inverse credibility distribution of \( \xi \) is \( \Psi^{-1}(\alpha) = G(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha)) \).

Furthermore, let us verify that \( \xi \) is an LR fuzzy number. Since \( \xi_1 \) and \( \xi_2 \) are LR fuzzy numbers, it follows from Theorem 1 that \( \Phi_1^{-1} \) and \( \Phi_2^{-1} \) are strictly increasing functions on \((0, 1)\). Considering that \( f(x_1, x_2) \) is strictly increasing with respect to \( x_1 \) and strictly decreasing with respect to \( x_2 \),
it is easy to deduce that $\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha))$ is a strictly increasing function with respect to $\alpha$. In other words, the inverse credibility distribution of $\xi$ exists and strictly increasing on $(0, 1)$. According to the third equivalent condition of LR fuzzy number (see Theorem 1), we obtain that $\xi$ is an LR fuzzy number.

Remark 3: If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an LR fuzzy number with inverse credibility distribution $\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))$.

Remark 4: If the function $f(x_1, x_2, \cdots, x_n)$ is strictly decreasing with respect to $x_1, x_2, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an LR fuzzy number with inverse credibility distribution $\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha))$.

Example 4.4: Let $\xi_1$ and $\xi_2$ be independent LR fuzzy numbers with credibility distributions $\Phi_1$ and $\Phi_2$, respectively. Since the function $f(x_1, x_2) = ax_1 - bx_2$ is strictly increasing with respect to $x_1$ and strictly decreasing with respect to $x_2$ for any constants $a > 0$ and $b > 0$, it follows from Theorem 3 that $a\xi_1 - b\xi_2$ is an LR fuzzy number, and its inverse credibility distribution is

$$\Psi^{-1}(\alpha) = a\Phi_1^{-1}(\alpha) - b\Phi_2^{-1}(1 - \alpha).$$

Providing that $\xi_1 = \mathcal{N}(3, 2)$ and $\xi_2 = \mathcal{N}(5, 3)$ are two Gaussian fuzzy numbers, and the parameters $a = 2$ and $b = 1$, the inverse credibility distribution of $2\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = 2\Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1 - \alpha) = \begin{cases} 2 \left(3 - 2\sqrt{-\ln(2\alpha)}\right) - \left(5 + 3\sqrt{-\ln(2 - 2(1 - \alpha))}\right), & \text{if } \alpha \leq 0.5 \\ 2 \left(3 + 2\sqrt{-\ln(2 - 2\alpha)}\right) - \left(5 - 3\sqrt{-\ln(2(1 - \alpha))}\right), & \text{if } \alpha > 0.5 \end{cases}$$

$$= \begin{cases} 1 - 7\sqrt{-\ln(2\alpha)}, & \text{if } \alpha \leq 0.5 \\ 1 + 7\sqrt{-\ln(2 - 2\alpha)}, & \text{if } \alpha > 0.5 \end{cases}$$

according to (18) for $\alpha \in [0, 1]$. It is obvious that $2\xi_1 - \xi_2$ is also a Gaussian fuzzy number, i.e., $2\xi_1 - \xi_2 = \mathcal{N}(1, 7)$.

Example 4.5: Let $\xi_1$ and $\xi_2$ be independent and positive LR fuzzy numbers with credibility distributions $\Phi_1$ and $\Phi_2$, respectively. Since $f(x_1, x_2) = x_1/x_2$ is a strictly monotone function for $x_1, x_2 > 0$, the quotient $\xi_1/\xi_2$ is an LR fuzzy number with the inverse credibility distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha)/\Phi_2^{-1}(1 - \alpha).$$

Providing that $\xi_1 = \mathcal{T}(1, 2, 3)$ and $\xi_2 = \mathcal{T}(3, 4, 5)$ are two triangular fuzzy numbers, it follows from (16) that $\Phi_1^{-1}(\alpha) = 2\alpha + 1$ and $\Phi_2^{-1}(\alpha) = 2\alpha + 3$. Thus the inverse credibility distribution of $\xi_1/\xi_2$
\[
\Psi^{-1}(\alpha) = (2\alpha + 1)/(2(1 - \alpha) + 3) = (2\alpha + 1)/(5 - 2\alpha).
\] (55)

5. Expected Value

In this section, we first give an equivalent form of the expected value of LR fuzzy numbers by the inverse credibility distribution, and then present a theorem for calculating the expected value of strictly monotone functions based on the equivalent form and the proposed operational law.

Expected value is the average value of a fuzzy variable in the sense of fuzzy measure. It has been defined in several ways. For instance, Dubois and Prade [9], Heilpern [19], Yager [36] gave the different definitions, respectively. In 2002, Liu and Liu [27] presented a general definition of expected value for fuzzy variables via the credibility distribution as follows.

**Definition 8.** (Liu and Liu [27]) Let \( \xi \) be a fuzzy variable. Then the expected value of \( \xi \) is defined by

\[
E[\xi] = \int_0^{+\infty} Cr\{\xi \geq x\} \, dx - \int_{-\infty}^0 Cr\{\xi \leq x\} \, dx
\]

provided that at least one of the two integrals is finite. (56)

In the following, we provide an equivalent form of the expected value of LR fuzzy numbers by means of the inverse credibility distribution.

**Theorem 4.** Let \( \xi \) be an LR fuzzy number. If its expected value exists, then

\[
E[\xi] = \int_0^1 \Phi^{-1}(\alpha) \, d\alpha
\]

where \( \Phi^{-1} \) is the inverse credibility distribution of \( \xi \). (57)

**Proof:** It follows from the definitions of expected value operator and credibility distribution that

\[
E[\xi] = \int_0^{+\infty} Cr\{\xi \geq x\} \, dx - \int_{-\infty}^0 Cr\{\xi \leq x\} \, dx
\]

\[
= \int_0^{+\infty} (1 - \Phi(x)) \, dx - \int_{-\infty}^0 \Phi(x) \, dx
\]

\[
= \int_{\Phi(0)}^1 \Phi^{-1}(\alpha) \, d\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha) \, d\alpha
\]

\[
= \int_0^1 \Phi^{-1}(\alpha) \, d\alpha.
\]

Theorem 4 implies that the value of the expected value \( E[\xi] \) is just the area surrounded by two axes, \( y = 1 \), and the curve of the inverse credibility distribution \( \Phi^{-1} \). Fig. 12 shows the geometric interpretation of the expected value of an LR fuzzy number.
Example 5.1: The expected value of a triangular fuzzy number $\xi = (a, b, c)$ can be calculated according to (16) and (57) as

$$E[\xi] = \int_0^{0.5} ((1 - 2\alpha)a + 2\alpha b) d\alpha + \int_{0.5}^1 ((2 - 2\alpha)b + (2\alpha - 1)c) d\alpha = \frac{a + 2b + c}{4}. \tag{59}$$

Example 5.2: The expected value of a Gaussian fuzzy number $\xi = \mathcal{N}(a, b)$ can be calculated according to (18) and (57) as

$$E[\xi] = \int_0^{0.5} (a - b\sqrt{-\ln(2\alpha)}) d\alpha + \int_{0.5}^1 (a + b\sqrt{-\ln(2 - 2\alpha)}) d\alpha$$

$$= 0.5a - b \int_0^{0.5} \sqrt{-\ln(2\alpha)} d\alpha + 0.5a + b \int_{0.5}^1 \sqrt{-\ln(2 - 2\alpha)} d\alpha$$

$$= a - 0.5b \int_0^1 \sqrt{-\ln t} dt - 0.5b \int_1^0 \sqrt{-\ln t} dt$$

$$= a. \tag{60}$$

According to the operational law and the equivalent form of the expected value, a theorem for calculating the expected value of strictly monotone functions is proved as follows.

**Theorem 5.** (Expected Value of Strictly Monotone Functions) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent L-R fuzzy numbers with credibility distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_m, x_{m+1}, x_{m+2}, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the expected value of the LR fuzzy number $\xi = f(\xi_1, \cdots, \xi_m, \xi_{m+1}, \cdots, \xi_n)$ is

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_{m-1}^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha)) d\alpha. \tag{61}$$

**Proof:** It follows immediately from Theorems 3 and 4. \qed

**Example 5.3:** Let $\xi$ be a nonnegative LR fuzzy number with credibility distribution $\Phi$. Since $f(x) = x^2$ is a strictly increasing function on $[0, +\infty)$, it follows from Theorem 3 that the square
$\xi^2$ is an LR fuzzy number with the inverse credibility distribution $(\Phi^{-1}(\alpha))^2$. Then its expected value is
\[
E[\xi^2] = \int_0^1 (\Phi^{-1}(\alpha))^2 d\alpha.
\]
Providing that $\xi$ is a triangular fuzzy number $T(1, 2, 3)$, the inverse credibility distribution of $\xi$ is $\Phi^{-1}(\alpha) = 1 + 2\alpha$ by means of (16). Hence the expected value of $\xi^2$ is
\[
E[\xi^2] = \int_0^1 (1 + 2\alpha)^2 d\alpha = 13/3.
\]

Example 5.4: Let $\xi_1$ and $\xi_2$ be independent and positive LR fuzzy numbers with credibility distributions $\Phi_1$ and $\Phi_2$, respectively. Since $f(x_1, x_2) = x_1/x_2$ is a strictly monotone function for $x_1, x_2 > 0$, the quotient $\xi_1/\xi_2$ is an LR fuzzy number with the inverse credibility distribution $\Upsilon(\alpha) = \Phi_1^{-1}(\alpha)/\Phi_2^{-1}(1 - \alpha)$. Then its expected value is
\[
E[\xi_1/\xi_2] = \int_0^1 \Phi_1^{-1}(\alpha)/\Phi_2^{-1}(1 - \alpha) d\alpha.
\]
Providing that $\xi_1 = T(1, 2, 3)$ and $\xi_2 = T(3, 4, 5)$ are two triangular fuzzy numbers, it is easy to have that the expected value of $\xi_1/\xi_2$ is
\[
E[\xi_1/\xi_2] = \int_0^1 \frac{2\alpha + 1}{5 - 2\alpha} d\alpha = -3\ln 0.6 - 1 \approx 0.5325
\]
according to (55) in Example 4.5.

Liu and Liu [28] has shown that the expected value operator defined in (57) has the linearity for general fuzzy numbers. However, the proof is somewhat complicated. Here, we show that the same conclusion for LR fuzzy numbers can be obtained through a relatively simple derivation from Theorems 4 and 5.

Theorem 6. (Linearity of Expected Value Operator) Let $\xi$ and $\eta$ be independent LR fuzzy numbers with finite expected values. Then for any real numbers $a$ and $b$, we have
\[
E[a\xi + b\eta] = aE[\xi] + bE[\eta].
\] (62)

Proof: Let $\xi$ and $\eta$ be independent LR fuzzy numbers with credibility distributions $\Phi$ and $\Psi$, respectively. If $a \geq 0$ and $b \geq 0$, it follows from Theorems 4 and 5 that
\[
E[a\xi + b\eta] = \int_0^1 a\Phi^{-1}(\alpha) + b\Psi^{-1}(\alpha) d\alpha
\]
\[
= a \int_0^1 \Phi^{-1}(\alpha) d\alpha + b \int_0^1 \Psi^{-1}(\alpha) d\alpha
\] (63)
\[
= aE[\xi] + bE[\eta].
\]
Similarly, if \( a \leq 0 \) and \( b \geq 0 \), we have
\[
E[\xi + \eta] = \int_0^1 a\Phi^{-1}(1 - \alpha) + b\Psi^{-1}(\alpha)d\alpha
\]
\[
= a\int_0^1 \Phi^{-1}(\alpha)d\alpha + b\int_0^1 \Psi^{-1}(\alpha)d\alpha
\]
\[
= aE[\xi] + bE[\eta].
\] (64)

For simplicity, we only proved the above two cases. It is easy to deduce that the other cases can be also verified similarly. That is, for any real numbers \( a \) and \( b \), the equality \( E[a\xi + b\eta] = aE[\xi] + bE[\eta] \) holds.

6. Fuzzy Programming

To handle the uncertain programming, Charnes and Cooper [1] initialized the chance-constrained programming which offers a powerful approach of modeling stochastic decision system. Following the idea of stochastic chance-constrained programming, a framework of fuzzy chance-constrained programming was presented by Liu and Iwamura [26]. As an extension of their work, in this section, we present a fuzzy programming model integrating the expected objective with some chance constraints, and then show that the model can be converted to a crisp equivalent mathematical model by using the proposed operational law.

6.1. Fuzzy programming model

Assume that \( \mathbf{x} \) is a decision vector, \( \xi = (\xi_1, \xi_2, \cdots, \xi_n) \) is an \( n \)-dimensional fuzzy vector, \( f(\mathbf{x}, \xi) \) is the objective function, and \( g_j(\mathbf{x}, \xi) \) is the constraint function for \( j = 1, 2, \cdots, p \). Since the objective function \( f(\mathbf{x}, \xi) \) is also a fuzzy variable, it cannot be minimized directly. Instead, we may minimize its expected value, i.e., \( E[f(\mathbf{x}, \xi)] \). Besides, considering that the fuzzy constraints \( g_j(\mathbf{x}, \xi) \leq 0, j = 1, 2, \cdots, p \), do not define a crisp feasible set, it is naturally desired that the fuzzy constraints hold with confidence levels \( \alpha_1, \alpha_2, \cdots, \alpha_p \). Then we have a set of chance constraints as follows,
\[
\text{Cr}\{g_j(\mathbf{x}, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p.
\] (65)

Therefore, in order to formulate decision systems with fuzzy parameters, we present the following fuzzy programming model,
\[
\begin{align*}
\min_{\mathbf{x}} & \quad E[f(\mathbf{x}, \xi)] \\
\text{subject to :} & \\
\text{Cr}\{g_j(\mathbf{x}, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p.
\end{align*}
\] (66)
The target of model (66) is to obtain a decision with the minimum expected objective value $E[f(x, \xi)]$ subject to a series of chance constraints.

**Definition 9.** A vector $x$ is called a feasible solution to the fuzzy programming model (66) if

$$\text{Cr}\{g_j(x, \xi) \leq 0\} \geq \alpha_j$$

holds for $j = 1, 2, \ldots, p$.

**Definition 10.** A feasible solution $x^*$ is called an optimal solution to the fuzzy programming model (66) if

$$E[f(x^*, \xi)] \leq E[f(x, \xi)]$$

holds for any feasible solution $x$.

### 6.2. Crisp equivalent model

For the fuzzy programming model (66), if the fuzzy vector $\xi$ consists of LR fuzzy numbers, a crisp equivalent form can be obtained by using the following theorems.

**Theorem 7.** Assume that the objective function $f(x, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent LR fuzzy numbers, then the expected objective function $E[f(x, \xi_1, \xi_2, \ldots, \xi_n)]$ in model (66) equals to

$$\int_0^1 f(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) d\alpha$$

where $\Phi_i^{-1}$ is the inverse credibility distribution of $\xi_i$ for $i = 1, 2, \ldots, n$.

**Proof:** It follows from Theorem 3 that the inverse credibility distribution of $f(x, \xi_1, \xi_2, \ldots, \xi_n)$ is

$$\Psi^{-1}(x, \alpha) = f(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Using Theorem 4, we have $E[f(x, \xi_1, \ldots, \xi_n)] = \int_0^1 \Psi^{-1}(x, \alpha) d\alpha$.

**Theorem 8.** Assume that the constraint function $g_j(x, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent LR fuzzy numbers, then the chance constraint

$$\text{Cr}\{g_j(x, \xi_1, \xi_2, \ldots, \xi_n) \leq 0\} \geq \alpha$$

holds if and only if

$$g_j(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) \leq 0$$

where $\Phi_i^{-1}$ is the inverse credibility distribution of $\xi_i$ for $i = 1, 2, \ldots, n$.  

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Proof: It follows from the operational law in Theorem 3 that the inverse credibility distribution of \( g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \) is

\[
\Psi^{-1}(x, \alpha) = g_j(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).
\] (73)

On the other hand, it is obvious that (71) holds if and only if \( \Psi^{-1}(x, \alpha) \leq 0 \).

Theorem 9. Assume that \( f(x, \xi_1, \xi_2, \cdots, \xi_n) \) is strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_m \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \cdots, \xi_n \), and \( g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \) is strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_k \) and strictly decreasing with respect to \( \xi_{k+1}, \xi_{k+2}, \cdots, \xi_n \) for \( j = 1, 2, \cdots, p \). If \( \xi_1, \xi_2, \cdots, \xi_n \) are independent LR fuzzy numbers, then the fuzzy programming model (66) is equivalent to the crisp mathematical programming

\[
\begin{align*}
\min_{x} & \quad \int_{0}^{1} f(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) d\alpha. \\
\text{subject to} : & \quad g_j(x, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \Phi_{k+1}^{-1}(1-\alpha_j), \cdots, \Phi_n^{-1}(1-\alpha_j)) \leq 0, \ j = 1, 2, \cdots, p
\end{align*}
\] (74)

where \( \Phi_i^{-1} \) is the inverse credibility distribution of \( \xi_i \) for \( i = 1, 2, \cdots, n \).

Proof: It follows from Theorems 7 and 8 immediately.

As a result, based upon Theorem 9, if the objective function \( f(x, \xi) \) is strictly monotone and \( \xi \) consists of independent LR fuzzy numbers, we can convert the fuzzy programming model (66) to the crisp model (74). After that, we may solve such type of fuzzy optimization problems within the framework of classic deterministic optimization requiring no particular solving techniques.

6.3. Numerical examples

In the following, a purchasing planning problem is given to illustrate the solution framework of fuzzy programming via the proposed fuzzy arithmetic operations.

Consider a company which plans to purchase some machines to build a new plant. This plant is to supply three types of components for its downstream plant in this company. Each type of component is produced by different machines, and thus three types of machines should be purchased. Denote by \( x_i \) the number of the \( i \)-th type of machine purchased for \( i = 1, 2, 3 \), respectively.

The price of the \( i \)-th type of machine is \( a_i \), and the total capital available for this procurement plan is \( a \). Then we have a constraint on the capital budgeting as

\[
a_1 x_1 + a_2 x_2 + a_3 x_3 \leq a.
\] (75)
Another constraint for this purchasing planning problem is the limitation of maximum space available for the machines. Denote by $b_i$ the space occupied by the $i$-th type of machine for $i = 1, 2, 3$, respectively, and by $b$ the total available space. Then we have the following constraint,

$$b_1 x_1 + b_2 x_2 + b_3 x_3 \leq b.$$  \hspace{1cm} (76)

The production capacity of the $i$-th type of machine is $\eta_i$, and the demand of the $i$-th type of component produced by the $i$-th type of machine from the downstream plant is $\xi_i$, $i = 1, 2, 3$. Since the demand should be fulfilled, that is, shortage is not allowed, we have $\eta_i x_i \geq \xi_i$, $i = 1, 2, 3$. In practice, the production capacity $\eta_i$ and the future demand $\xi_i$ are usually uncertain. Here we suppose that they are fuzzy variables. In this case, $\eta_i x_i \geq \xi_i$ does not define a crisp constraint. If the manager sets $\alpha_i$ as the confidence level to be achieved of meeting the demands of the $i$-the type of component, then we have the following chance constraints,

$$\text{Cr}\{\eta_i x_i \geq \xi_i\} \geq \alpha_i, \hspace{0.5cm} i = 1, 2, 3.$$  \hspace{1cm} (77)

Assume that the profit produced by per $i$-th type of machine is $\tau_i$, $i = 1, 2, 3$, then the total profit is $\tau_1 x_1 + \tau_2 x_2 + \tau_3 x_3$. The profits $\tau_i$, $i = 1, 2, 3$, are allocated by the company according to the sales of its final products, which are usually affected by season, competitors and other factors. Consequently, the profits are assumed as fuzzy variables in this paper, and the objective is to maximize the expected value of the total profit, i.e.,

$$E[\tau_1 x_1 + \tau_2 x_2 + \tau_3 x_3].$$  \hspace{1cm} (78)

In the end, we have the following integer programming model for this purchasing planning problem,

$$\begin{cases}
\text{max } E[\tau_1 x_1 + \tau_2 x_2 + \tau_3 x_3] \\
\text{subject to :} \\
\hspace{1cm} a_1 x_1 + a_2 x_2 + a_3 x_3 \leq a \\
\hspace{1cm} b_1 x_1 + b_2 x_2 + b_3 x_3 \leq b \\
\hspace{1cm} \text{Cr}\{\eta_i x_i \geq \xi_i\} \geq \alpha_i, \hspace{0.5cm} i = 1, 2, 3 \\
\hspace{1cm} x_i \text{ are nonnegative integers, } i = 1, 2, 3.
\end{cases}$$  \hspace{1cm} (79)

Assume that $\xi_i$, $\eta_i$ and $\tau_i$ are independent LR fuzzy numbers with credibility distribution $\Upsilon_i$, $\Psi_i$ and $\Phi_i$, respectively, $i = 1, 2, 3$. Then we can convert the above model (79) to the following
deterministic form according to Theorem 9,

\[
\begin{align*}
\max & \ x_1 \int_0^1 \Phi_1^{-1}(\alpha) \, d\alpha + x_2 \int_0^1 \Phi_2^{-1}(\alpha) \, d\alpha + x_3 \int_0^1 \Phi_3^{-1}(\alpha) \, d\alpha \\
\text{subject to :} & \\
& a_1 x_1 + a_2 x_2 + a_3 x_3 \leq a \\
& b_1 x_1 + b_2 x_2 + b_3 x_3 \leq b \\
& \Upsilon_i^{-1}(\alpha_i) - x_i \Psi_1^{-1}(1 - \alpha_i) \leq 0, \ i = 1, 2, 3 \\
& x_i \text{ are nonnegative integers, } i = 1, 2, 3,
\end{align*}
\]

which can be easily solved by classical numerical methods or intelligent algorithms.

For example, we apply model (80) to the data listed in Table 1, and obtain the following linear integer programming model,

\[
\begin{align*}
\max & \ 4.25 x_1 + 6.25 x_2 + 4.5 x_3 \\
\text{subject to :} & \\
& 5 x_1 + 6 x_2 + 4 x_3 - 600 \leq 0 \\
& 7 x_1 + 6 x_2 + 8 x_3 - 800 \leq 0 \\
& 204.8990 - 17.1283 x_1 \leq 0 \\
& 186.1237 - 25.2138 x_2 \leq 0 \\
& 207.3485 - 21.1711 x_3 \leq 0 \\
& x_i \text{ are nonnegative integers, } i = 1, 2, 3.
\end{align*}
\]

By using LINGO, the optimal total profit is obtained as 627.5, and the optimal solution is

\[
(x_1^*, x_2^*, x_3^*) = (12, 62, 42).
\]

Obviously, model (79) with parameters listed in Table 1 is hard to be solved by traditional approaches but fuzzy simulation (which has been suggested in [22]), since three types of fuzzy numbers, i.e., Gaussian fuzzy numbers, Cauchy fuzzy numbers, and triangular fuzzy numbers, simultaneously appear in the same model, which makes it more complicated. However, as shown in models (80) and (81), it can be transformed to a deterministic form and then be easily solved by classical methods following from the proposed solution framework of fuzzy programming with parameters of LR fuzzy numbers.
Table 1: Parameter values of the purchasing planning problem

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_1$</td>
<td>$N(20,3)$</td>
<td>$\xi_1$</td>
<td>$C(200,4)$</td>
<td>$\tau_1$</td>
<td>$T(3,4,6)$</td>
</tr>
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<td>$N(30,5)$</td>
<td>$\xi_2$</td>
<td>$C(180,5)$</td>
<td>$\tau_2$</td>
<td>$T(5,6,8)$</td>
</tr>
<tr>
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<td>$N(25,4)$</td>
<td>$\xi_3$</td>
<td>$C(210,6)$</td>
<td>$\tau_3$</td>
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<td>$b$</td>
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</tr>
</tbody>
</table>

7. Conclusion

In this paper, we mainly concentrated on the fuzzy arithmetic for LR fuzzy numbers. The major results of this study include the following aspects: 1) the notion of LR fuzzy number was defined, and it is proved that a fuzzy number is an LR fuzzy number if and only if it has a regular credibility distribution; 2) an operational law on independent LR fuzzy numbers was proposed for fuzzy arithmetic; 3) an equivalent definition of the expected value operator and a theorem for calculating the expected value of strictly monotone functions were represented; 4) a solution framework of fuzzy programming was constructed based on the proposed operational law.

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References


